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Hardy–Littlewood–Sobolev and Stein–Weiss inequalities and integral systems on the Heisenberg group

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ABSTRACT

In this paper, we study two types of weighted Hardy–Littlewood–Sobolev (HLS) inequalities, also known as Stein–Weiss inequalities, on the Heisenberg group. More precisely, we prove the $|u|$ weighted HLS inequality in Theorem 1.1 and the $|z|$ weighted HLS inequality in Theorem 1.5 (where we have denoted $u = (z, t)$ as points on the Heisenberg group). Then we provide regularity estimates of positive solutions to integral systems which are Euler–Lagrange equations of the possible extremals to the Stein–Weiss inequalities. Asymptotic behavior is also established for integral systems associated to the $|u|$ weighted HLS inequalities around the origin. By these a priori estimates, we describe asymptotically the possible optimizers for sharp versions of these inequalities.

1. Introduction

The n-dimensional Heisenberg group is $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ with group structure given by

$$uv = (z, t)(z', t') = (z + z', t + t' + 2Im(z \cdot \bar{z}))$$

for any two points $u = (z, t)$, $v = (z', t') \in \mathbb{H}^n$, where $z, z' \in \mathbb{C}^n$, $t, t' \in \mathbb{R}$ and $z \cdot \bar{z} = \sum_{j=1}^n z_j \overline{z_j'}$. Haar measure on $\mathbb{H}^n$ is the Lebesgue measure $du = dzdt$, in which $z = x + iy$ with $x, y \in \mathbb{R}^n$.

The Lie algebra of $\mathbb{H}^n$ is generated by the left invariant vector fields

$$T = \frac{\partial}{\partial t}, \qquad X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}.$$

For each real number $d \in \mathbb{R}$, we denote the dilation $\delta_d u = \delta_d(z, t) = (dz, d^2 t)$, the homogeneous norm on $\mathbb{H}^n$ as $|u| = |(z, t)| = (|z|^4 + t^2)^{1/4}$, and $Q = 2n + 2$ as the homogeneous dimension.

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Now we recall the famous Hardy–Littlewood–Sobolev inequality on $\mathbb{R}^N$. Let $1 < r, s < \infty$ and $0 < \lambda < N$ such that $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$, then

$$\left| \int\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\overline{f(x)} g(y)}{|x-y|^\lambda} dxdy \right| \le C_{r,\lambda,N} \|f\|_r \|g\|_s, \tag{1.1}$$

for any $f \in L^r(\mathbb{R}^N)$ and $g \in L^s(\mathbb{R}^N)$, where $\|\cdot\|_r$ and $\|\cdot\|_s$ are the L^r and L^s norms on $\mathbb{R}^N$, respectively. And $0 < C_{r,\lambda,N} < \infty$ is a constant depending on r, λ and N only.

This inequality was introduced by Hardy and Littlewood in $\mathbb{R}^1$ [1–3] and generalized by Sobolev [4] to $\mathbb{R}^N$. However, none of them is in its sharp form. Namely, neither the sharp constant $C_{r,\lambda,N}$ nor the extremal function such that the inequality (1.1) holds with the sharp constant was known. For a special case when $r = s = 2N/(2N - \lambda)$, Lieb [5] gave the sharp version of (1.1), i.e., he proved the inequality with sharp (best) constant $C_{\lambda,r,N}$ and showed its optimizers (functions for which the equality (1.1) holds with the smallest constant $C_{\lambda,r,N}$). For general case when $r \neq s$, neither sharp constant $C_{r,\lambda,N}$ nor optimizers[1] are known yet. See more details[2] about Hardy–Littlewood–Sobolev inequality in $\mathbb{R}^N$, we refer the reader to Lieb and Loss's monograph [6]. Throughout this paper, the Hardy–Littlewood–Sobolev inequalities are simply denoted as HLS inequalities.

In 1950s, Stein and Weiss introduced the weighted HLS inequality in [8], that is,

$$\left| \int\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\overline{f(x)} g(y)}{|x|^\alpha |x-y|^\lambda |y|^\beta} dxdy \right| \le C_{\alpha,\beta,r,\lambda,N} \|f\|_r \|g\|_s, \tag{1.2}$$

where $1 < r, s < \infty$, $0 < \lambda < N$ and $\alpha + \beta \ge 0$ such that $\lambda + \alpha + \beta \le N$, $\alpha < N/r'$, $\beta < N/s'$ and $\frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{N} = 2$ (where we have used the notation: For an index $1 \le r \le \infty$, we let r' denote its conjugate index, that is, $r + r' = rr'$.) The sharp constant in the Stein–Weiss inequality (1.2) is still unknown as far as we are aware of, even in the special case when $r = s$. When $\lambda = N - 2$, the Euler–Lagrange system of (1.2) consists of two Poisson's equations. In Chen and Li's paper [9], they studied the integral systems in this case. In the work of Caristi, D'Ambrosio and E. Mitidieri [10], they also studied the integral systems (inequalities) associated with the Stein–Weiss inequalities and nonexistence of solutions to such systems. We refer the reader to their work and the references therein.

Much less is known on the Heisenberg group $\mathbb{H}^n$. With the settings on the Heisenberg group $\mathbb{H}^n$ introduced in the beginning, we shall move our attention to the analogous weighted HLS inequality, namely the Stein–Weiss inequality, on $\mathbb{H}^n$, and consider both $|u|$ weights (Theorem 1.1) and $|z|$ weights (Theorem 1.5).

Our first result in this paper is the following $|u|$ weighted HLS inequality on $\mathbb{H}^n$.

Theorem 1.1 (*$|u|$ Weighted HLS Inequality*). *For $1 < r, s < \infty$, $0 < \lambda < Q = 2n+2$ and $\alpha + \beta \ge 0$ such that $\lambda + \alpha + \beta \le Q$, $\alpha < Q/r'$, $\beta < Q/s'$, and $\frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{Q} = 2$, there exists a positive constant $C_{\alpha,\beta,r,\lambda,n}$ independent of the functions f and g such that*

$$\left| \int\int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\overline{f(u)} g(v)}{|u|^\alpha |u^{-1}v|^\lambda |v|^\beta} dudv \right| \le C_{\alpha,\beta,r,\lambda,n} \|f\|_r \|g\|_s. \tag{1.3}$$

Here $u = (z, t)$ and $v = (z', t')$, $u^{-1} = (-z, -t)$ and $d(u, v) := |u^{-1}v| = |v^{-1}u|$ is a left-invariant metric.

Remark. It is easy to see if the above inequality (1.3) holds with a finite constant $C_{\alpha,\beta,r,\lambda,n}$ independent of the functions f and g, then $\frac{1}{r} + \frac{1}{s} + \frac{\lambda+\alpha+\beta}{Q} = 2$ must hold.

Finding and studying sharp constants $C_{\alpha,\beta,r,\lambda,n}$ and its optimizers have attracted a great attention of many people. The non-weighted version of the inequality of (1.3) (i.e., $\alpha = \beta = 0$) was proved by Folland and Stein [11] in terms of fractional integral (Proposition 8.7 and Lemma 15.3 in [11]).

$$\left| \int\int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\overline{f(u)} g(v)}{|u^{-1}v|^\lambda} dudv \right| \le C_{r,\lambda,n} \|f\|_r \|g\|_s. \tag{1.4}$$

In conjunction with the CR Yamabe problem on the CR manifolds (see [12,13]), Jerison and Lee [13] proved the sharp version and gave the optimizer of (1.4) for $\lambda = Q - 2$ and $r = s = 2Q/(2Q - \lambda) = 2Q/(Q + 2)$. This is equivalent to the sharp constant and extremal problem of the L^2 to $L^{\frac{2Q}{Q-2}}$ Sobolev inequality on the Heisenberg group. It is worth mentioning

[1] They are also referred as maximizers or extremals in the literature.

[2] Lieb used rearrangement method in [5], and recently, Frank and Lieb developed a new method to prove the sharp version of the same special case of (1.1) ($r = s = 2N/(2N - \lambda)$) without using rearrangement, see [7].

that the sharp constant and extremal problem for general $1 \le p < Q$ is still widely open. Nevertheless, the sharp constant for the borderline case when $p = Q$ was established by Cohn and the second author [14], namely the sharp Moser–Trudinger inequality on domains of finite measure.

Recently, Lam and the second author have established in [15] the sharp Moser–Trudinger inequality on the whole Heisenberg group: There exists a positive constant

$$\alpha_Q = Q\left[\frac{2\pi^n \Gamma(\frac{1}{2})\Gamma(\frac{Q-1}{2})}{\Gamma(Q2)\Gamma(n)}\right]^{Q'-1}$$

such that for any pair β and α satisfying $0 \le \beta < Q, 0 < \alpha \le \alpha_Q(1-\frac{\beta}{Q})$ there holds

$$\sup_{\|f\|_{1,\tau}\le 1}\int_{\mathbb{H}^n}\frac{1}{|u|^\beta}\left[\exp(\alpha|f(u)|^{Q/(Q-1)}) - \sum_{k=0}^{Q-2}\frac{\alpha^k}{k!}|f(u)|^{kQ/(Q-1)}\right]du \le C_{\beta,\tau,Q} < \infty.$$

The constant $\alpha_Q(1-\frac{\beta}{Q})$ is best possible in the sense that the supremum is infinite if $\alpha > \alpha_Q(1-\frac{\beta}{Q})$. Here τ is any positive number, and

$$\|f\|_{1,\tau} = \left[\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n} f|^Q + \tau\int_{\mathbb{H}^n}|f|^Q\right]^{1/Q}.$$

The above result of [15] sharpened the Moser–Trudinger inequality on unbounded domains of [16] where such inequality was studied for the subcritical case $\alpha < \alpha_Q(1-\frac{\beta}{Q})$.

Returning to the sharp HLS inequality on the Heisenberg group, Frank and Lieb [17] have succeeded in extending the result of Jerison and Lee [13] to all $0 < \lambda < Q$. We state the result of Frank and Lieb in [17] as the following theorem.

Theorem 1.2 (*Frank and Lieb, Theorem 2.1 in [17]*). *Let $0 < \lambda < Q$ and $r = 2Q/(2Q-\lambda)$. Then for any $f, g \in L^r(\mathbb{H}^n)$,*

$$\left|\int\int_{\mathbb{H}^n\times\mathbb{H}^n}\frac{\overline{f(u)}g(v)}{|u^{-1}v|^\lambda}dudv\right| \le \left(\frac{\pi^{n+1}}{2^{n-1}n!}\right)^{\frac{\lambda}{Q}}\frac{n!\Gamma((Q-\lambda)/2)}{\Gamma^2((2Q-\lambda)/4)}\|f\|_r\|g\|_r,$$

with equality if and only if

$$f(u) = cH(\delta(a^{-1}u)), \qquad g(v) = c'H(\delta(a^{-1}v))$$

for some $c, c' \in \mathbb{C}$, $\delta > 0$, $a \in \mathbb{H}^n$ (unless $f \equiv 0$ or $g \equiv 0$), and

$$H = [(1+|z|^2)^2 + t^2]^{-\frac{2Q-\lambda}{4}}.$$

Their results also justified Branson et al. natural guess in [18] about the optimizer H. However, little about sharp constants and optimizers has been known so far when $r \ne s$ in (1.4). Therefore, it is widely open for the sharp constants and extremal functions for the weighted HLS inequalities, or also known as the Stein–Weiss inequalities (1.3) on the Heisenberg group $\mathbb{H}^n$. Recently, the first author has established in [19] the existence of extremal functions of the HLS inequality (1.4) on the Heisenberg group in all the case of r and s using the concentrated compactness of Lions [20,21].

Our second main purpose of this paper is to initiate the investigation of the weighted HLS inequalities (1.3) on $\mathbb{H}^n$ by studying the regularity estimates and asymptotic behavior of solutions to the Euler–Lagrange equations to the Stein–Weiss inequalities.

Let

$$J(f,g) = \int\int_{\mathbb{H}^n\times\mathbb{H}^n}\frac{\overline{f(u)}g(v)}{|u|^\alpha|u^{-1}v|^\lambda|v|^\beta}dudv. \tag{1.5}$$

Maximizing the functional $J(f,g)$ in (1.5) under the constraints that $\|f\|_r = \|g\|_s = 1$, one can derive the Euler–Lagrange system of equations for $f, g \ge 0$ corresponding to (1.5),

$$\begin{cases}\lambda_1 r f(u)^{r-1} = \dfrac{1}{|u|^\alpha}\displaystyle\int_{\mathbb{H}^n}\frac{g(v)}{|v|^\beta|u^{-1}v|^\lambda}dv,\\ \lambda_2 s g(u)^{s-1} = \dfrac{1}{|u|^\beta}\displaystyle\int_{\mathbb{H}^n}\frac{f(v)}{|v|^\alpha|u^{-1}v|^\lambda}dv,\end{cases} \tag{1.6}$$

where $\lambda_1 r = \lambda_2 s = J(f, g)$. Setting $F(u) = c_1 f(u)^{r-1}$, $G(u) = c_2 G(u)^{s-1}$ and taking $p = \frac{1}{r-1}$ and $q = \frac{1}{s-1}$, and choosing c_1 and c_2 appropriately, (1.6) becomes

$$\begin{cases} F(u) = \dfrac{1}{|u|^{\alpha}} \displaystyle\int_{\mathbb{H}^n} \dfrac{G^q(v)}{|v|^{\beta}|u^{-1}v|^{\lambda}} dv, \\ G(u) = \dfrac{1}{|u|^{\beta}} \displaystyle\int_{\mathbb{H}^n} \dfrac{F^p(v)}{|v|^{\alpha}|u^{-1}v|^{\lambda}} dv, \end{cases} \tag{1.7}$$

where

$$\begin{cases} 0 < p, q < \infty,\ 0 < \lambda < Q,\ \alpha + \beta \geq 0, \\ \lambda + \alpha + \beta \leq Q,\ \dfrac{1}{p+1} > \dfrac{\alpha}{Q},\ \dfrac{1}{q+1} > \dfrac{\beta}{Q},\ \dfrac{1}{p+1} + \dfrac{1}{q+1} = \dfrac{\lambda + \alpha + \beta}{Q}. \end{cases} \tag{1.8}$$

Remark. Since $\alpha + \beta \geq 0$ in (1.8), we shall consider two cases that $\alpha, \beta \geq 0$ and $\alpha \geq 0, \beta \leq 0$ without loss of generality.

We are interested in regularity estimates and asymptotic behavior of the solutions F and G of (1.7) and (1.8), which give characterization of the possible optimizers for (1.3). The systems of integral equations are of independent interest as well, e.g., the system when $\lambda = N - 2$ corresponds to "weighted" Lane–Emden system in critical case. See Li and Lim [22] for more information about these systems in Euclidean space and the singularity analysis they carried out.

For the reader's convenience, we will abuse the notations and use f, g to replace F and G in (1.7) respectively. Our main regularity estimate theorem about solutions (f, g) to the Euler–Lagrange equations (1.7) to $|u|$ weighted HLS inequality states

Theorem 1.3 (*Regularity Estimates for $|u|$ Weighted HLS Inequality*). *Let $(f, g) \in L^{p+1}(\mathbb{H}^n) \times L^{q+1}(\mathbb{H}^n)$ be a pair of positive solutions of the system (1.7) and (1.8). Suppose that $p, q > 1$ and denote $\overline{\lambda} = \lambda + \alpha + \beta$.*

(**1.3.i**) *If $\alpha, \beta \geq 0$, then $(f, g) \in L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$ for all r and s such that*

$$\frac{1}{r} \in \left(\frac{\alpha}{Q}, \frac{\lambda + \alpha}{Q}\right) \cap \left(\frac{\beta}{Q} - \frac{1}{q+1} + \frac{1}{p+1}, \frac{\lambda + \beta}{Q} - \frac{1}{q+1} + \frac{1}{p+1}\right)$$

and

$$\frac{1}{s} \in \left(\frac{\beta}{Q}, \frac{\lambda + \beta}{Q}\right) \cap \left(\frac{\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1}, \frac{\lambda + \alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1}\right).$$

(**1.3.ii**) *If $\alpha \geq 0$ and $\beta \leq 0$, then $(f, g) \in L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$ for all r and s such that*

$$\frac{1}{r} \in \left(\frac{\alpha}{Q}, \frac{\overline{\lambda}}{Q}\right) \cap \left(-\frac{1}{q+1} + \frac{1}{p+1}, \frac{\lambda + \beta}{Q} - \frac{1}{q+1} + \frac{1}{p+1}\right)$$

and

$$\frac{1}{s} \in \left(0, \frac{\lambda + \beta}{Q}\right) \cap \left(\frac{\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1}, \frac{\overline{\lambda}}{Q} - \frac{1}{p+1} + \frac{1}{q+1}\right).$$

Remark. The above regularity estimate is an "extrapolation" theorem, take (**1.3.i**) for instance, it asserts that if $(f, g) \in L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$ for one pair of $(r, s) \in (\frac{\alpha}{Q}, \frac{\lambda+\alpha}{Q}) \times (\frac{\beta}{Q}, \frac{\lambda+\beta}{Q})$, then $(f, g) \in L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$ for all pairs of $(r, s) \in (\frac{\alpha}{Q}, \frac{\lambda+\alpha}{Q}) \times (\frac{\beta}{Q}, \frac{\lambda+\beta}{Q})$ (with proper restrictions of course).

While our main asymptotic behavior theorem about optimizers of $|u|$ weighted HLS inequality states

Theorem 1.4 (*Asymptotic Behavior for $|u|$ Weighted HLS Inequality*). *Let $(f, g) \in L^{p+1}(\mathbb{H}^n) \times L^{q+1}(\mathbb{H}^n)$ be a pair of positive solutions of the system (1.7) and (1.8). Suppose that $p, q > 1$, then if $|u| \sim 0$,*

(**1.4.i**)

$$f(u) \sim \begin{cases} \dfrac{A_1}{|u|^{\alpha}} & \text{if } \lambda + \beta(q+1) < Q, \\ \dfrac{A_2 |\ln |u\,||}{|u|^{\alpha}} & \text{if } \lambda + \beta(q+1) = Q, \\ \dfrac{A_3}{|u|^{\beta(q+1)+\alpha+\lambda-Q}} & \text{if } \lambda + \beta(q+1) > Q, \end{cases}$$

where $A_1 = \int_{\mathbb{H}^n} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv$, $A_2 = |\Sigma_1| (\int_{\mathbb{H}^n} \frac{f^p(v)}{|v|^{\lambda+\alpha}} dv)^q$ and $A_3 = (\int_{\mathbb{H}^n} \frac{f^p(v)}{|v|^{\lambda+\alpha}} dv)^q \int_{\mathbb{H}}^n \frac{1}{|v|^{\beta(q+1)}|e^{-1}v|^{\lambda}} dv$.

(**1.4.ii**)

$$g(u) \sim \begin{cases} \dfrac{B_1}{|u|^{\beta}} & \text{if } \lambda+\alpha(p+1)<Q, \\ \dfrac{B_2|\ln|u\,||}{|u|^{\beta}} & \text{if } \lambda+\alpha(p+1)=Q, \\ \dfrac{B_3}{|u|^{\alpha(p+1)+\beta+\lambda-Q}} & \text{if } \lambda+\alpha(p+1)>Q, \end{cases}$$

where $B_1=\int_{\mathbb{H}^n}\frac{f^p(v)}{|v|^{\lambda+\alpha}}dv$, $B_2=|\Sigma_1|(\int_{\mathbb{H}^n}\frac{g^q(v)}{|v|^{\lambda+\beta}}dv)^p$ *and* $B_3=(\int_{\mathbb{H}^n}\frac{g^q(v)}{|v|^{\lambda+\beta}}dv)^p\int_{\mathbb{H}^n}\frac{1}{|v|^{\alpha(p+1)}|e^{-1}v|^{\lambda}}dv$.

Remark. Combining (1.8) with further conditions $p, q > 1$, we have $\overline{\lambda} < Q$ and either $\lambda+\beta(q+1) < Q$ or $\lambda+\alpha(p+1) < Q$.

We also refer the reader to [23] for asymptotic analysis for weighted HLS inequality in Euclidean spaces.

Let us switch our attention to the weighted HLS inequalities with different weights, i.e., $|z|$ weights. More precisely,

Theorem 1.5 ($|z|$ *Weighted HLS Inequality*). *For* $1 < r, s < \infty$, $0 < \lambda < Q = 2n+2$ *and* $0 \le \alpha+\beta \le n\lambda$ *such that* $\lambda+\alpha+\beta \le Q$, $\alpha < 2n/r'$, $\beta < 2n/s'$ *and* $\frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{Q}=2$,

$$\left|\int\int_{\mathbb{H}^n\times\mathbb{H}^n}\frac{\overline{f(u)}g(v)}{|z|^{\alpha}|u^{-1}v|^{\lambda}|z'|^{\beta}}dudv\right| \le C_{\alpha,\beta,r,\lambda,n}\|f\|_r\|g\|_s. \tag{1.9}$$

Here, $u=(z,t)$ *and* $v=(z',t')$.

Remark. 1. Comparing the conditions in Theorems 1.1 and 1.5, the reason why we need stronger assumptions on α and β as $\alpha < 2n/r'$ and $\beta < 2n/s'$ is explained in Section 2, and $\alpha+\beta \le n\lambda$ is a direct consequence of $\alpha < 2n/r'$, $\beta < 2n/s'$ and $\frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{Q}=2$.

2. It is not hard to deduce that if the above inequality (1.9) holds with a finite constant $C_{\alpha,\beta,r,\lambda,n}$ independent of the functions f and g, then $\frac{1}{r}+\frac{1}{s}+\frac{\lambda+\alpha+\beta}{Q}=2$ must hold.

For the above weighted HLS inequality with $|z|$ weights on the Heisenberg group, Beckner gave the sharp constant (Theorem 3 in [24]) when $\alpha=\beta=(Q-\lambda)/2$ and $r=s=2$, and proved nonexistence of optimizers under these same conditions. However, it is completely *open* in other general cases including existence questions.

One can also derive the Euler–Lagrange system associated to the $|z|$ weighted inequality (1.9), and it can be formulated after renormalization as we did in (1.7)

$$\begin{cases} f(u)=\dfrac{1}{|z|^{\alpha}}\displaystyle\int_{\mathbb{H}^n}\frac{g^q(v)}{|z'|^{\beta}|u^{-1}v|^{\lambda}}dv, \\ g(u)=\dfrac{1}{|z|^{\beta}}\displaystyle\int_{\mathbb{H}^n}\frac{f^p(v)}{|z'|^{\alpha}|u^{-1}v|^{\lambda}}dv, \end{cases} \tag{1.10}$$

where

$$\begin{cases} 0<p,q<\infty,\ 0<\lambda<Q,\ 0\le\alpha+\beta\le n\lambda, \\ \lambda+\alpha+\beta\le Q,\ \dfrac{1}{p+1}>\dfrac{\alpha}{2n},\ \dfrac{1}{q+1}>\dfrac{\beta}{2n},\ \dfrac{1}{p+1}+\dfrac{1}{q+1}=\dfrac{\lambda+\alpha+\beta}{Q}, \\ u=(z,t),\qquad v=(z',t'). \end{cases} \tag{1.11}$$

The regularity estimates of solutions of (1.10) and (1.11) are as follows.

Theorem 1.6 (*Regularity Estimates for* $|z|$ *Weighted HLS Inequality*). *Let* $(f,g)\in L^{p+1}(\mathbb{H}^n)\times L^{q+1}(\mathbb{H}^n)$ *be a pair of positive solutions of the system* (1.10) *and* (1.11). *Suppose that* $p, q > 1$.

(**1.6.i**) *If* $\alpha,\beta \ge 0$, *then* $(f,g)\in L^r(\mathbb{H}^n)\times L^s(\mathbb{H}^n)$ *for all r and s such that*

$$\frac{1}{r}\in\left(\frac{\alpha}{2n},\frac{\overline{\lambda}}{Q}-\frac{\beta}{2n}\right)\cap\left(\frac{\beta}{2n}-\frac{1}{q+1}+\frac{1}{p+1},\frac{\overline{\lambda}}{Q}-\frac{\alpha}{2n}-\frac{1}{q+1}+\frac{1}{p+1}\right)$$

and

$$\frac{1}{s}\in\left(\frac{\beta}{2n},\frac{\overline{\lambda}}{Q}-\frac{\alpha}{2n}\right)\cap\left(\frac{\alpha}{2n}-\frac{1}{p+1}+\frac{1}{q+1},\frac{\overline{\lambda}}{Q}-\frac{\beta}{2n}-\frac{1}{p+1}+\frac{1}{q+1}\right).$$

(**1.6.ii**) *If $\alpha \geq 0$ and $\beta \leq 0$, then $(f, g) \in L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$ for all r and s such that*

$$\frac{1}{r} \in \left(\frac{\alpha}{2n}, \frac{\overline{\lambda}}{Q}\right) \cap \left(-\frac{1}{q+1} + \frac{1}{p+1}, \frac{\overline{\lambda}}{Q} - \frac{\alpha}{2n} - \frac{1}{q+1} + \frac{1}{p+1}\right)$$

and

$$\frac{1}{s} \in \left(0, \frac{\overline{\lambda}}{Q} - \frac{\alpha}{2n}\right) \cap \left(\frac{\alpha}{2n} - \frac{1}{p+1} + \frac{1}{q+1}, \frac{\overline{\lambda}}{Q} - \frac{1}{p+1} + \frac{1}{q+1}\right).$$

Finally, we make the following remarks. All of Theorems 1.1, 1.3 and 1.4, namely the $|u|$ weighted inequality (1.3), and the regularity estimates and asymptotic behavior for solutions to the Euler–Lagrange equations to the $|u|$ weighted inequality hold in the more general setting of stratified groups (we refer to [25,26] for an introduction of stratified groups). The proofs of these theorems given in this paper work out without any substantial changes on the stratified groups. Nevertheless, Theorems 1.5 and 1.6 for $|z|$ weighted HLS inequalities do involve the partial weight $|z|$ and have to be formulated carefully in the general settings. We should not discuss in details here.

We organize this paper as follows: In Section 2, we prove two weighted HLS inequalities in Theorems 1.1 and 1.5. We then establish regularity estimates Theorems 1.3 and 1.6 in Section 3, while asymptotic behavior Theorem 1.4 is proved in Section 4. We shall mention that although we focus on verifying $|u|$ weighted HLS inequality and related theories, $|z|$ weighted version can be carried out mutatis mutandis in most of the discussion, and essential difference between them is pointed out when it is necessary.

2. Proofs of weighted HLS inequalities (Stein–Weiss inequalities)

In this section, we prove two types of weighted HLS inequalities, that is, Theorems 1.1 and 1.5. One is with the weight of the form of the power of $|u|$, the other one is with the weight of the form of power of $|z|$. We begin with $|u|$ weighted version, and state two alternative recordings of Theorem 1.1, thus we can prove and apply them by our convenience.

Theorem 2.1. *Let $1 < p \leq q < \infty, 0 < \lambda < Q = 2n+2$ and $\alpha+\beta \geq 0$ such that $\alpha < Q/q, \beta < Q/p'$ and $\frac{1}{q} = \frac{1}{p} + \frac{\lambda+\alpha+\beta}{Q} - 1$, then*

$$\|Sf\|_q \leq C\|f\|_p,$$

in which $C = C_{\alpha,\beta,p,\lambda,n}$ is independent of f, and

$$Sf(u) = S_{\lambda,\alpha,\beta}f(u) = \int_{\mathbb{H}^n} \frac{f(v)dv}{|u|^{\alpha}|u^{-1}v|^{\lambda}|v|^{\beta}}.$$

Before we introduce the other equivalent theorem, let us set up the notations. We define the $|u|$ weighted L^p norm as

$$\|f\|_{L^p_{\gamma}} = \left(\int_{\mathbb{H}^n} |f(u)|^p|u|^{\gamma}du\right)^{\frac{1}{p}},$$

and $L^p_{\gamma}(\mathbb{H}^n)$ as the space of all the measurable functions with finite L^p_{γ} norm for all $\gamma \in \mathbb{R}$.

Theorem 2.2. *Let $1 < p \leq q < \infty, 0 < \lambda < Q = 2n+2$ and $\alpha+\beta \geq 0$ such that $\alpha < Q/q, \beta < Q/p'$ and $\frac{1}{q} = \frac{1}{p} + \frac{\lambda+\alpha+\beta}{Q} - 1$, then*

$$\|Tf\|_{L^q_{-\alpha q}} \leq C\|f\|_{L^p_{\beta p}}, \tag{2.1}$$

in which $C = C_{\alpha,\beta,p,\lambda,n}$ is independent of f, and

$$Tf(u) = T_{\lambda}f(u) = \int_{\mathbb{H}^n} \frac{f(v)dv}{|u^{-1}v|^{\lambda}}. \tag{2.2}$$

Thus (2.1) becomes $\||u|^{-\alpha}\, Tf(u)\|_q \leq C\||u|^{\beta}f(u)\|_p$, or

$$\left(\int_{\mathbb{H}^n} |Tf(u)|^q|u|^{-\alpha q}du\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{H}^n} |f(u)|^p|u|^{\beta p}du\right)^{\frac{1}{p}}. \tag{2.3}$$

Remark. 1. Some basic computations and theorems on the Heisenberg group, including quasi-triangular inequality, volume of balls and integrability of certain integrals etc. can be found in [11], Chapter 1 in [26] (More generally, homogeneous groups are considered therein.) and Section 4 in [27]; Calculations using polar coordinates to evaluate some integrals can be done similarly to those used in [14].

2. The conditions $p \leq q$ and $\frac{1}{q} = \frac{1}{p} + \frac{\lambda+\alpha+\beta}{Q} - 1$ in Theorems 2.1 and 2.2 are equivalent with $\lambda + \alpha + \beta \leq Q$ and $\frac{1}{p} + \frac{1}{q'} + \frac{\lambda+\alpha+\beta}{Q} = 2$ in Theorem 1.1.
3. The analogous theorem in Euclidean space as Theorem 2.1 was first introduced by Hardy and Littlewood [1] in $\mathbb{R}^1$, and generalized by Stein and Weiss in [8] to $\mathbb{R}^n$.
4. The constants given in Theorems 2.1 and 2.2 above are obviously not sharp here. It is a challenging problem to find best constants to these inequalities. In $\mathbb{R}^n$, some work concerning the sharp bound has been done by Beckner [28], Eilertsen [29].
5. Lieb studied the optimizers of the (double) weighted fractional integral in Euclidean spaces (Theorem 5.1 in [5]), under a stronger assumption that $\alpha, \beta \geq 0$ instead of $\alpha + \beta \geq 0$. (The stronger condition is needed to use rearrangement method.) His results conclude that the optimizers exist when $p < q$, and do not exist when $p = q$.
6. Sawyer and Wheeden in [30] gave some results about (2.3) with general weights on both Euclidean and homogeneous spaces.

The equivalences between Theorems 1.1, 2.1 and 2.2 are not difficult to be verified, for reader's convenience, we provide a simple observation of the equivalence between Theorems 1.1 and 2.1 here.

Proposition 2.3. *Theorems 1.1 and 2.1 are equivalent.*

Proof of Proposition 2.3. Note that the conditions for indices are equivalent in both theorems by (1) in the remark above. To prove sufficiency, compute that

$$\begin{aligned}
\|Sf\|_q &\leq \sup_{\|g\|_{q'}=1} \left| \int_{\mathbb{H}^n} \overline{Sf(u)} g(u)du \right| \\
&= \sup_{\|g\|_{q'}=1} \left| \int\int_{\mathbb{H}^n\times\mathbb{H}^n} \frac{\overline{f(v)}g(u)}{|u|^\alpha |u^{-1}v|^\lambda |v|^\beta} dudv \right| \\
&\leq \sup_{\|g\|_{q'}=1} C\|f\|_p \|g\|_{q'} \\
&= C\|f\|_p.
\end{aligned}$$

To prove necessity, compute that

$$\begin{aligned}
\left| \int\int_{\mathbb{H}^n\times\mathbb{H}^n} \frac{\overline{f(u)}g(v)}{|u|^\alpha |u^{-1}v|^\lambda |v|^\beta} dudv \right| &= \left| \int_{\mathbb{H}^n} \overline{f(u)} \left(\int_{\mathbb{H}^n} \frac{g(v)dv}{|u|^\alpha |u^{-1}v|^\lambda |v|^\beta} \right) du \right| \\
&= \left| \int_{\mathbb{H}^n} \overline{f(u)} Sg(u)du \right| \\
&\leq \|f\|_r \|Sg\|_{r'} \\
&\leq C\|f\|_r \|g\|_s. \quad \square
\end{aligned}$$

We denote $K(u, v) = |u^{-1}v|^{-\lambda} = d(u, v)^{-\lambda}$, thus we get $Tf(u) = \int_{\mathbb{H}^n} K(u, v)f(v)dv$ in (2.1). $d(u, v)$ satisfies the triangular inequality,

$$d(u_1, u_3) \leq d(u_1, u_3) + d(u_3, u_2). \tag{2.4}$$

Then we state a lemma from Sawyer and Wheeden's paper (Theorem 4 in [30]), a simplified version is sufficient here to prove Theorem 2.2, thus Theorems 1.1 and 2.1 follow immediately. We shall mention that the original theorem in [30] is more general that it covers the cases for all the quasi-metric spaces with a doubling measure and properly equipped group structure.

Lemma 2.4 (*Sawyer and Wheeden*)**.** *The operator T defined in (2.2) is bounded from $L^p_{\beta p}(\mathbb{H}^n)$ to $L^q_{-\alpha q}(\mathbb{H}^n)$, if both of the following two statements are true.*

1. *There exists $\varepsilon > 0$ such that for any pair of balls B and B' with radius r and r' satisfying $B' \subseteq 4B$,*

$$\left(\frac{r'}{r}\right)^{Q-\varepsilon} \left(\frac{\varphi(B')}{\varphi(B)}\right) \leq C_\varepsilon. \tag{2.5}$$

2. *There exists $t > 1$ such that for any ball $B \subseteq \mathbb{H}^n$,*

$$\varphi(B)|B|^{\frac{1}{p'}+\frac{1}{q}} \left(\frac{1}{|B|}\int_B |u|^{-\alpha q t}du\right)^{\frac{1}{qt}} \left(\frac{1}{|B|}\int_B |u|^{-\beta p' t}du\right)^{\frac{1}{p't}} \leq C_t. \tag{2.6}$$

Here $\varphi(B) = \sup\{K(u, v)|u, v \in B, d(u, v) \geq 2^{-24}r\}$ for a ball $B \in \mathbb{H}^n$ with radius r. C_ε and C_t are two finite constants depending only on ε and t, respectively.

Next we apply the above lemma to verify Theorem 2.2, that is, we only need to show that conditions (2.5) and (2.6) of Lemma 2.4 are satisfied.

Proof of Theorem 2.2. To show (2.5), note that for $K(u, v) = |u^{-1}v|^{-\lambda} = d(u, v)^{-\lambda}$ here, $\varphi(B) = 2^{24\lambda}r^{-\lambda}$ and thus the left hand side of (2.5) becomes

$$\left(\frac{r'}{r}\right)^{Q-\varepsilon}\left(\frac{\varphi(B')}{\varphi(B)}\right) = \left(\frac{r'}{r}\right)^{Q-\lambda-\varepsilon}.$$

Since $0 < \lambda < Q$, we choose $\varepsilon > 0$ small such that $Q - \lambda - \varepsilon > 0$, then for $B' \subseteq 4B$,

$$\left(\frac{r'}{r}\right)^{Q-\lambda-\varepsilon} \le 4^{Q-\lambda-\varepsilon} := C_\varepsilon < \infty,$$

which shows (2.5).

To show (2.6), rewrite the left hand side of (2.6) as

$$\varphi(B)|B|^{\frac{1}{p'}+\frac{1}{q}}\left(\frac{1}{|B|}\int_B |u|^{-\alpha qt}du\right)^{\frac{1}{qt}}\left(\frac{1}{|B|}\int_B |u|^{-\beta p't}du\right)^{\frac{1}{p't}} = M_1 \times M_2 \times M_3,$$

in which

$$M_1 = \varphi(B)|B|^{\frac{1}{q}+\frac{1}{p'}} = Cr^{-\lambda}r^{Q(\frac{1}{q}+\frac{1}{p'})} = r^{\bar{\lambda}-\lambda},$$

since $\frac{1}{q} = \frac{1}{p} + \frac{\bar{\lambda}}{Q} - 1$ and $\frac{1}{q} + \frac{1}{p'} = \frac{\bar{\lambda}}{Q}$.

$$M_2 = \left(\frac{1}{|B|}\int_B |u|^{-\alpha qt}du\right)^{\frac{1}{qt}} \le \left(C\frac{r^{Q-\alpha qt}}{r^Q}\right)^{\frac{1}{qt}} = Cr^{-\alpha}, \tag{2.7}$$

if $\alpha qt < Q$, and

$$M_3 = \left(\frac{1}{|B|}\int_B |u|^{-\beta p't}du\right)^{\frac{1}{p't}} \le \left(C\frac{r^{Q-\beta p't}}{r^Q}\right)^{\frac{1}{p't}} = Cr^{-\beta}, \tag{2.8}$$

if $\beta p't < Q$. Because $\alpha < Q/q$ and $\beta < Q/p'$, we have $\min\{\frac{Q}{\alpha q}, \frac{Q}{\beta p'}\} > 1$, and it ensures the existence of t such that $1 < t < \min\{\frac{Q}{\alpha q}, \frac{Q}{\beta p'}\}$, thus $\alpha qt < Q$ and $\beta p't < Q$, furthermore $M_2 < \infty$ and $M_3 < \infty$. Then

$$\varphi(B)|B|^{\frac{1}{p'}+\frac{1}{q}}\left(\frac{1}{|B|}\int_B |u|^{-\alpha qt}du\right)^{\frac{1}{qt}}\left(\frac{1}{|B|}\int_B |u|^{-\beta p't}du\right)^{\frac{1}{p't}} = Cr^{\bar{\lambda}-\lambda-\alpha-\beta} = C_t < \infty,$$

which shows (2.6), then by Lemma 2.4, the proof is completed. □

To prove Theorem 1.5, we also introduce two equivalent recordings.

Theorem 2.5. *Let $1 < p \le q < \infty$, $0 < \lambda < Q = 2n + 2$ and $0 \le \alpha + \beta \le n\lambda$ such that $\alpha < 2n/q$, $\beta < 2n/p'$ and $\frac{1}{q} = \frac{1}{p} + \frac{\lambda+\alpha+\beta}{Q} - 1$, then*

$$\|\widetilde{S}f\|_q \le C\|f\|_p,$$

in which $C = C_{\alpha,\beta,p,\lambda,n}$ is independent of f, and

$$\widetilde{S}f(u) = \widetilde{S}_{\lambda,\alpha,\beta}f(u) = \int_{\mathbb{H}^n} \frac{f(v)dv}{|z|^\alpha|u^{-1}v|^\lambda|z'|^\beta}. \tag{2.9}$$

Recall T defined in (2.2), we have

$$\left(\int_{\mathbb{H}^n} |Tf(u)|^q|z|^{-\alpha q}du\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{H}^n} |f(u)|^p|z|^{\beta p}du\right)^{\frac{1}{p}}. \tag{2.10}$$

Here, $u = (z, t)$ and $v = (z', t')$. As a result, Theorem 1.5 follows.

Proof of Theorem 2.5. We only need to modify and verify (2.5) and (2.6) in the proof of Theorem 2.2. (2.5) stays the same as the kernel $K(u, v) = |u^{-1}v|^{-\lambda}$ does not change. Then, we need to verify the revised version of (2.6).

There exists $t > 1$ such that for any ball $B \subseteq \mathbb{H}^n$,

$$\varphi(B)|B|^{\frac{1}{p'}+\frac{1}{q}}\left(\frac{1}{|B|}\int_B |z|^{-\alpha qt}du\right)^{\frac{1}{qt}}\left(\frac{1}{|B|}\int_B |z|^{-\beta p't}du\right)^{\frac{1}{p't}} \le C_t. \tag{2.11}$$

Similarly we prove (2.11) as we did for (2.6) except that we need to replace (2.7) and (2.8) by computing the following

$$M_2 = \left(\frac{1}{|B|}\int_B |z|^{-\alpha qt}du\right)^{\frac{1}{qt}}$$

and

$$M_3 = \left(\frac{1}{|B|}\int_B |z|^{-\beta p't}du\right)^{\frac{1}{p't}}.$$

M_2 and M_3 are finite if we require $\alpha qt < 2n$ and $\beta p't < 2n$,[3] and these are guaranteed by the conditions $\alpha < 2n/q$ and $\beta < 2n/p'$. □

3. Regularity estimates

In this section, we prove regularity estimates in Theorems 1.3 and 1.6. Let us begin with a theorem of regularity lifting by contracting operators, that is, suppose V is a topological vector space with two extended norms,

$$\|\cdot\|_X, \|\cdot\|_Y : V \to [0, \infty],$$

let $X := \{v \in V : \|v\|_X < \infty\}$ and $Y := \{v \in V : \|v\|_Y < \infty\}$. The operator $T : X \to Y$ is said to be contracting if

$$\|Tf - Th\|_Y \le \eta\|f - h\|_X,$$

$\forall f, h \in X$ and some $0 < \eta < 1$. And T is said to be shrinking if

$$\|Tf\|_Y \le \theta\|f\|_X,$$

$\forall f \in X$ and some $0 < \theta < 1$.

Remark. It is obvious that for a linear operator T, these two conditions above are equivalent. Thus the following theorem is also true for linear shrinking operators.

Theorem 3.1 (*Regularity Lifting by Contracting Operators*)**.** *Let T be a contracting operator from X to itself and from Y to itself, and assume that X, Y are both complete. If $f \in X$, and there exists $g \in Z := X \cap Y$ such that $f = Tf + g$ in X, then $f \in Z$.*

We omit the proof here since it is easy and can be found in [31]. One can also find in [32] some application of Theorem 3.1 on integral equations associated with Bessel potentials.

In the following work of this section, the proof of Theorem 1.3 is divided into two subsections as in Section 3.1 ($\alpha, \beta \ge 0$) and in Section 3.2 ($\alpha \ge 0, \beta \le 0$), by applying the above regularity lifting. While we present the outline of the proof for Theorem 1.6 in Section 3.3.

3.1. Proof of Theorem 1.3 if $\alpha, \beta \ge 0$

For a fixed real number $a > 0$, define

$$g_a(u) = \begin{cases} g(u) & \text{if } |g(u)| > a, \text{ or } |u| > a, \\ 0 & \text{otherwise.} \end{cases}$$

Let $g_b(u) = g(u) - g_a(u)$, and similarly we define f_a and f_b, then g_b and f_b are uniformly bounded by a in $B_a(0)$ obviously. It is evident that $g_a \cdot g_b = 0$ and $g^t = (g_a + g_b)^t = g_a^t + g_b^t$ for all $t > 0$. Define the linear operator T_1,

$$T_1h(u) = \frac{1}{|u|^\alpha}\int_{\mathbb{H}^n} \frac{g_a^{q-1}(v)h(v)}{|v|^\beta|u^{-1}v|^\lambda}dv = S_{\lambda,\alpha,\beta}(g_a^{q-1}h)(u).$$

[3] See Cohn and Lu's computation of $\int_{|u|<1} |z|^{-t}dt$ for $t < 2n$ on p. 1574 in [14].

Since f satisfies (1.5), we have

$$\begin{aligned} f(u) &= \frac{1}{|u|^\alpha}\int_{\mathbb{H}^n}\frac{g^q(v)}{|v|^\beta|u^{-1}v|^\lambda}dv \\ &= \frac{1}{|u|^\alpha}\int_{\mathbb{H}^n}\frac{((g_a+g_b)^{q-1}g)(v)}{|v|^\beta|u^{-1}v|^\lambda}dv \\ &= \frac{1}{|u|^\alpha}\int_{\mathbb{H}^n}\frac{(g_a^{q-1}g+g_b^{q-1}(g_a+g_b))(v)}{|v|^\beta|u^{-1}v|^\lambda}dv \\ &= T_1g(u)+\frac{1}{|u|^\alpha}\int_{\mathbb{H}^n}\frac{g_b^q(v)}{|v|^\beta|u^{-1}v|^\lambda}dv, \end{aligned}$$

and $f = T_1g + F$, in which

$$F(u) = \frac{1}{|u|^\alpha}\int_{\mathbb{H}^n}\frac{g_b^q(v)}{|v|^\beta|u^{-1}v|^\lambda}dv = S_{\lambda,\alpha,\beta}(g_b^q)(u).$$

Similarly, we define

$$T_2h(u) = \frac{1}{|u|^\beta}\int_{\mathbb{H}^n}\frac{f_a^{p-1}(v)h(v)}{|v|^\alpha|u^{-1}v|^\lambda}dv = S_{\lambda,\beta,\alpha}(f_a^{p-1}h)(u)$$

and

$$G(u) = \frac{1}{|u|^\beta}\int_{\mathbb{H}^n}\frac{f_b^q(v)}{|v|^\alpha|u^{-1}v|^\lambda}dv = S_{\lambda,\beta,\alpha}(f_b^p)(u).$$

Then we have $g = T_2f + G$. Define the operator $T(h_1, h_2) = (T_1h_2, T_2h_1)$, equip the product space $L^{p+1}(\mathbb{H}^n) \times L^{q+1}(\mathbb{H}^n)$ with norm $\|(h_1, h_2)\|_{p+1,q+1} = \|h_1\|_{p+1} + \|h_2\|_{q+1}$, and $L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$ with norm $\|(h_1, h_2)\|_{r,s} = \|h_1\|_r + \|h_2\|_s$. It is easy to see they are both complete under these norms respectively.

Thus we immediately observe that (f, g) solves the equation $(h_1, h_2) = T(h_1, h_2) + (F, G)$. In order to apply regularity lifting by contracting operators (Theorem 3.1), we fix the indices r and s satisfying

$$\frac{1}{r}-\frac{1}{s}=\frac{1}{p+1}-\frac{1}{q+1} \tag{3.1}$$

Note that the interval conditions in (**1.3.i**) of Theorem 1.3 guarantee the existence of such pairs (r, s). Then to arrive at the conclusion that $(f, g) \in L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$, we need to verify the following conditions, for sufficiently large a. (Here T is linear, by the remark above we only need to verify that it is shrinking.)

(1) T is shrinking from $L^{p+1}(\mathbb{H}^n) \times L^{q+1}(\mathbb{H}^n)$ to itself.
(2) T is shrinking from $L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$ to itself.
(3) $(F, G) \in L^{p+1}(\mathbb{H}^n) \times L^{q+1}(\mathbb{H}^n) \cap L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$, i.e., $F \in L^{p+1}(\mathbb{H}^n) \cap L^r(\mathbb{H}^n)$ and $G \in L^{q+1}(\mathbb{H}^n) \cap L^r(\mathbb{H}^n)$.

(1) T is shrinking from $L^{p+1}(\mathbb{H}^n) \times L^{q+1}(\mathbb{H}^n)$ to itself.

First, we show that $\|T_1h\|_{p+1} \le \frac{1}{2}\|h\|_{q+1}$ for all $h \in L^{q+1}(\mathbb{H}^n)$. Observe that $\alpha < Q/(p+1)$, $\beta < Q/(q+1) = Q/(\frac{q+1}{q})'$ from (1.6), and

$$\frac{q}{q+1}+\frac{\overline{\lambda}}{Q}-1=\frac{\overline{\lambda}}{Q}-\frac{1}{q+1}=\frac{1}{p+1}.$$

By $|u|$ weighted fractional integral inequality in Theorem 2.1, together with Hölder inequality,

$$\begin{aligned} \|T_1h\|_{p+1} &= \|S_{\lambda,\alpha,\beta}(g_a^{q-1}h)\|_{p+1} \\ &\le C\|g_a^{q-1}h\|_{\frac{q+1}{q}} \\ &\le C\|g_a^{q-1}\|_{\frac{q+1}{q-1}}\|h\|_{q+1} \\ &= C\|g_a\|_{q+1}^{q-1}\|h\|_{q+1}, \end{aligned}$$

in which we choose a sufficiently large that $C\|g_a\|_{q+1}^{q-1} \le \frac{1}{2}$, since $g \in L^{q+1}(\mathbb{H}^n)$. Thus $\|T_1h\|_{p+1} \le \frac{1}{2}\|h\|_{q+1}$ is verified. Similarly we can prove that $\|T_2h\|_{q+1} \le \frac{1}{2}\|h\|_{p+1}$ for all $h \in L^{q+1}(\mathbb{H}^n)$ by choosing a large enough. Combining them together, we have no difficulty to get

$$\begin{aligned}\|T(h_1,h_2)\|_{p+1,q+1} &= \|T_1h_2\|_{p+1}+\|T_2h_1\|_{q+1}\\ &\le \frac{1}{2}(\|h_2\|_{q+1}+\|h_1\|_{p+1})\\ &= \frac{1}{2}\|(h_1,h_2)\|_{p+1,q+1},\end{aligned}$$

and this shows that T is shrinking from $L^{p+1}(\mathbb{H}^n)\times L^{q+1}(\mathbb{H}^n)$ to itself.

(2) T is shrinking from $L^r(\mathbb{H}^n)\times L^s(\mathbb{H}^n)$ to itself.

We use the same tool as we did in (1), that is, $|u|$ weighted fractional integral inequality in Theorem 2.1 with assistance of Hölder inequality, by properly choosing the indices. Here, we prove that $\|T_2h\|_s\le\frac{1}{2}\|h\|_r$ first,

$$\begin{aligned}\|T_2h\|_s &= \|S_{\lambda,\beta,\alpha}(f_a^{p-1}h)\|_s\\ &\le C\|f_a^{p-1}h\|_t\\ &\le C\|f_a^{p-1}\|_{\frac{p+1}{p-1}}\|h\|_r\\ &= C\|f_a\|_{p+1}^{p-1}\|h\|_r,\end{aligned}$$

in which we choose a sufficiently large that $C\|g_a\|_{q+1}^{q-1}\le\frac{1}{2}$, since $g\in L^{q+1}(\mathbb{H}^n)$. Thus, $\|T_2h\|_s\le\frac{1}{2}\|h\|_r$ for all $h\in L^r(\mathbb{H}^n)$. The indices r, s and t above satisfy

$$\frac{1}{t}=\frac{p-1}{p+1}+\frac{1}{r}$$

and by (3.1),

$$\begin{aligned}\frac{1}{t}+\frac{\overline{\lambda}}{Q}-1 &= \frac{p-1}{p+1}+\frac{1}{r}+\frac{\overline{\lambda}}{Q}-1\\ &= \frac{p-1}{p+1}+\frac{1}{r}+\frac{1}{p+1}+\frac{1}{q+1}-1\\ &= \frac{1}{r}-\frac{1}{p+1}+\frac{1}{q+1}\\ &= \frac{1}{s}.\end{aligned}$$

It is also easy to check that $\beta<Q/s$, since $1/s>\beta/Q$, and $\alpha<Q/t'$, since

$$\frac{1}{t'}=1-\frac{1}{t}=\frac{\overline{\lambda}}{Q}-\frac{1}{s}>\frac{\overline{\lambda}}{Q}-\frac{\lambda+\beta}{Q}=\frac{\alpha}{Q}.$$

Similarly we estimate T_1 for $h\in L^s(\mathbb{H}^n)$, and easily pass the results to $L^r(\mathbb{H}^n)\times L^s(\mathbb{H}^n)$, i.e.,

$$\|T(h_1,h_2)\|_{r,s}\le\frac{1}{2}\|(h_1,h_2)\|_{r,s},$$

which shows that T is shrinking from $L^r(\mathbb{H}^n)\times L^s(\mathbb{H}^n)$ to itself.

(3) $F\in L^{p+1}(\mathbb{H}^n)\cap L^r(\mathbb{H}^n)$ and $G\in L^{p+1}(\mathbb{H}^n)\cap L^r(\mathbb{H}^n)$.

It is evident once one notices that g_b and f_b are uniformly bounded by a in $B_a(0)$.

Applying regularity lifting we finish the proof of (**1.3.i**) in Theorem 1.3.

3.2. *Proof of Theorem 1.3 if $\alpha\ge 0$, $\beta\le 0$*

We repeat the same procedure as we did in Section 3.1, that is, we fix the pair r and s by (3.1), and then verify the same three conditions, in order to apply regularity lifting. We adapt all the settings in Section 3.1 and only need to verify (2) to make sure the interval conditions in (**1.3.ii**) of Theorem 1.3 are sufficient.

(2) T is shrinking from $L^r(\mathbb{H}^n)\times L^s(\mathbb{H}^n)$ to itself.

First, for T_1,

$$\begin{aligned}\|T_1h\|_r &= \|S_{\lambda,\alpha,\beta}(g_a^{q-1}h)\|_r\\ &\le C\|g_a^{q-1}h\|_t\\ &\le C\|g_a^{q-1}\|_{\frac{q+1}{q-1}}\|h\|_s\\ &= C\|g_a\|_{q+1}^{q-1}\|h\|_s.\end{aligned}$$

We have $\alpha < Q/r$, since $1/r > \alpha/Q$, and $\beta < Q/t'$, since $\beta \leq 0$ and

$$\frac{1}{t'} = 1 - \frac{1}{t} = \frac{\overline{\lambda}}{Q} - \frac{1}{r} > 0.$$

Then, for T_2,

$$\begin{aligned}\|T_2 h\|_s &= \|S_{\lambda,\beta,\alpha}(f_a^{p-1}h)\|_s \\ &\leq C\|f_a^{p-1}h\|_t \\ &\leq C\|f_a^{p-1}\|_{\frac{p+1}{p-1}}\|h\|_r \\ &= C\|f_a\|_{p+1}^{p-1}\|h\|_r.\end{aligned}$$

We have $\beta < Q/s$, since $\beta \leq 0$, and $\alpha < Q/t'$, since

$$\frac{1}{t'} = 1 - \frac{1}{t} = \frac{\overline{\lambda}}{Q} - \frac{1}{s} > \frac{\overline{\lambda}}{Q} - \frac{\lambda+\beta}{Q} = \frac{\alpha}{Q}.$$

Therefore we finish the proof of (2), and thus complete Theorem 1.3. Let us prove a proposition of the intervals in Theorem 1.3.

Proposition 3.2. *We assume the same conditions as in Theorem 1.3, then*

(1) *If $\alpha, \beta \geq 0$, and assume further that $\frac{\alpha}{Q} - \frac{\beta}{Q} \geq \frac{1}{p+1} - \frac{1}{q+1}$, then $(f, g) \in L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$ for all r and s such that*

$$\frac{1}{r} \in \left(\frac{\alpha}{Q}, \frac{\lambda+\beta}{Q} - \frac{1}{q+1} + \frac{1}{p+1}\right)$$

and

$$\frac{1}{s} \in \left(\frac{\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1}, \frac{\lambda+\beta}{Q}\right).$$

(2) *If $\alpha \geq 0, \beta \leq 0$, and assume further that $\frac{\alpha}{Q} \geq \frac{1}{p+1} - \frac{1}{q+1}$, then $(f, g) \in L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$ for all r and s such that*

$$\frac{1}{r} \in \left(\frac{\alpha}{Q}, \frac{\lambda+\beta}{Q} - \frac{1}{q+1} + \frac{1}{p+1}\right)$$

and

$$\frac{1}{s} \in \left(\frac{\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1}, \frac{\lambda+\beta}{Q}\right).$$

Proof of Proposition 3.2. The proof is trivial and we observe that in (1), $\frac{\alpha}{Q} - \frac{\beta}{Q} \geq \frac{1}{p+1} - \frac{1}{q+1}$ if and only if $(\lambda+2\alpha)(p+1) \geq 2Q$ (or equivalently, $(\lambda + 2\beta)(q + 1) \leq 2Q$). While in (2), $\frac{\alpha}{Q} \geq \frac{1}{p+1} - \frac{1}{q+1}$ if and only if $(\lambda + 2\alpha + \beta)(p + 1) \geq 2Q$ (or equivalently, $(\lambda + \beta)(q + 1) \leq 2Q$). Similarly we can develop the contrary case that

(1') If $\alpha, \beta \geq 0$, and assume further that $\frac{\alpha}{Q} - \frac{\beta}{Q} < \frac{1}{p+1} - \frac{1}{q+1}$, then $(f, g) \in L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$ for all r and s such that

$$\frac{1}{r} \in \left(\frac{\beta}{Q} - \frac{1}{q+1} + \frac{1}{p+1}, \frac{\lambda+\alpha}{Q}\right)$$

and

$$\frac{1}{s} \in \left(\frac{\beta}{Q}, \frac{\lambda+\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1}\right).$$

(2') If $\alpha \geq 0, \beta \leq 0$, and assume further that $\frac{\alpha}{Q} < \frac{1}{p+1} - \frac{1}{q+1}$, then $(f, g) \in L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$ for all r and s such that

$$\frac{1}{r} \in \left(-\frac{1}{q+1} + \frac{1}{p+1}, \frac{\overline{\lambda}}{Q}\right)$$

and

$$\frac{1}{s} \in \left(0, \frac{\overline{\lambda}}{Q} - \frac{1}{p+1} + \frac{1}{q+1}\right).$$

We also observe that in (1'), $\frac{\alpha}{Q} - \frac{\beta}{Q} < \frac{1}{p+1} - \frac{1}{q+1}$ if and only if $(\lambda+2\alpha)(p+1) > 2Q$ (or equivalently, $(\lambda+2\beta)(q+1) > 2Q$). While in (2'), $\frac{\alpha}{Q} < \frac{1}{p+1} - \frac{1}{q+1}$ if and only if $(\lambda + 2\alpha + \beta)(p + 1) < 2Q$ (or equivalently, $(\lambda + \beta)(q + 1) > 2Q$). $\square$

3.3. Proof of Theorem 1.6

The proof of Theorem 1.6 follows similarly as we did in Section 3.1 and Section 3.2: First we define

$$\widetilde{T}_1 h(u) = \frac{1}{|z|^\alpha}\int_{\mathbb{H}^n}\frac{g_a^{q-1}(v)h(v)}{|z'|^\beta|u^{-1}v|^\lambda}dv = \widetilde{S}_{\lambda,\alpha,\beta}(g_a^{q-1}h)(u),$$

$$\widetilde{F}(u) = \frac{1}{|z|^\alpha}\int_{\mathbb{H}^n}\frac{g_b^{q}(v)}{|z'|^\beta|u^{-1}v|^\lambda}dv = \widetilde{S}_{\lambda,\alpha,\beta}(g_b^{q})(u),$$

and

$$\widetilde{T}_2 h(u) = \frac{1}{|z|^\beta}\int_{\mathbb{H}^n}\frac{f_a^{p-1}(v)h(v)}{|z'|^\alpha|u^{-1}v|^\lambda}dv = \widetilde{S}_{\lambda,\beta,\alpha}(f_a^{p-1}h)(u),$$

$$\widetilde{G}(u) = \frac{1}{|z|^\beta}\int_{\mathbb{H}^n}\frac{f_b^{q}(v)}{|z'|^\alpha|u^{-1}v|^\lambda}dv = \widetilde{S}_{\lambda,\beta,\alpha}(f_b^{p})(u).$$

Thus we immediately observe that (f, g) solves the equation $(h_1, h_2) = \widetilde{T}(h_1, h_2) + (F, G)$, in which $\widetilde{T}(h_1, h_2) = (\widetilde{T}_1 h_2, \widetilde{T}_2 h_1)$. We also fix the indices r and s satisfying (3.1) and only need to verify that $\widetilde{T}$ is shrinking from $L^r(\mathbb{H}^n) \times L^s(\mathbb{H}^n)$ to itself, that is, to prove the regularity estimates intervals (**1.6.i**) and (**1.6.ii**) in Theorem 1.6 are sufficient to guarantee that

$$\|\widetilde{T}(h_1, h_2)\|_{r,s} \le \frac{1}{2}\|(h_1, h_2)\|_{r,s}$$

for a large. We omit the details here.

4. Asymptotic behavior

In this section, we give the proof of Theorem 1.4, asymptotic behavior for $|u|$ weighted HLS inequality. Since the estimates become trivial when $\alpha = 0$ or $\beta = 0$, we only consider the nontrivial cases $\alpha, \beta > 0$ in Section 4.1 and $\alpha > 0, \beta < 0$ in Section 4.2. It is worthwhile to note here that our asymptotic behavior estimate is near the origin, to get its counterpart around infinite, one might need to consider the change of the system under Kelvin-type transform, we refer the reader to [33] for the study of Kelvin transform on the Heisenberg group and leave this for further investigation.

4.1. $|u| \sim 0$ if $\alpha, \beta > 0$

(**1.4.i**) If $|u| \sim 0$, we show asymptotic behavior of f under three conditions $\lambda + \beta(q+1) <, =, > Q$. A lemma is necessary to justify the eligibility of the constants in the theorem.

Lemma 4.1. *The constants A_1, A_2 and A_3 in (**i**) of Theorem 1.4 are all finite.*

Proof of Lemma 4.1. (1) If $\lambda + \beta(q + 1) < Q$, to show $A_1 = \int_{\mathbb{H}^n}\frac{g^q(v)}{|v|^{\lambda+\beta}}dv < \infty$, observe that

$$A_1 = \int_{\mathbb{H}^n}\frac{g^q(v)}{|v|^{\lambda+\beta}}dv = \int_{B_\delta}\frac{g^q(v)}{|v|^{\lambda+\beta}}dv + \int_{\mathbb{H}^n\setminus B_\delta}\frac{g^q(v)}{|v|^{\lambda+\beta}}dv := I_1 + I_2$$

in which B_δ is the Heisenberg ball in $\mathbb{H}^n$ centered at the origin with radius δ, i.e., $B_\delta = \{v \in \mathbb{H}^n \parallel v| < \delta\}$. Then by Hölder inequality,

$$I_1 = \int_{B_\delta}\frac{g^q(v)}{|v|^{\lambda+\beta}}dv \le \left(\int_{B_\delta} g^{qt}(v)dv\right)^{\frac{1}{t}}\left(\int_{B_\delta}\frac{1}{|v|^{(\lambda+\beta)t'}}dv\right)^{\frac{1}{t'}},$$

where $\int_{B_\delta}\frac{1}{|v|^{(\lambda+\beta)t'}}dv < \infty$ if and only if $(\lambda + \beta)t' < Q$, that is,

$$\frac{1}{t} < 1 - \frac{\lambda+\beta}{Q},$$

and thus,

$$\frac{1}{qt} < \frac{1}{q} - \frac{\lambda+\beta}{qQ}.$$

From (**1.3.i**) in Theorem 1.3, we have $g \in L^s(\mathbb{H}^n)$ for all s such that

$$\frac{1}{s} \in \left(\frac{\beta}{Q}, \frac{\lambda+\beta}{Q}\right) \cap \left(\frac{\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1}, \frac{\lambda+\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1}\right).$$

Given that $\frac{1}{q+1} < \frac{\lambda+\beta}{Q}$, one can easily verify that

$$\frac{1}{q} - \frac{\lambda+\beta}{qQ} > \frac{\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1},$$

and $\lambda + \beta(q+1) < Q$ guarantees that

$$\frac{1}{q} - \frac{\lambda+\beta}{qQ} > \frac{\beta}{Q}.$$

Then we are able to choose t such that $(\lambda+\beta)t' < Q$ and $\|g\|_{qt} < \infty$, thus $I_1 < \infty$ follows. To estimate I_2,

$$I_2 = \int_{\mathbb{H}^n\setminus B_\delta} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv \le \left(\int_{\mathbb{H}^n\setminus B_\delta} g^{qt}(v)dv\right)^{\frac{1}{t}} \left(\int_{\mathbb{H}^n\setminus B_\delta} \frac{1}{|v|^{(\lambda+\beta)t'}} dv\right)^{\frac{1}{t'}},$$

we use the method with the same fashion, i.e., choosing $t > 0$ such that $(\lambda+\beta)t' > Q$, that is,

$$\frac{1}{qt} > \frac{1}{q} - \frac{\lambda+\beta}{qQ}$$

and

$$\frac{1}{qt} \in \left(\frac{\beta}{Q}, \frac{\lambda+\beta}{Q}\right) \cap \left(\frac{\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1}, \frac{\lambda+\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1}\right).$$

We have $I_2 < \infty$ and therefore $A_1 < \infty$.

(2) If $\lambda + \beta(q+1) = Q$, by the remark below Theorem 1.4, we know that $\lambda + \alpha(p+1) < Q$. Similar to the proof of $A_1 < \infty$, we obtain $B_1 = \int_{\mathbb{H}^n} \frac{f^p(v)}{|v|^{\lambda+\alpha}} dv < \infty$. Then simply note that $A_2 = |\Sigma_1|(\int_{\mathbb{H}^n} \frac{f^p(v)}{|v|^{\lambda+\alpha}} dv)^q = |\Sigma_1|B_1^q < \infty$.

(3) If $\lambda + \beta(q+1) > Q$, by the same reason as in (2) above, $B_1 < \infty$. Compute that

$$A_3 = \left(\int_{\mathbb{H}^n} \frac{f^p(v)}{|v|^{\lambda+\alpha}} dv\right)^q \int_{\mathbb{H}^n} \frac{1}{|v|^{\beta(q+1)}|e^{-1}v|^\lambda} dv = B_1^q \int_{\mathbb{H}^n} \frac{1}{|v|^{\beta(q+1)}|e^{-1}v|^\lambda} dv,$$

we only need to show

$$\int_{\mathbb{H}^n} \frac{1}{|v|^{\beta(p+1)}|e^{-1}v|^\lambda} dv < \infty.$$

It is evident since $|v|^{\beta(q+1)}|e^{-1}v|^\lambda \sim |v|^{\beta(p+1)}$, when $|v| \sim 0$ and $\beta(q+1) < Q$. While $|v|^{\beta(q+1)}|e^{-1}v|^\lambda \sim |v|^{\lambda+\beta(p+1)}$, when $|v| \sim \infty$ and $\lambda + \beta(p+1) > Q$, and we finish the proof of this lemma. $\square$

Now we proceed to prove the asymptotic behavior of f around the origin.

Case 1. If $\lambda + \beta(q+1) < Q$, we show $f(u) \sim A_1/|u|^\alpha$ as $|u| \sim 0$.

Since A_1 is finite from Lemma 4.1, we only need to prove that

$$\lim_{|u|\to 0} \left| \int_{\mathbb{H}^n} \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} dv - \int_{\mathbb{H}^n} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv \right| = 0. \tag{4.1}$$

Given $\delta > 0$,

$$\begin{aligned}\left| \int_{\mathbb{H}^n} \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} dv - \int_{\mathbb{H}^n} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv \right| &\le \int_{\mathbb{H}^n} \left| \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} - \frac{g^q(v)}{|v|^{\lambda+\beta}} \right| dv \\ &\le \int_{B_\delta} \left(\frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} + \frac{g^q(v)}{|v|^{\lambda+\beta}} \right) dv + \int_{\mathbb{H}^n\setminus B_\delta} \left| \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} - \frac{g^q(v)}{|v|^{\lambda+\beta}} \right| dv \\ &:= J_1(u) + J_2(u).\end{aligned}$$

Here, we apply Young's inequality

$$ab \le \frac{a^k}{k} + \frac{b^{k'}}{k'},$$

where $a, b \ge 0$ and $\frac{1}{k} + \frac{1}{k'} = 1$. By letting $a = \frac{1}{|v|^\beta}$, $b = \frac{1}{|u^{-1}v|^\lambda}$, $k = \frac{\lambda+\beta}{\beta}$ and $k = \frac{\lambda+\beta}{\lambda}$, we have

$$\frac{1}{|v|^\beta |u^{-1}v|^\lambda} \le \frac{1}{k|v|^{k\beta}} + \frac{1}{k'|u^{-1}v|^{k'\lambda}} \le \frac{\beta}{(\lambda+\beta)|v|^{\lambda+\beta}} + \frac{\lambda}{(\lambda+\beta)|u^{-1}v|^{\lambda+\beta}}, \tag{4.2}$$

and hence,

$$
\begin{aligned}
J_1(u) &= \int_{B_\delta} \left(\frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} + \frac{g^q(v)}{|v|^{\lambda+\beta}} \right) dv \\
&\le \frac{\beta}{\lambda+\beta} \int_{B_\delta} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv + \frac{\lambda}{\lambda+\beta} \int_{B_\delta} \frac{g^q(v)}{|u^{-1}v|^{\lambda+\beta}} dv + \int_{B_\delta} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv \\
&\to 0,
\end{aligned}
$$

as $\delta \to 0$, since each of the three integrals above over $\mathbb{H}^n$ is finite according to the computation of I_1 in Lemma 4.1. Next we fix δ small so that $J_1(u)$ is small. For the same reason we can get

$$
\begin{aligned}
J_2(u) &= \int_{\mathbb{H}^n \setminus B_\delta} \left| \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} - \frac{g^q(v)}{|v|^{\lambda+\beta}} \right| dv \\
&\le \int_{\mathbb{H}^n \setminus B_\delta} \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} dv + \int_{\mathbb{H}^n \setminus B_\delta} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv \\
&\le \frac{\beta}{\lambda+\beta} \int_{\mathbb{H}^n \setminus B_\delta} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv + \frac{\lambda}{\lambda+\beta} \int_{\mathbb{H}^n \setminus B_\delta} \frac{g^q(v)}{|u^{-1}v|^{\lambda+\beta}} dv + \int_{\mathbb{H}^n \setminus B_\delta} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv \\
&< \infty,
\end{aligned}
$$

for any $u \in \mathbb{H}^n$. By letting $u \to 0$, note that $|v| \ge \delta \gg |u|$ for $|u|$ small, we have $J_2(u) \to 0$ from Lebesgue bounded convergence theorem, then (4.4) is verified and (**1.4.i**) is proved under $\lambda + \beta(q+1) < Q$.

Case 2. If $\lambda + \beta(q+1) = Q$, we show $f(u) \sim A_2 |\ln|u\|/|u|^\alpha$ as $|u| \sim 0$.

We prove that

$$
\lim_{|u|\to 0} \frac{|u|^\alpha f(u)}{|\ln|u\|} = \lim_{|u|\to 0} \frac{|u|^\alpha f(u)}{-\ln|u|} = A_2.
$$

Since $B_1 < \infty$ from (2) of Lemma 4.1. We obtain $g(u) \sim B_1/|u|^\beta$ as $|u| \sim 0$ in (**1.4.ii**) by a similar argument as in Case 1. That is, $g(u) = \frac{B_1 + o(1)}{|u|^\beta}$ as $|u| \to 0$. For $\delta > 0$ small we compute that

$$
\begin{aligned}
\frac{|u|^\alpha f(u)}{-\ln|u|} &= \frac{1}{-\ln|u|} \int_{\mathbb{H}^n} \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} dv \\
&= \frac{1}{-\ln|u|} \int_{B_\delta} \frac{(B_1+o(1))^q}{|v|^{\beta(q+1)} |u^{-1}v|^\lambda} dv + \frac{1}{-\ln|u|} \int_{\mathbb{H}^n \setminus B_\delta} \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} dv \\
&= \frac{(B_1+o(1))^q}{-\ln|u|} \int_{B_\delta} \frac{1}{|v|^{Q-\lambda} |u^{-1}v|^\lambda} dv + \frac{1}{-\ln|u|} \int_{\mathbb{H}^n \setminus B_\delta} \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} dv.
\end{aligned}
$$

Note that $\beta(q+1) = Q - \lambda$, and the second term above tends to 0 when $|u|$ is small since the integral in it is bounded. Then,

$$
\begin{aligned}
\lim_{|u|\to 0} \frac{|u|^\alpha f(u)}{-\ln|u|} &= \lim_{|u|\to 0} \frac{(B_1+o(1))^q}{-\ln|u|} \int_{B_\delta} \frac{1}{|v|^{Q-\lambda} |u^{-1}v|^\lambda} dv \\
&= B_1^q \lim_{|u|\to 0} \frac{1}{-\ln|u|} \int_{B_\delta} \frac{1}{|v|^{Q-\lambda} |u^{-1}v|^\lambda} dv \\
&= B_1^q \lim_{|u|\to 0} \frac{1}{-\ln|u|} \int_{B_{\delta/|u|}} \frac{1}{|w|^{Q-\lambda} |e^{-1}w|^\lambda} dw,
\end{aligned}
$$

in which $w = v/|u|$, $e = u/|u|$ is a unit vector in $\mathbb{H}^n$. Then we fix $\delta > 0$, we apply a result concerning polar coordinates on $\mathbb{H}^n$, see e.g. Proposition on p. 1574 in [14].

$$
\begin{aligned}
\frac{1}{-\ln|u|} \int_{B_{\delta/|u|}} \frac{1}{|w|^{Q-\lambda} |e^{-1}w|^\lambda} dw &= \frac{1}{-\ln|u|} \int_0^{\delta/|u|} dr \int_{\Sigma_1} \frac{r^{Q-1} dv}{r^{Q-\lambda} |e^{-1} r v|^\lambda} \\
&= \frac{1}{-\ln|u|} \int_{\Sigma_1} dv \left(\int_0^R \frac{r^{\lambda-1}}{|e^{-1} r v|^\lambda} dr + \int_R^{\delta/|u|} \frac{r^{\lambda-1}}{|e^{-1} r v|^\lambda} dr \right) \\
&= \frac{1}{-\ln|u|} \int_{\Sigma_1} dv \left(O(1) + \int_R^{\delta/|u|} \frac{1}{r} dr \right) \\
&\to |\Sigma_1|.
\end{aligned}
$$

Thus,

$$\lim_{|u|\to 0}\frac{|u|^{\alpha}f(u)}{-\ln|u|}=|\Sigma_1|B_1^q=A_2,$$

and (**1.4.i**) is proved under $\lambda+\beta(q+1)=Q$.

Case 3. If $\lambda+\beta(q+1)>Q$, we show $f(u)\sim\frac{A_3}{|u|^{\beta(q+1)+\alpha+\lambda-Q}}$ as $|u|\sim 0$.

We prove that

$$\lim_{|u|\to 0}|u|^t f(u)=A_3,$$

in which $t=\beta(q+1)+\alpha+\lambda-Q$. As in Case 2, we use $g(u)=\frac{B_1+o(1)}{|u|^{\beta}}$ as $|u|\to 0$. For $\delta>0$ small we compute that

$$\begin{aligned}
|u|^t f(u) &= \frac{|u|^t}{|u|^{\alpha}}\int_{\mathbb{H}^n}\frac{g^q(v)}{|v|^{\beta}|u^{-1}v|^{\lambda}}dv\\
&= |u|^{t-\alpha}\int_{B_\delta}\frac{(B_1+o(1))^q}{|v|^{\beta(q+1)}|u^{-1}v|^{\lambda}}dv+|u|^{t-\alpha}\int_{\mathbb{H}^n\setminus B_\delta}\frac{g^q(u)}{|v|^{\beta}|u^{-1}v|^{\lambda}}dv\\
&= |u|^{t-\alpha}(B_1+o(1))^q\int_{B_\delta}\frac{1}{|v|^{\beta(q+1)}|u^{-1}v|^{\lambda}}dv+|u|^{t-\alpha}\int_{\mathbb{H}^n\setminus B_\delta}\frac{g^q(u)}{|v|^{\beta}|u^{-1}v|^{\lambda}}dv
\end{aligned}$$

and the second term above tends to 0 when $|u|$ is small since the integral in it is bounded (See the proof in Case 1.) and $t-\alpha=\lambda+\beta(q+1)-Q>0$. Then,

$$\begin{aligned}
\lim_{|u|\to 0}|u|^t f(u) &= \lim_{|u|\to 0}|u|^{t-\alpha}(B_1+o(1))^q\int_{B_\delta}\frac{1}{|v|^{\beta(q+1)}|u^{-1}v|^{\lambda}}dv\\
&= B_1^q\lim_{|u|\to 0}|u|^{t-\alpha}\int_{B_\delta}\frac{1}{|v|^{\beta(q+1)}|u^{-1}v|^{\lambda}}dv\\
&= B_1^q\lim_{|u|\to 0}\int_{B_{\delta/|u|}}\frac{1}{|w|^{\beta(q+1)}|e^{-1}w|^{\lambda}}dw\\
&= B_1^q\int_{\mathbb{H}^n}\frac{1}{|v|^{\beta(q+1)}|e^{-1}v|^{\lambda}}dv\\
&= A_3,
\end{aligned}$$

and (**1.4.i**) is proved under $\lambda+\beta(q+1)>Q$.

(**1.4.ii**) If $|u|\sim 0$, repeat the same process above, we are able to get the asymptotic behavior estimates for $g(u)$ as $|u|\sim 0$, and thus finish Section 4.1, $\alpha,\beta>0$.

4.2. $|u|\sim 0$ if $\alpha>0$, $\beta<0$

We apply the same method with modified approach to estimate f and g near the origin, under its corresponding regularity estimate (**1.3.ii**) in Theorem 1.3. Since f and g are not similar as they are in Section 4.1, we provide the proof for both of them.

(**1.4.i**) Here $\beta<0$ and $\lambda<Q$ shows that $\lambda+\beta(q+1)<Q$, and we only need to show $f(u)\sim A_1/|u|^{\alpha}$ as $|u|\sim 0$. We shall begin with a lemma similar to Lemma 4.1, to provide the finiteness of the constant A_1.

Lemma 4.2. *The constant A_1 in* (**1.4.i**) *of Theorem 1.4 is finite.*

Proof of Lemma 4.2. To show $A_1=\int_{\mathbb{H}^n}\frac{g^q(v)}{|v|^{\lambda+\beta}}dv<\infty$, observe that

$$A_1=\int_{\mathbb{H}^n}\frac{g^q(v)}{|v|^{\lambda+\beta}}dv=\int_{B_\delta}\frac{g^q(v)}{|v|^{\lambda+\beta}}dv+\int_{\mathbb{H}^n\setminus B_\delta}\frac{g^q(v)}{|v|^{\lambda+\beta}}dv:=I_1+I_2.$$

Then by Hölder inequality,

$$\begin{aligned}
I_1 &= \int_{B_\delta}\frac{g^q(v)}{|v|^{\lambda+\beta}}dv\\
&\le \frac{1}{\delta^{\beta}}\int_{B_\delta}\frac{g^q(v)}{|v|^{\lambda}}dv\\
&\le \frac{1}{\delta^{\beta}}\left(\int_{B_\delta}g^{qt}(v)dv\right)^{\frac{1}{t}}\left(\int_{B_\delta}\frac{1}{|v|^{\lambda t'}}dv\right)^{\frac{1}{t'}},
\end{aligned}$$

where $\int_{B_\delta} \frac{1}{|v|^{\lambda t'}} dv < \infty$ if and only if $\lambda t' < Q$, that is,

$$\frac{1}{t} < 1 - \frac{\lambda}{Q},$$

and thus,

$$\frac{1}{qt} < \frac{1}{q} - \frac{\lambda}{qQ}.$$

From (**ii**) in Theorem 1.3, we have $g \in L^s(\mathbb{H}^n)$ for all s such that

$$\frac{1}{s} \in \left(0, \frac{\lambda+\beta}{Q}\right) \cap \left(\frac{\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1}, \frac{\bar{\lambda}}{Q} - \frac{1}{p+1} + \frac{1}{q+1}\right).$$

One can easily verify that

$$\frac{1}{q} - \frac{\lambda}{qQ} > \frac{\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1},$$

and $\lambda < Q$ implies that

$$\frac{1}{q} - \frac{\lambda}{qQ} > 0.$$

Then we are able to choose t such that $(\lambda+\beta)t' < Q$ and $\|g\|_{qt} < \infty$, thus $I_1 < \infty$ follows. To estimate I_2,

$$I_2 = \int_{\mathbb{H}^n \setminus B_\delta} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv \le \left(\int_{\mathbb{H}^n \setminus B_\delta} g^{qt}(v) dv\right)^{\frac{1}{t}} \left(\int_{\mathbb{H}^n \setminus B_\delta} \frac{1}{|v|^{(\lambda+\beta)t'}} dv\right)^{\frac{1}{t'}}, \tag{4.3}$$

choosing $t > 0$ such that $(\lambda+\beta)t' > Q$, that is,

$$\frac{1}{qt} > \frac{1}{q} - \frac{\lambda+\beta}{qQ}$$

and

$$\frac{1}{s} \in \left(0, \frac{\lambda+\beta}{Q}\right) \cap \left(\frac{\alpha}{Q} - \frac{1}{p+1} + \frac{1}{q+1}, \frac{\bar{\lambda}}{Q} - \frac{1}{p+1} + \frac{1}{q+1}\right).$$

We have $I_2 < \infty$ and therefore $A_1 < \infty$. □

Now notice that Young's inequality (4.2) we used in Section 4.1 when $\alpha, \beta > 0$ is not valid here, thus a different approach is needed. To show

$$\lim_{|u|\to 0} \left| \int_{\mathbb{H}^n} \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} dv - \int_{\mathbb{H}^n} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv \right| = 0. \tag{4.4}$$

Given $\delta > 0$,

$$\begin{aligned} \left| \int_{\mathbb{H}^n} \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} dv - \int_{\mathbb{H}^n} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv \right| \le & \int_{B_\delta} \left(\frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} + \frac{g^q(v)}{|v|^{\lambda+\beta}} \right) dv + \int_{\mathbb{H}^n \setminus B_\delta} \left| \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} - \frac{g^q(v)}{|v|^{\lambda+\beta}} \right| dv \\ & := J_1(u) + J_2(u). \end{aligned}$$

In which,

$$\begin{aligned} J_1(u) &= \int_{B_\delta} \left(\frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} + \frac{g^q(v)}{|v|^{\lambda+\beta}} \right) dv \\ &\le \frac{1}{\delta^\beta} \left(\int_{B_\delta} \frac{g^q(v)}{|u^{-1}v|^\lambda} dv + \int_{B_\delta} \frac{g^q(v)}{|v|^\lambda} dv \right) \\ &\to 0, \end{aligned}$$

as $\delta \to 0$, since both integrals above over $\mathbb{H}^n$ is finite according to the computation of I_1 in Lemma 4.2. Next we fix δ small so that $J_1(u)$ is small. Note that as $u \to 0$, $|v| \ge \delta \gg |u|$, we have $|u^{-1}v| \sim |v|$. In fact, applying the quasi-triangular inequality (2.3), if $|u| \le \frac{1}{3}\delta \le \frac{1}{3}|v|$, then

$$|v| \le 2(|u| + |u^{-1}v|) \le \frac{2}{3}|v| + 2|u^{-1}v|,$$

thus $|u^{-1}v| \geq \frac{1}{6}|v|$, and

$$\begin{aligned} J_2(u) &= \int_{\mathbb{H}^n\setminus B_\delta} \left| \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} - \frac{g^q(v)}{|v|^{\lambda+\beta}} \right| dv \\ &\leq \int_{\mathbb{H}^n\setminus B_\delta} \frac{g^q(v)}{|v|^\beta |u^{-1}v|^\lambda} dv + \int_{\mathbb{H}^n\setminus B_\delta} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv \\ &\leq 6\int_{\mathbb{H}^n\setminus B_\delta} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv + \int_{\mathbb{H}^n\setminus B_\delta} \frac{g^q(v)}{|v|^{\lambda+\beta}} dv \\ &< \infty. \end{aligned}$$

By letting $u \to 0$, we have $J_2(u) \to 0$ from Lebesgue bounded convergence theorem, then (4.4) is verified and (**1.4.i**) is proved.

(**1.4.ii**) If $|u| \sim 0$, we show the asymptotic behavior of g near the origin. Let us prove $B_1, B_2, B_3 < \infty$ first.

Lemma 4.3. *The constant B_1, B_2 and B_3 in* (**1.4.ii**) *of Theorem 1.4 are all finite.*

Proof of Lemma 4.3. If $\lambda + \alpha(p+1) \geq Q$, then $\lambda + \beta(q+1) < Q$ and $A_1 < \infty$ from Lemma 4.2. Thus B_2 and B_3 are finite, since $B_2 = |\Sigma_1|A_1^p$ and $B_3 = A_1^q \int_{\mathbb{H}^n} \frac{1}{|v|^{\alpha(p+1)}|e^{-1}v|^\lambda} dv$. We just concentrate on $B_1 = \int_{\mathbb{H}^n} \frac{f^p(v)}{|v|^{\lambda+\alpha}}$ if $\lambda + \alpha(p+1) < Q$.

As we did in Lemmas 4.1 and 4.2, we break the integral to the one around the origin and the one away from the origin.

$$B_1 = \int_{\mathbb{H}^n} \frac{f^p(v)}{|v|^{\lambda+\alpha}} dv = \int_{B_\delta} \frac{f^p(v)}{|v|^{\lambda+\alpha}} dv + \int_{\mathbb{H}^n\setminus B_\delta} \frac{f^p(v)}{|v|^{\lambda+\alpha}} dv := K_1 + K_2.$$

A dedicate calculation shows that the regularity interval in (**ii**) of Theorem 1.3

$$\frac{1}{r} \in \left(\frac{\alpha}{Q}, \frac{\overline{\lambda}}{Q}\right) \cap \left(-\frac{1}{q+1} + \frac{1}{p+1}, \frac{\lambda+\beta}{Q} - \frac{1}{q+1} + \frac{1}{p+1}\right)$$

is sufficient to ensure $K_1 < \infty$. For the integral away from the origin K_2,

$$\begin{aligned} K_2 &= \int_{\mathbb{H}^n\setminus B_\delta} \frac{g^q(v)}{|v|^{\lambda+\alpha}} dv \\ &= \int_{\mathbb{H}^n\setminus B_\delta} \frac{|v|^\beta g^q(v)}{|v|^{\overline{\lambda}}} dv \\ &\leq \frac{1}{\delta^\beta} \int_{\mathbb{H}^n\setminus B_\delta} \frac{g^q(v)}{|v|^{\overline{\lambda}}} dv \\ &\leq \frac{1}{\delta^\beta} \left(\int_{\mathbb{H}^n\setminus B_\delta} g^{qt}(v) dv \right)^{\frac{1}{t}} \left(\int_{\mathbb{H}^n\setminus B_\delta} \frac{1}{|v|^{\overline{\lambda}t'}} dv \right)^{\frac{1}{t'}}. \end{aligned}$$

One can easily verify the existence of t such that $\overline{\lambda}t' > Q$ and

$$\frac{1}{qt} \in \left(\frac{\alpha}{Q}, \frac{\overline{\lambda}}{Q}\right) \cap \left(-\frac{1}{q+1} + \frac{1}{p+1}, \frac{\lambda+\beta}{Q} - \frac{1}{q+1} + \frac{1}{p+1}\right).$$

We have $K_2 < \infty$ and therefore $B_1 < \infty$. □

As $\alpha > 0$ here, Young's inequality is available and (**1.4.ii**) can be proved in the same way as in Section 4.1.

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