Nodal Patterns in Vibrations

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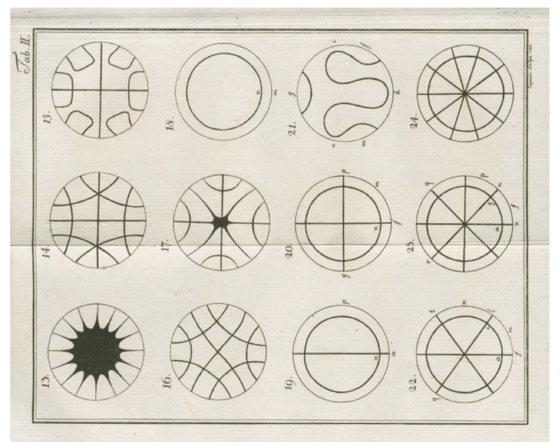
Introduction

Nodal patters in vibrating plates can be modeled by Chaldni Patterns. Initially, these patterns were introduced using sand and metal plates. When a plate was struck, the sand would gravitate to areas without vibration, which led to an arrangement of patterns.

Our Research Focus

Our research focus dives deeper into the nodal patterns. The two important research questions we were most concerned with include: How many nodal lines or curves are created, and how many nodal lines or curves intersect the boundary? This project observes the vibrations in guitar strings, while focusing on the vibration of a circular drumhead and answering the questions above.



Guitar String

The vibration of guitar string is represented by the ordinary differential equation

$$\begin{cases} -y''(x) = \lambda^2 y(x) \quad x \in (0,1), \\ \frac{\partial y}{\partial x}(0) = \frac{\partial y}{\partial x}(1) = 0, \end{cases}$$

where λ is the frequency. Solving the equation by trying the solutions $y(x) = e^{rx}$, we arrive at

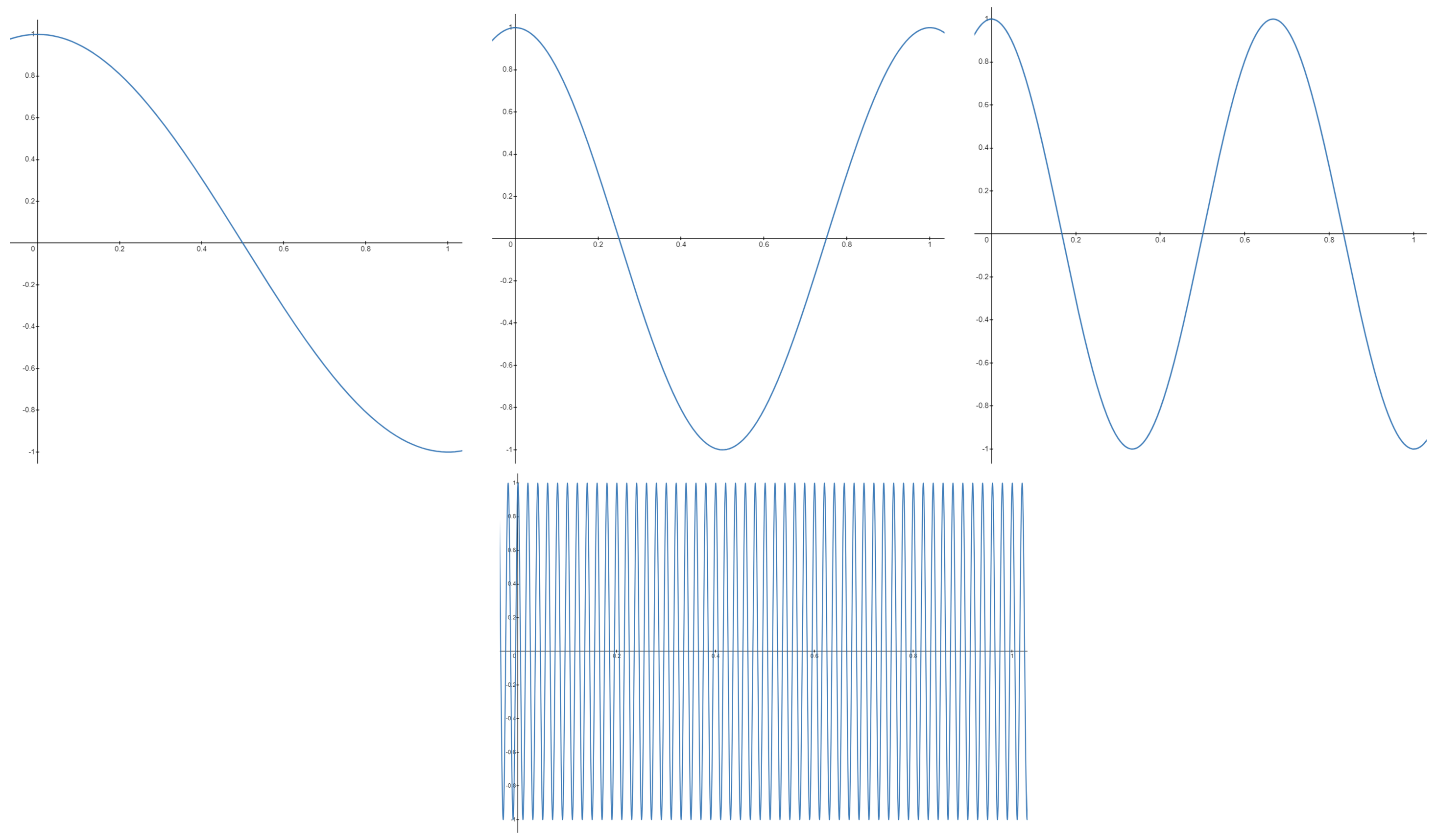
$$y(x) = c_1 cos(\lambda x) + c_2 sin(\lambda x). \tag{1}$$

The boundary conditions lead to $y'(x) = c_2 cos(n\pi x) = 0$ in $[0,1]$ for eigenvalues $\lambda = n$ and $n = 1, 2, 3, ... \to \infty$. The specific nodal points $cos(n\pi x) = 0$ are

$$x = \frac{1}{2n}, \frac{3}{2n}, \frac{5}{2n}, ..., \frac{2n-1}{2n}.$$

The number of nodal points are

$$H^0\{x \in (0,1) | \frac{\partial y}{\partial x} = 0\} = n - 1 \leq C\lambda. \tag{2}$$



Chlandi Pattern: Playing Guitar for n = 1, 2, 3, and 100, respectively

Two-Dimensional Circular drumhead and its Solutions

The vibration of drumhead is model by the Neumann eigenvalue problems

$$\begin{cases} -\Delta u(x,y) = \lambda u(x,y) \quad (x,y) \in \Omega, \\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \quad (x,y) \in \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. It is interesting and fundamental to study the sizes of nodal sets in the vibration. It was conjectured by the Fields medalist Shing-Tung Yau [3] that the sizes of interior nodal sets is bounded as

$$c\sqrt{\lambda} \leq H^{n-1}\{z \in \Omega | u(z) = 0\} \leq C\sqrt{\lambda}. \tag{3}$$

The interior nodal sets touch the boundary $\partial\Omega$ to form boundary nodal sets. It was shown by Zhu [4] that the sharp upper bound of boundary critical sets is

$$H^{n-2}\{z \in \partial\Omega || \nabla u(z)| = 0\} \leq C\lambda. \tag{4}$$

Our goal is to verify that the upper bound in (3) and (4) are optimal. We exam the vibration of two-dimensional Chlandi Pattern in the ball.

$$\begin{cases} -\Delta u(x,y) = \lambda u(x,y) \quad (x,y) \in \mathbb{B}_1 \\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \quad (x,y) \in \partial\mathbb{B}_1 \end{cases}$$

where $\mathbb{B}_1 = \{(x,y) | x^2 + y^2 < 1\}$ and $\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. To solve this equation, we used polar coordinates (r, θ) where $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Utilizing our boundary condition, we began to solve the two-dimensional aspect of Chlandi Pattern concerning a disc shaped Chlandi plate as follows:

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}. \tag{5}$$

The next step in solving for the nodal lines and curves is to use the separation of variables method as shown below:

$$u(r,\theta) = R(r)\Phi(\theta)$$

To summarize the steps above, we simplified the original equation by separating the variables given in the equation. Now that we have a simplified version, we can substitute this value into the original equation to give us Bessel's Equation:

$$r^2 R''(r) + rR'(r) + (\lambda r^2 - n^2)R(r) = 0 \quad , n = 1, 2, 3..... \to \infty. \tag{6}$$

$$\Phi''(\theta) + n^2\Phi(\theta) = 0$$

The equation (6) is called Bessel's Equation. Using $R'(1) = 0$ we get $J_n'(\sqrt{\lambda}) = 0$. The solution of (6) is given as the $n - th$ Bessel function $J_n'(\sqrt{\lambda}r)$. The solution is as follows:

$$u'(r,\theta)_{nm} = A_1 J_n'(j_{n,m}r) sin(n\theta) \quad or \quad u'(r,\theta) = A_2 J_n'(j_{n,m}r) cos(n\theta)$$

where $\sqrt{\lambda} = j_{n,m}$ is the root of $J_m'(\sqrt{\lambda}) = 0$.

Analysis of Nodal curves

If $cos(n\theta) = 0$, then $\theta = \frac{m\pi}{2n}$ with $m = 1, 2, \cdots, \frac{2n-1}{n}$. Thus the nodal sets of $u(r,\theta)$ is a collection of and $s - 1$ concentric circles with radius $\frac{j_{n,l}}{j_{n,m}}$, $l = 1, \cdots, m - 1$.
The length of the nodal line from n diameters is $2n$.

Now the zeros of $j'_{n,m}$ and $j_{n,m}$ interlace according to

$$n \leq j'_{n,m} \leq j_{n,m} \leq j'_{n,m+1} \leq j_{n,m+1} \tag{7}$$

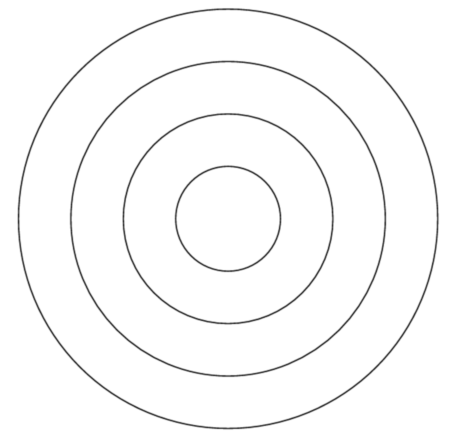
To study the nodal $n - 1$ concentric circles, we consider the following two cases:

1. *Case 1:* n tends to infinity and m is bounded
2. *Case 2:* m tends to infinity and n is bounded

Case 1: As $n \to \infty$ and $m \leq M$ for positive constant M, $\lambda = j'_{n,m} \to \infty$. Using the fact that

$$j'_{n,M} = n + o(n) \quad \text{as } n \to \infty, \tag{8}$$

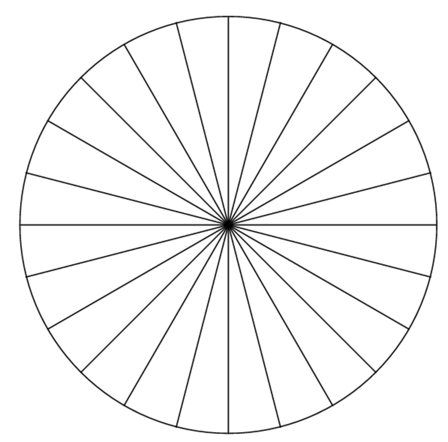
$$H^1\{(x,y) \in \mathbb{B}_1 | u(x,y) = 0\} = 2n + 2\pi \sum_{l=1}^{m-1} \frac{j'_{n,l}}{j'_{n,m}} \geq 2n \approx C\sqrt{\lambda}. \tag{9}$$



Case 2: As $m \to \infty$ and $n \leq M$ for positive constant M, $\lambda = j'_{n,m} \to \infty$. Using the fact that

$$j'_{M,m} = (m + \frac{M}{2} - \frac{1}{4})\pi + O(m^{-1}) \text{ as } m \to \infty, \tag{10}$$

$$\begin{aligned} H^1\{(x,y) \in \mathbb{B}_1 | u(x,y) = 0\} &= 2n + 2\pi \sum_{l=1}^{m-1} \frac{j'_{n,l}}{j'_{n,m}} \geq 2\pi \sum_{l=\frac{m}{2}}^{m-1} \frac{j'_{n,l}}{j'_{n,m}} \\ &\approx 2\pi \frac{\sum_{l=\frac{m}{2}}^{m-1}(l + \frac{M}{2} - \frac{1}{4})}{(m + \frac{M}{2} - \frac{1}{4})} \\ &\approx C\sqrt{\lambda}. \end{aligned} \tag{11}$$



From (12) and (11), we learn that the lower bounds and upper bounds in (3) are optimal, which can be achieved at the two-dimensional disc.
The intersection of the k diameters with the boundary in the ball are points $\{(x,y) \in \partial\mathbb{B}_1 | \, |\nabla u| = |u'(1) = 0\}$. From case 1,

$$H^0\{(x,y) \in \partial\mathbb{B}_1 | \, |\nabla u(x,y)| = 0\} \approx C\sqrt{\lambda}. \tag{12}$$

References

References

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Acknowledgments

1. Professor Jiuyi Zhu, Department of Mathematics, LSU.