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Periodicity versus Chaos in One-Dimensional Dynamics*

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Abstract. We survey recent results in one-dimensional dynamics and, as an application, we derive rigorous basic dynamical facts for two standard models in population dynamics, the Ricker and the Hassell families. We also informally discuss the concept of chaos in the context of one-dimensional discrete time models.

First we use the model case of the quadratic family for an informal exposition. We then review precise generic results before turning to the population models.

Our focus is on typical asymptotic behavior, seen for most initial conditions and for large sets of maps. Parameter sets corresponding to different types of attractors are described. In particular it is shown that maps with strong chaotic properties appear with positive frequency in parameter space in our population models. Natural measures (asymptotic distributions) and their stability properties are considered.

Key words. interval dynamics, attractors, chaos, parameter dependence, population models

AMS subject classifications. Primary, 37-02; Secondary, 37E05, 37N25, 92D25

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Part I. An Informal Overview

1. Introduction. The object of this paper is to survey some aspects of what is presently known about the asymptotic dynamics of interval mappings and also to give applications to two standard models in population dynamics. We consider one-dimensional systems whose state space is an interval I, with a discrete time evolution given by a mapping $f : I \to I$; if the system is in state x_n at time n, it will be in state $x_{n+1} = f(x_n)$ at time $n+1$. Our main theme is, *What is the typical long-time behavior for such a system?* Here, *typical* means something that is seen for many initial conditions and many parameters. The requirements on f will vary from theorem to theorem in order to avoid being unnecessarily restrictive, but our main concern is unimodal (one-humped) mappings with some smoothness and parameterized families of such mappings. A particularly well understood case is the quadratic (alias the logistic) family

$$Q_\lambda(x) = \lambda x(1-x), \qquad 0 < \lambda \leq 4, \tag{1.1}$$

mapping the interval $[0, 1]$ into itself.

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In population dynamics one often considers the *Ricker* family [Ric54]

$$R_{\lambda,\beta}(x) = \lambda x e^{-\beta x}, \qquad \lambda > 1, \quad \beta > 0, \tag{1.2}$$

and the *Hassell* family [Has74]

$$H_{\lambda,\beta}(x) = \frac{\lambda x}{(1+x)^{\beta}}, \qquad \lambda > 1, \quad \beta > 1. \tag{1.3}$$

Both these models have $[0,\infty)$ as state space, but all nontransient dynamics takes place on a bounded interval.

Robert May's famous article [May76] was one of the first to call broad attention to the complexity of the dynamics produced by systems such as these. Background on the use of these models in population dynamics can be found in the above-mentioned sources and in [HLM74] and [MO76]. Surveys also appear in the books by Renshaw [Ren91] and Murray [Mur93]. A discussion of chaos in ecological modeling with an extensive bibliography can be found in [HHE$^+$93]. Recent work includes [Mor90], where problems with fitting models to real data was discussed; [GHK94] and [HV97] considered the case of stochastically varying parameters; in [GHL97] certain models were studied that combine discrete and continuous time modeling. Nice introductions to the mathematical aspects of unimodal dynamics can be found in [CE80] and in [Dev89].

The present article is an attempt to complement, update, and refine the above-mentioned expositions. The case of deterministic one-dimensional dynamics is today fairly well understood, and even though there is a considerable technical machinery involved, many theorems that are relevant for applications can be stated and understood in a standard context.

We will discuss attractors (there is a unique one for each of the systems above) and so-called natural measures which describe the asymptotic distributions on the attractors. A strong form of chaotic maps, which are characterized by a natural measure with an integrable density and exponential sensitivity to initial conditions, is recognized, and it is shown that such maps appear with positive probability in typical families.

We particularly want to stress that the parameter dependence is much more subtle and complicated (and interesting) than the much oversimplified picture sometimes put forth of a sequence of period-doubling bifurcations leading to a "chaotic regime." This "chaotic regime" is in fact densely interspersed with "periodic regimes," some corresponding to short cycles, where typical orbits are absorbed into one single periodic motion. Also, asymptotic distributions depend in a nontrivial way on the parameter. We consider both deterministic and stochastic perturbations.

This paper is divided into three parts. Part I continues with a few remarks on one-dimensional models and on the concept of chaos. Next follows an informal survey of what can be expected in one-parameter families of systems, using (1.1) as a model case. Then in Part II follow a few sections containing precisely formulated theorems. In Part III this theory is applied to the Ricker and Hassell models.

This is a survey. Of the theorems stated in Part II, only parts of Theorem 20 on the parameter dependence of natural measures are original to the author. The conclusions for the population models in Part III are in most cases easy consequences of the theorems of Part II, requiring only routine calculations. The proof of Theorems 28 and 29 (abundance of chaotic maps) is the least trivial part. Most of the statements of Part III have been part of the working assumptions in population dynamics for many years, but only recently have tools become available to back them up with rigorous results.

1.1. The Visible versus the Invisible: Period Three versus Chaos. When modeling with dynamical systems, one is primarily interested in phenomena that are visible, numerically and in real life. This means focusing on dynamics that are stable under perturbations and that are realized for large sets of initial conditions and parameters. Catchy as it might be, the title "Period Three Implies Chaos" of the often cited paper [LY75] by Li and Yorke causes confusion when this dichotomy between visible and invisible dynamics is neglected. Sometimes chaos is declared to exist in a certain system, because a period-3 orbit has been detected. But the chaos alluded to in the theorem of Li and Yorke might very well take place on a set of Lebesgue measure zero and thus be completely invisible. For example, there is an open set of parameters for which the quadratic maps Q_λ have a globally attracting 3-cycle, attracting almost all (in the sense of Lebesgue measure) initial conditions. So these maps will model a highly regular behavior, even though there is some weak type of chaos taking place on an invisible set. The same is true for the families (1.2) and (1.3): For each fixed β there is an open interval of λ values such that the corresponding map has a 3-cycle attracting almost all initial conditions. See Figure 1 for an example, taken from the Hassell family. These period-3 windows in parameter space are clearly visible in Figures 2 (logistic), 3 (Hassell), and 5 (Ricker). Note that this is deep into "the chaotic regime." (This coexistence of period-3 chaos and a global periodic attractor in fact persists on a dense set of λ values to the right of these period-3 windows, since maps with periodic attractors are dense in parameter space and the period-3 orbit persists as an unstable one.)

1.2. Long-Periodic Motions versus Chaos. In real phenomena, a long stable cycle will be indistinguishable from nonperiodic motion, simply because one never observes more than a finite number of generations. Whether or not an attractor is truly nonperiodic is probably of purely academic interest; in this respect, the asymptotic theory of dynamical systems might not seem very relevant. But for the overall picture, theory is important. For example, we will see that the asymptotic distribution of orbits, though almost independent of initial conditions, will be very sensitive to perturbations in parameter space when we are close to certain chaotic systems. One should also be aware of the following scenario, which could be realized in any of the models above.

Suppose we are given a system f, which has stable cycle of length p, but this has not been revealed to us. We choose an initial condition x_0 and observe its orbit for, say, $p-1$ steps. So far we have not detected any periodic motion, and furthermore it can happen that small perturbations of x_0 grow rapidly during these first $p-1$ iterates. If p is large we may then be tempted to declare the motion to be chaotic, but following the system for one more step we see the orbit closing, and the sensitivity to perturbations is killed by a small derivative in this last step of the cycle. Using the word *chaos* for a possibly longer-than-observed stable periodic motion could thus be very misleading.

It is true that the transient phase could be long and complicated before settling into a (possibly short) periodic cycle. In the presence of a 3-cycle, stable or unstable, Šarkovs̆kiĭ's theorem [Šar64] implies that there are cycles of all orders and that these will influence the transient dynamics. But to avoid verbal inflation, we suggest sticking to the word *complicated*, and saving *chaos* for systems that are truly and visibly chaotic in the sense hinted at above and to be described below.

Unfortunately, in a parameterized model it is almost never possible to pinpoint a truly chaotic parameter, even though one knows that they are there (in great abun-

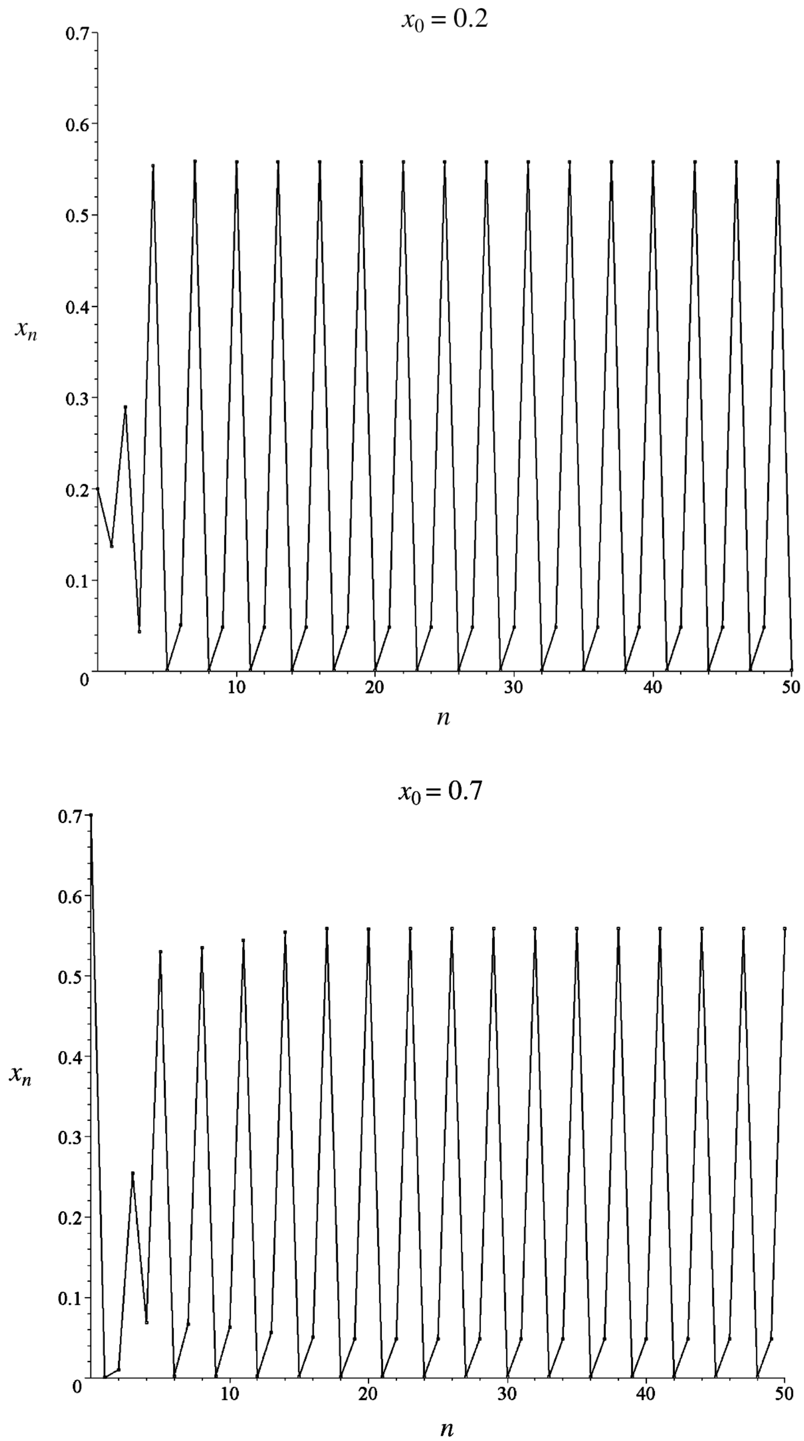

Fig. 1 *Two orbits, x_n versus n, of the Hassell map $H_{31,20.914}$ of (1.3), which has an attracting 3-cycle $0.0017 \mapsto 0.0485 \mapsto 0.5584 \mapsto 0.0017$, attracting almost all orbits.*

dance), and it is not always true that a chaotic map and nearby maps with long stable cycles have the same asymptotics even on a coarse-grained level. But on the other hand, we will see that small random perturbations of a chaotic map often roughly preserve the asymptotics and that strongly chaotic maps typically occupy a positive measure set in parameter space. In this sense, chaotic maps are visible and stochastically robust.

1.3. Deterministic One-Dimensional Models in Population Dynamics. There are several motivations to study deterministic time-discrete one-dimensional models in the context of population dynamics:

1. Such models are indeed used in population dynamics; see, for example, [Ric54], [May76], [HLM74], [Ren91], [Mur93], and references therein.
2. Higher dimensional systems can sometimes be approximated by, or reduced to, one-dimensional systems. This could be through different scales of relaxation times, or by considering return maps for continuous time systems to some lower dimensional subset. How models like (1.2) could be derived from models with structured competition is discussed in [GHL97] and in the references therein. See also [BS99] for an example of a Ricker-like map as a first return map for amplitudes in a certain continuous-time predator-prey model.
3. For many of the results described in this paper there are analogous statements about more complex systems, some of them proven and some of them expected to be true in generic situations. So the fairly well understood one-dimensional case (and in particular the quadratic family) can be seen as a model of models, exhibiting various features one could expect to find in more complicated (realistic) systems.
4. Even when a stochastic model is preferred, the study of such a model is often facilitated by a good understanding of its deterministic counterpart. See, for example, [Hög97], where Ricker maps and their asymptotic distributions appear as a limiting case of certain stochastic branching models.

2. An Informal Summary. We now give an informal summary, using the quadratic family (1.1) as an example. We will give signposts to the full theorems appearing in the following sections, where the reader can see in which situations the various properties are rigorously known. When we say that a set is "large" we mean that it has positive Lebesgue measure; "almost all" means full Lebesgue measure. f^n will as usual denote the nth iterate of f.

2.1. The Attractor. Let us imagine the following experiment. For each parameter λ we pick a random initial point $x_0 \in [0,1]$ and start to compute its orbit

$$\{x_n(\lambda)\}_{n=0}^{\infty}, \qquad \text{where } x_n(\lambda) := Q_\lambda^n(x_0).$$

What is the long-time behavior of this sequence? Denote by $\omega_\lambda(x_0)$ the set of accumulation points of the orbit of x_0 under Q_λ, that is, the set where the orbit tends to settle down once transients have died out. It is conceivable that the different initial points x_0 will have different asymptotic orbits, but it turns out that this is typically not the case. For each λ there exists a unique set $\Omega_\lambda \subset [0,1]$, the *metric attractor* (Definition 1) of Q_λ, such that $\omega_\lambda(x_0) = \Omega_\lambda$ for almost all x_0. In other words, if we choose x_0 from a uniform distribution on $[0,1]$, the orbit of x_0 will be attracted to and fill out Ω_λ with full probability. For different λ values we see different attractors. If we plot Ω_λ versus λ we obtain the well-known bifurcation diagram of Figure 2.

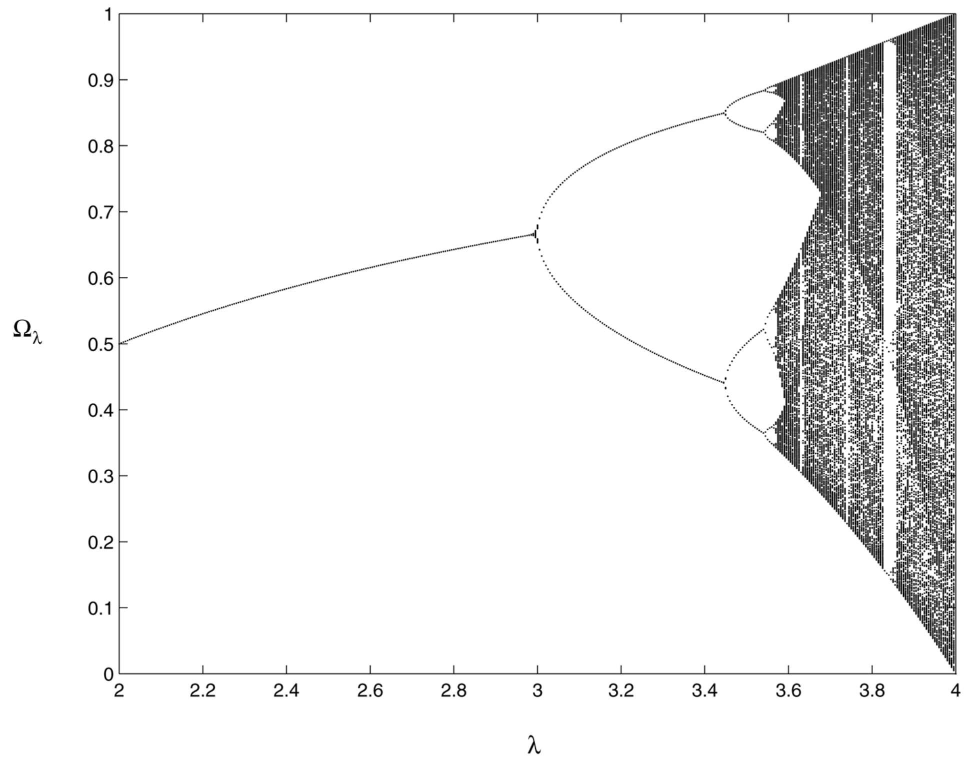

Fig. 2 *The attractor Ω_λ versus λ for the logistic map $Q_\lambda(x) = \lambda x(1-x)$ of (1.1). For 400 equally spaced λ values we have computed 900 iterates of $x_0 = 0.5$. For each λ, iterates x_{301}–x_{900}, an approximation of Ω_λ, are shown. For a dense set of parameters, Ω_λ is a (possibly very long) periodic cycle. Notice the parameter interval around $\lambda \approx 3.85$, where we see an attractive 3-cycle followed by a cascade of attractive $(3 \cdot 2^n)$-cycles.*

The attractor Ω_λ comes in only three different flavors (see Theorem 6):

(1) a periodic cycle,
(2) an attracting Cantor set of zero Lebesgue measure, and
(3) a finite union of intervals;

all of them do appear. Chaos in almost any sense (except for "period-3 chaos") is only possible when we have an interval attractor.

2.2. The Structure in Parameter Space. A natural task is to describe the set of parameters corresponding to these three different types. In the quadratic case the following is true.

(1) $\mathcal{P} := \{\lambda \mid \Omega_\lambda \text{ is a periodic cycle}\}$ is dense in parameter space (see Theorems 14 and 17) and consists of countably infinitely many nontrivial intervals. Moving the parameter inside one connected component of $\mathcal{P}$, we see the period-doubling scenario, with universal scaling in parameter space (see section 2.6).
(2) $\mathcal{C} := \{\lambda \mid \Omega_\lambda \text{ is a Cantor set}\}$ is a completely disconnected set of Lebesgue measure zero (see Theorem 10).
(3) $\mathcal{I} := \{\lambda \mid \Omega_\lambda \text{ is a union of intervals}\}$ is a completely disconnected set of positive Lebesgue measure (see Theorems 9 and 18).

So the type of attractor we are likely to find depends on how we restrict our parameter. Any nontrivial parameter interval will contain maps with stable cycles, and inside one of the components of $\mathcal{P}$ we find stable periodic maps with full probability. But close to a $\lambda \in \mathcal{I}$ we are likely to find interval attractors. Recall that this is a general property of a set of positive Lebesgue measure: almost all points of the set are so-called Lebesgue density points, where measure accumulates.

2.3. The Distribution of Orbits. Knowing that for a given system Q_λ almost all orbits tend to the same attractor, we might ask how different orbits distribute themselves on the attractor. Except for when Ω_λ is a periodic cycle, it is not a priori clear that they have the same distribution. We put point masses along the orbits and normalize in order to obtain a probability measure: we form the Birkhoff sums

$$\mu_{\lambda,n}(x_0) = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{x_k(\lambda)}.$$

Hoping for convergence, we take the limit (in the weak* sense[1])

$$\mu_\lambda(x_0) = \lim_{n\to\infty} \mu_{\lambda,n}(x_0)$$

in order to obtain a probability measure describing the asymptotic distribution of the orbit of x_0 under Q_λ. For an interval J, $\mu_\lambda(x_0)(J)$ is then the asymptotic frequency with which the orbit of x_0 under Q_λ visits J. Note that $\mu_\lambda(x_0)$ will be *invariant*:

$$\mu_\lambda(x_0)(Q_\lambda^{-1}E) = \mu_\lambda(x_0)(E) \qquad \text{for any measurable set } E.$$

Our hope is that $\mu_\lambda(x_0)$ should exist and also be essentially independent of x_0, so that we can speak of a common invariant measure μ_λ describing the asymptotics for a large set of initial conditions. In this case μ_λ is called a *natural* (*physical*, *observable*, *SRB*, *SBR*, etc.[2]) measure (see Definition 8). This is indeed true for almost all quadratic maps (but fails for uncountably many parameters λ). More precisely, we have the following (see Theorems 9 and 10):

(1) If $\lambda \in \mathcal{P}$, there exists a natural measure μ_λ and $\mu_\lambda = \mu_\lambda(x_0)$ for almost all x_0. μ_λ consists of normalized point masses on the periodic cycle Ω_λ.

(2) If $\lambda \in \mathcal{C}$, there exists a natural measure μ_λ and $\mu_\lambda = \mu_\lambda(x_0)$ for almost all x_0. The support of μ_λ equals the Cantor attractor Ω_λ, and so μ_λ is singular with respect to Lebesgue measure.

(3) (a) There is a full-measure subset $\mathcal{S} \subset \mathcal{I}$ such that for all $\lambda \in \mathcal{S}$, there exists a natural measure μ_λ that equals $\mu_\lambda(x_0)$ for almost all x_0. Furthermore, μ_λ is absolutely continuous with respect to Lebesgue measure; that is, μ_λ has an integrable density, and the support of μ_λ equals the interval attractor Ω_λ.

(b) There are uncountably many parameters in $\mathcal{I}\backslash\mathcal{S}$. For these maps various singular phenomena can occur: Q_λ may lack a finite natural measure altogether, or the distribution on the interval attractor could be extremely nonuniform, leading to a measure with a nonintegrable density.

If we start off with a distribution of initial conditions rather than a single initial point, then the natural measures are attracting stationary states. This point of view is suitable when comparing with stochastic models.

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[1] Recall that μ is the weak* limit of μ_n if $\int_I f d\mu = \lim_{n\to\infty} \int_I f d\mu_n$ for any function f continuous on I.

[2] This concept was introduced by Ya. G. Sinai in the context of so-called Anosov systems and was applied by D. Ruelle and R. Bowen in the setting of Axiom A systems.

2.4. Periodic Cycles or Chaos. We have just seen that for almost all parameters λ, Q_λ has a unique natural measure μ_λ describing the asymptotics of almost all orbits under Q_λ, where μ_λ is either

- [the periodic case $\mathcal{P}$] a finite union of point masses sitting on a periodic attractor, or
- [the chaotic case $\mathcal{S}$] a finite measure absolutely continuous with respect to Lebesgue measure, supported on an interval attractor.

Cases (2) and (3b) above are avoided with full probability in parameter space.

We call the second case *chaotic* for the following reason: not only do orbits distribute themselves in a seemingly random way, but these maps also have positive Lyapunov exponents almost everywhere; that is, there are constants $\gamma > 0$ and $C > 0$ such that

$$|D_x Q_\lambda^n(x_0)| \geq Ce^{\gamma n}$$

for almost all x_0 when $\lambda \in \mathcal{S}$ (Theorem 12). The map will have a strong form of sensitive dependence on initial conditions, with nearby orbits separating at a uniform exponential speed. Any uncertainty in the observed initial state will make long-time predictions of the orbit impossible, but the statistics of the orbit are accessible. Orbits distributes themselves *stochastically* according to μ_λ. Figures 7 and 8 show the Lyapunov exponent and the natural measure for $R_{16.999,2.0}$.

2.5. Parameter Dependence of Asymptotic Distributions. We now address the question of how the natural measure varies with the parameter. First observe that a stable periodic attractor persists and moves continuously with the parameter in any smooth family, so μ_λ will depend (weak*) continuously on λ at points in the interior of $\mathcal{P}$.

For chaotic maps the situation is much more complicated. Recall that $\mathcal{P}$, the set of parameters corresponding to maps with periodic attractors, is dense in parameter space, so arbitrarily small perturbations of a chaotic map can lead to a map with a periodic attractor. The natural measure also exhibits a sensitive dependence on the parameter. In, say, the logistic family, it is known that there is a set $\mathcal{A} \subset \mathcal{S}$ of positive measure such that for any $\lambda^* \in \mathcal{A}$,

- $\lambda \mapsto \mu_\lambda$ is discontinuous at $\lambda = \lambda^*$. Any measure consisting of point masses on an unstable periodic orbit of Q_{λ^*}, as well as the natural measure μ_{λ^*}, can be approximated with measures supported on periodic attractors of nearby maps. To avoid this discontinuity, either open intervals $\subset \mathcal{P}$ or the entire set $\mathcal{A}$ must be deleted.

Still there are some continuity properties:

- As mentioned, μ_{λ^*} can be approximated with natural measures for certain nearby maps with periodic attractors;
- $\lambda \mapsto \mu_\lambda$ restricted to $\mathcal{A}$ is continuous;
- μ_{λ^*} is stochastically stable: if we add a stochastic perturbation after each iteration we get a Markov chain with an equilibrium measure. When the size of the perturbation goes to zero, this equilibrium measure tends to the measure μ_{λ^*}.

See Theorems 20 and 21. In this context we would like to stress the first and the last of these properties: typically, the natural measure of a chaotic map may be destroyed under arbitrarily small deterministic perturbations, but it will be approximately preserved under small stochastic perturbations. Noise is in this way a savior when modeling in the chaotic regime. We now give a simple example of this idea.

Example. The map $Q_4(x) = 4x(1-x)$ is known to be of interval attractor type with an absolutely continuous natural measure $\mu = \mu_4$. This is one of the rare instances when such a measure can be computed explicitly, and μ turns out to have a density symmetric with respect to the critical point $c = 1/2$.

For $\lambda = 4$, we have that

$$1/2 \xrightarrow{Q_4} 1 \xrightarrow{Q_4} 0 \overset{Q_4}{\circlearrowleft} .$$

So for λ close to 4, the critical orbit will first move close to 1 and then spend a long time on the left-hand side of the interval climbing up from 0. By tuning λ we can arrange for the critical orbit to finally land on the critical point $c = 1/2$. We thus find a sequence $\{\lambda_n\}$, $\lim_{n\to\infty} \lambda_n = 4$, such that Q_{λ_n} has a periodic attractor of length n containing the critical point and such that each of these periodic attractors has just one point in the right half of the interval $[0, 1]$.

Thus $\mu(1/2, 1] = 1/2$ but $\mu_{\lambda_n}(1/2, 1] = 1/n$, so $\lim_{n\to\infty} \mu_{\lambda_n}(0, 1] = 0$. By continuity on $\mathcal{P}$ the same holds for parameter sequences $\{\gamma_n\}$, when γ_n is sufficiently close to λ_n and, in particular, in the same periodic window. So arbitrarily small deterministic perturbations of $\lambda = 4$ can lead to completely different asymptotics.

However, the probability of this is small. It is known that the set $\mathcal{A}$, within which measures do vary continuously, has a Lebesgue density point at $\lambda = 4$. So if we add small random perturbations at each iteration step, we are most likely to end up composing a chain of maps in which most maps belong to the set $\mathcal{A}$. Intuitively at least, this explains why stochastic perturbations roughly preserve the asymptotic distribution.

We remark that by changing coordinates, the quadratic family can be written in the form

$$x \mapsto x^2 + c.$$

In this form perturbations in the parameter c are the same as perturbations in phase space.

2.6. Period Doubling at Universal Rates. For completeness we end this overview with a few words on the period-doubling scenario, even though we will not go into any details.

As can be seen from Figure 2, for $\lambda < \lambda_1 = 3$, the attractor Ω_λ is a fixed point. When λ increases beyond 3, the attractor bifurcates into a 2-cycle, followed by a 4-cycle, and then through the whole sequence of 2^n-cycles. Let λ_n be the parameter where the 2^n-cycle is created. If one studies the rate of change of the distance in parameter space between successive bifurcations one finds that

$$\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} = \delta \approx 4.669. \tag{2.1}$$

In other words, the distances between successive bifurcation points scale asymptotically in a geometric way, as λ_n increase to the finite limit λ_∞. (The attractor Ω_{λ_∞} is a Cantor attractor.)

As mentioned, the set $\mathcal{P}$ is dense in parameter space. In fact it consists of intervals $[\tilde\lambda, \tilde\lambda_\infty)$, where a periodic attractor of length, say, k is born when λ passes $\tilde\lambda$, and period doubling to $(k \cdot 2^n)$-cycles occurs as λ passes certain $\tilde\lambda_n$. Once again the sequence $\{\tilde\lambda_n\}$ accumulates at some finite $\tilde\lambda_\infty$, corresponding to a Cantor attractor, with *the same* asymptotic geometric rate δ.

In fact, this is *independent* of the actual analytic form of the function family. The same scaling with the same δ is found in any one-humped family, where the critical point c is nondegenerate ($f''(c) \neq 0$), and the hump moves monotonically (in the vertical direction) with the parameter. For families with critical point of higher but finite order, there are other universal rates.

A key feature of this is that from observations of a few period-doubling bifurcations one can predict successive bifurcation values and their accumulation point. This can be done for real-life systems, assuming nothing about the underlying model other than a one-humped transition function with nondegenerate critical point.

One of the first researchers to observe this phenomenon and provide a mathematical explanation was Mitchel Feigenbaum [Fei79]. (The number $\delta = 4.669\ldots$ is known as Feigenbaum's constant.) This scenario has been rigorously verified in a number of cases, including the logistic family [Lan82]. See also [CE80], [EE90], [EW87], [Sul88], Chapter VI in [MS93], [McM96], [Lyu97], [Mar98], and ongoing work by de Melo, Pinto, and de Faria.

Period doubling has also been observed in numerical experiments in higher dimensions with both discrete and continuous time systems, as well as in physical systems.

Part II. Rigorous Results

We will now introduce some concepts and theorems that give the scenario above for the logistic family (1.1), and which to a large extent are applicable to the Ricker family (1.2) and the Hassell family (1.3).

We consider continuous maps of an interval I into itself. f^n will denote the nth iterate of f, and by the (forward) orbit of x (under f) we mean the set $\{f^k(x)\}_{k=0}^{\infty}$.

3. Attractors for S-Unimodal Maps. The asymptotic behavior of the orbit of point x under f is described by the set of accumulation points, the *ω-limit set of x,* $\omega(x)$, defined by

$$\omega(x) = \omega_f(x) = \left\{ y \in I \mid \exists \{n_j\}_{j=1}^{\infty} \text{ such that } y = \lim_{j\to\infty} f^{n_j}(x) \right\}.$$

A point x is *periodic* with period p if $f^p(x) = x$, and p is the smallest positive integer with this property. A periodic orbit P is *attracting* if there is a neighborhood U of P such that $\omega(y) = P$ for all $y \in U$. If f is C^1, we may define the multiplier $\lambda(P)$ of a periodic orbit P: by the chain rule, $(f^p)'$ is constant on a periodic orbit of length p. Let $\lambda(P) = (f^p)'(x_0)$, where $x_0 \in P$. We then classify periodic orbits as follows:

- P is *superstable* if $\lambda(P) = 0$. This is equivalent to the condition that P contains a critical point.
- P is *stable* if $0 < |\lambda(P)| < 1$.
- P is *neutral* if $|\lambda(P)| = 1$.
- P is *unstable* if $|\lambda(P)| > 1$.

It is clear that stable and superstable periodic orbits are attracting. A neutral periodic orbit may or may not be attracting.

A set Γ is called *forward invariant* if $f(\Gamma) = \Gamma$.

Let $B(\Gamma)$ denote *the basin of attraction* of a forward invariant set Γ, that is,

$$B(\Gamma) = \{x \mid \omega(x) \subset \Gamma\}.$$

In other words, $B(\Gamma)$ consists of all points that asymptotically end up in Γ.

By an *attractor* we mean a set that has a large basin, such that all parts of the attractor attract something substantial.

DEFINITION 1 (metric attractor [Mil85]). *A forward invariant set* Ω *is called a* metric attractor *if* $B(\Omega)$ *satisfies*

(1) $B(\Omega)$ *has positive Lebesgue measure;*

(2) *if* Ω' *is another forward invariant set, strictly contained in* Ω*, then* $B(\Omega) \setminus B(\Omega')$ *has positive measure.*

An attracting fixed point or an attracting periodic cycle are of course attractors in this sense.

We will mainly consider *unimodal* (one-humped) maps.

DEFINITION 2 (unimodal). *A continuous interval map* $f\colon I = [a,b] \to I$ *is* unimodal *if there is a unique maximum* c *in the interior of* I *such that* f *is strictly increasing on* $[a,c)$ *and strictly decreasing on* $(c,b]$*. For simplicity, unimodal will also require that either* a *be a fixed point with* b *as its other preimage, or that* $I = \left[f^2(c), f(c)\right]$.

Remark. The conditions on the boundary behavior guarantee uniqueness of attractors for the class of maps we are about to study. See Theorem 4, Corollary 5, and Theorem 6.

For the maps we have in mind, $f(c) \le c$ will always imply the existence of a globally attracting fixed point. If $c \le f^2(c) < f(c)$, then there is a globally attracting fixed point or 2-cycle in $(c, f(c))$. These cases need not be unimodal according to our definition, although they have the right shape, but on the other hand there is not much to say about the dynamics.

If $f^2(c) < c < f(c)$, we can often reduce the study to the so-called *dynamical core* $[f^2(c), f^1(c)]$, which is mapped onto itself and absorbs all initial conditions (except $x_0 = a$ when $a \in \partial I$ is a fixed point).

We consider differentiable maps; in particular the turning point will be a critical point, $f'(c) = 0$.

If $f''(c) = 0$, we say that the critical point c is *degenerate*. If the nth derivative vanishes at c, $f^{(n)}(c) = 0$ for all $n > 0$, we say that c is a *flat critical point*. Here we will always assume at least nonflatness.

DEFINITION 3 (negative Schwarzian). *An interval map* $f\colon I \to I$ *has* negative Schwarzian derivative *if* f *is of class* C^3 *and*

$$Sf(x) := \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 < 0 \quad \text{for all } x \in I \setminus \{c \mid f'(c) = 0\}.$$

A unimodal map with negative Schwarzian will be referred to as an S-unimodal map.

The importance of the Schwarzian derivative in interval dynamics comes from the following theorem.

THEOREM 4 (Singer's theorem [Sin78]). *Let* $f\colon I \to I$ *be a* C^3 *interval map with negative Schwarzian derivative. Then each attracting periodic orbit attracts at least one critical point or boundary point. Furthermore, each neutral periodic point is attracting.*

COROLLARY 5. *An S-unimodal map can have at most one periodic attractor, and it will attract the critical point.*

Unimodal maps with negative Schwarzian and nonflat critical point come in only a few different flavors according to the next theorem, due to Blokh and Lyubich.

THEOREM 6 (see [BL91]). *Let* $f\colon I \to I$ *be an S-unimodal map with nonflat critical point. Then* f *has a unique metric attractor* Ω*, such that* $\omega(x) = \Omega$ *for Lebesgue almost all* $x \in I$*. The attractor* Ω *is of one of the following types:*

(1) *an attracting periodic orbit;*
(2) *a Cantor set of measure zero;*
(3) *a finite union of intervals with a dense orbit.*

In the first two cases, $\Omega = \omega(c)$.

We will also need the following concept.

DEFINITION 7 (Misiurewicz map). *A map f is called a* Misiurewicz map *if it has no periodic attractors and if critical orbits do not accumulate on critical points, that is, if*

$$\mathcal{C} \cap \omega_f(\mathcal{C}) = \emptyset,$$

where $\mathcal{C}$ denotes the set of critical points of f.

Example. Consider $Q_4(x) = 4x(1-x)$. One verifies that this is an S-unimodal map on $[0,1]$ and that $Q_4(0) = Q_4(1) = 0$. Furthermore, we know that for the critical point $c = 1/2$ we have

$$1/2 \xrightarrow{Q_4} 1 \xrightarrow{Q_4} 0 \overset{Q_4}{\circlearrowleft} .$$

Since 0 is a repelling fixed point, we apply Singer's theorem and conclude that Q_4 has no periodic attractors. We thus see that Q_4 is a Misiurewicz map, and from this it follows that Q_4 has strong chaotic properties (see below).

4. Natural Measures.

A Borel measure μ is invariant for $f\colon I \to I$ if $\mu(f^{-1}(E)) = \mu(E)$ for every measurable set $E \subset I$. One looks for invariant measures that describe the asymptotic distribution under iteration for a large set of initial points.

DEFINITION 8 (natural measure). *An invariant measure μ is called a* natural measure *for f if*

$$\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} \tag{4.1}$$

for all x in a set of positive Lebesgue measure. (Here δ_x denotes a Dirac mass in x, and the limit is in the weak sense.)*

Natural measures are also known by the names *physical measures* and *Sinai–Ruelle–Bowen (SRB) measures*. One could think of them as *absorbing stationary distributions*. In many cases, as we will see, such measures exist and describe *almost all* orbits so that they are unique.

For a map with a periodic attractor, point masses of equal weight on the points of the attractor constitute a natural measure.

We will also be interested in the case when the natural measure has a density; this we interpret as a sign of chaos. The following abbreviations will be used:

- *acim* stands for *absolutely continuous* (with respect to Lebesgue measure) *invariant measure*;
- *acip* stands for *absolutely continuous invariant probability measure*, a finite and normalized *acim*.

The following theorem (see Chapter V.1 in [MS93]) will be relevant for our discussion.

THEOREM 9. *Let f be an S-unimodal map with nonflat critical point. If f has a periodic attractor, or a Cantor attractor, then f admits a unique natural measure supported on the attractor.*

If f admits an acip μ, then
(1) *μ is a natural measure;*
(2) *the attractor Ω of f is an interval attractor;*
(3) $\operatorname{supp}(\mu) = \Omega$, *in particular, μ is equivalent to the Lebesgue measure on Ω.*

In all these cases the natural measure describes the distribution for almost all initial conditions.

Remark. Observe that the theorem does not guarantee the existence of a natural measure in the case of an interval attractor. Indeed there are uncountably many parameters in the logistic family, for which the corresponding maps have interval attractors and lack natural measures altogether, or have natural measures with weird properties. See [Joh87] and [HK90].

But, at least in the logistic family, both such singular phenomena and Cantor attractors are rare in the sense of Lebesgue measure.

THEOREM 10 (Lyubich [Lyu97], [Lyu98]). *For almost all $\lambda \in (0,4]$, the logistic map Q_λ has either a periodic attractor or an interval attractor supporting an acip.*

A generalization of this theorem to unimodal, real-analytic families with quadratic critical point has recently been announced by Ávila, Lyubich, and de Melo.

5. Chaos. In [Guc79], Guckenheimer made the following definition.

DEFINITION 11 (sensitive dependence). *An interval map f has* sensitive dependence on initial conditions *if there exists a set K of positive Lebesgue measure with the property that there exists a $\delta > 0$ such that for every $x \in K$ and every interval neighborhood J of x, there is an n such that $f^n(J)$ has length larger than δ.*

This means that with positive probability we find points with arbitrarily small neighborhoods which sooner or later expand to macroscopic scale.

If f is as in Theorem 6, Guckenheimer proved that

- if f has a periodic attractor, then f does not have sensitive dependence;
- if f has an interval attractor, then f has sensitive dependence.

Remark. Cantor attractors can be of two types. In [Lyu94], Lyubich showed that an S-unimodal map with nondegenerate critical point that also has a Cantor attractor has to be what is called infinitely renormalizable. Guckenheimer's work implies that there is no sensitive dependence in this case. So for S-unimodal maps with nondegenerate critical point, sensitive dependence on initial conditions is equivalent to the presence of an interval attractor.

In families of unimodal maps with critical point of sufficiently high order, there also exist attractors worthy of the name *strange attractors*: Cantor attractors with sensitive dependence on initial conditions. This was proved by Bruin et al. in [BKNvS96].

There are several definitions in use for what it means for a map to be chaotic, where sensitive dependence is one of the weakest. Even weaker is the type of chaos implied by the existence of a 3-cycle [LY75]; in this case we have sensitive dependence on an invariant uncountable set, which could be of measure zero. As mentioned in the introduction, such "period-3 chaos" could peacefully coexist with a stable periodic attractor whose basin of attraction has full measure.

One other possibility is to say that *f is chaotic if f admits an acip.* In the case of nonflat S-unimodal maps, almost all orbits distribute themselves according to this measure over entire intervals. This follows from Theorem 9 and is intimately connected with strong expansion properties. In fact it is equivalent to having a positive Lyapunov exponent almost everywhere.

THEOREM 12 (Keller [Kel90]). *Let $f : I \to I$ be an S-unimodal map with nonflat critical point. Then f admits an acip if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)| = \kappa > 0$$

for almost all $x \in I$.

Refer to Figures 7 and 8 for an illustration using a certain Ricker map of Misiurewicz type.

For nonflat S-unimodal maps, some weak expansion along the critical orbit of f is a sufficient condition for f to admit an acip. This is sometimes a good way to prove that f is chaotic.

THEOREM 13 (Nowicki and van Strien [NvS91]). *Let f be an S-unimodal map with nonflat critical point c of order ℓ; i.e., assume that there are constants C_1 and C_2 such that*

$$C_1 |x-c|^{\ell-1} \leq Df(x) \leq C_2 |x-c|^{\ell-1}.$$

Also assume that

$$\sum_{n=0}^{\infty} \frac{1}{|Df^n(f(c))|^{1/\ell}} < \infty.$$

Then f admits an acip, with a density that belongs to L^p for all $p < \frac{\ell}{\ell-1}$.

Remark. Typically one would expect exponential growth of the derivative to be associated with an acip, but not even the weak condition in the theorem above is necessary [Bru94a], [Bru94b].

It is easy to see that an acip will have singularities along the critical orbit $\{c_1, c_2, \ldots\}$ and that these will imply that the density of the acip is not in L^p for $p \geq \ell/(\ell-1)$, whenever the critical point is of order ℓ. These singularities are clearly visible in Figure 8.

Any Misiurewicz map with negative Schwarzian admits an acip under mild nonflatness conditions. See [Mis81] and [BM89]. In generic one-parameter families, there are uncountably many Misiurewicz maps, but they form a set of measure zero in parameter space [San98]. But as we will see in the next section, Misiurewicz parameters are Lebesgue density points of parameters corresponding to acips. In particular we find maps admitting acips for a positive measure set of parameters.

6. The Structure in Parameter Space. For S-unimodal maps with nonflat critical point, Theorem 6 gives three possible different types of asymptotic behavior. A fundamental problem presents itself: Describe the subsets in parameter space corresponding to different types of attractors.

6.1. Periodic Maps Are Dense. For the quadratic family we have the following theorem.

THEOREM 14. *The set of parameters λ such that Q_λ has a stable periodic attractor is dense and open in* $(0,4]$.

This theorem was first announced by Świątek in [Sa92], and a complete proof was given in [GŚ99]. In his thesis, Kozlovski [Koz97] solved the famous problem of structural stability and Axiom A for unimodal maps, and he generalized the theorem above to real-analytic families.

DEFINITION 15 (Axiom A). *An interval map f satisfies* Axiom A *if*

(1) *f has finitely many period attractors, and*
(2) *Df^n is uniformly exponentially large in n outside the basins of the periodic attractors.*

Note that an S-unimodal map satisfies Axiom A iff it has a stable periodic attractor.

THEOREM 16 (Kozlovski [Koz97]). *Axiom A maps are dense in the space of all C^k unimodal maps in the C^k topology, $k = 1, 2, \ldots, \infty, \omega$.*

THEOREM 17 (Kozlovski [Koz97]). *If $f_{\lambda_1,\dots,\lambda_l}$ is an l-parameter family of real-analytic unimodal maps of an interval, depending in a real-analytic fashion on the parameter(s), such that the family contains at least one Axiom A map and at least two nonconjugated maps, then the Axiom A maps are dense in the set of parameters $\{(\lambda_1,\dots,\lambda_l)\}$.*

6.2. Chaotic Maps Are Common in the Sense of Measure. Maps with periodic attractors are thus predominant from a topological point of view. On the other hand, one knows that chaotic maps are common in the sense of Lebesgue measure. This is a theorem that exists in many versions. The first result in this direction is due to Jakobson [Jak81]. The formulation below is taken from [MS93]. A slightly different version of Theorem 18 below was first presented in [TTY92]. The proof in [TTY94], as well as the proof in [MS93], uses the techniques introduced by Benedicks and Carleson in [BC85] and [BC91], where they proved an abundance of chaotic maps in the quadratic family.

The setup is as follows:

H1: f_λ is a one-parameter family of C^2 unimodal maps of an interval I.

H2: Each f_λ has a nondegenerate critical point c (we assume c is independent of λ; this could always be achieved by a smooth change of coordinates).

H3: Each f_λ has a repelling fixed point on the boundary of I.

H4: The map $(x,\lambda)\mapsto \big(f_\lambda(x), D_x f_\lambda(x), D_x^2 f_\lambda(x)\big)$ is C^1.

H5: λ^* is a parameter value such that f_{λ^*} is a Misiurewicz map; for simplicity we assume that the critical orbit is mapped onto an unstable periodic cycle P^* in a finite number of steps.

By general theory, an unstable periodic orbit (in fact any *hyperbolic set*) persists and moves smoothly under small perturbations of the map; for λ sufficiently close to λ^* we can find a point $x(\lambda)\in I$ and an unstable periodic orbit $P(\lambda)$ such that

- $\lambda\mapsto x(\lambda)$ is differentiable;
- $x(\lambda^*) = f_{\lambda^*}(c)$;
- $P(\lambda)$ moves continuously with λ and $P(\lambda^*) = P^*$;
- $x(\lambda)$ is mapped onto $P(\lambda)$ in the same number of steps as $x(\lambda^*)$, and onto the corresponding point of $P(\lambda)$.

One needs to know that the map really moves with the parameter at $\lambda=\lambda^*$ in the following sense:

H6: $\frac{d}{d\lambda}\left(x(\lambda) - f_\lambda(c)\right)\big|_{\lambda=\lambda^*} \neq 0$.

THEOREM 18. *If f_λ satisfies* H1–H6*, there exist constants $\gamma>0$ and $C>0$ and a positive measure set E of parameters with λ^* as a Lebesgue density point, such that*

$$|D_x f_\lambda^n(f_\lambda(c))| \geq Ce^{\gamma n} \qquad \text{for all } \lambda\in E \text{ and all } n\geq 1.$$

Using Theorems 9, 12, and 13 one also immediately obtains the following corollary.

COROLLARY 19. *If f_λ is also an S-unimodal family, then for all $\lambda\in E$ one also has that*

(1) *f_λ admits an acip μ_λ, with a density that is in L^p for any $p<2$;*

(2) *μ_λ is a natural measure describing asymptotics for almost all orbits;*

(3) *f_λ has positive Lyapunov exponent almost everywhere.*

Other papers with similar results and/or different proofs include [CE80], [Ryc88], [BY92], and [Luz00]. Following his paper [Tsu93b], Tsujii gave a generalization to polymodal families in [Tsu93a]. The hypothesis H2 can be relaxed to a nonflat critical point. Under some extra conditions the theorem in fact generalizes to certain families with flat critical point [Thu99]. We mention that one of the key points in the construction of the good parameter set E is that for $\lambda \in E$, the critical orbit should not accumulate too fast on the critical point c. For some small positive number α one requires that

$$|f_\lambda^n(c) - c| \geq e^{-\alpha n} \qquad \text{for all } \lambda \in E \quad \text{and all } n \geq 1. \tag{6.1}$$

We summarize the picture in parameter space for real-analytic one-parameter families of maps satisfying H1–H6. Stable periodic attractors are found for a dense and open set of parameters, and acips for a nowhere dense set of positive measure. These two cases (acips and period attractors) have full measure in parameter space. For this last property, nondegenerate critical point H2 is essential to the best of our knowledge; cf. Theorem 10 and its generalization to real-analytic unimodal families.

7. Parameter Dependence and Stochastic Perturbations. According to Theorem 10, the mapping

$$\Psi : \lambda \mapsto \mu_\lambda =: \text{the natural measure of the logistic map } Q_\lambda$$

is well defined for almost all λ in $(0, 4]$. Of course, Ψ is continuous on the interior of $\mathcal{P}$, the set of parameters for which Q_λ has a stable periodic orbit. In what follows we discuss the structure in parameter space and the parameter dependence of μ_λ near certain maps admitting an acip. From Theorem 18 and its corollary, we get a positive measure set $\mathcal{A}$, such that for $\lambda \in \mathcal{A}$, Q_λ admits an acip μ_λ. Combining the results from several papers, we see that we can choose $\mathcal{A}$ to have several other features, which we believe are typical for maps with an acip. In particular one finds that the parameter dependence of natural measures has severe singularities at any parameter in $\mathcal{A}$.

THEOREM 20. *There is a set of parameters $\mathcal{A}$ of positive measure, such that for each $\lambda \in \mathcal{A}$, one has the following properties.*

(1) *Q_λ fulfills the conclusions of Theorem* 18 *and Corollary* 19*. In particular Q_λ admits a natural acip μ_λ.*
(2) *There is an open set $\mathcal{O} \subset \mathcal{P}$ of parameters accumulating on λ, such that if $\mathcal{O} \ni \lambda_n \to \lambda$, then $\mu_{\lambda_n} \to \mu_\lambda$; i.e., the acip μ_λ is the weak* limit of certain measures on periodic attractors of nearby maps* [Thu96], [Thu98].
(3) *For any unstable periodic orbit P of Q_λ, there is an open set $\mathcal{O}_P \subset \mathcal{P}$ of parameters accumulating on λ, such that if $\mathcal{O}_P \ni \lambda_n \to \lambda$, then $\mu_{\lambda_n} \to \mu_P^{\text{sing}}$, where μ_P^{sing} is the equidistributed singular measure on P; i.e., any invariant measure of Q_λ sitting on an unstable periodic orbit can be approximated in the weak* sense by measures on periodic attractors of certain nearby maps* [Thu98].
(4) *It follows that $\Psi : \lambda \mapsto \mu_\lambda$ is discontinuous at any point of $\mathcal{A}$; to recover continuity open intervals of periodic maps, or the entire set $\mathcal{A}$, must be deleted. Ψ is thus not continuous on any set of parameters of full measure* [Thu98].
(5) *Ψ restricted to $\mathcal{A}$ is weak* continuous; in fact the densities vary continuously in L^p for $p < 2$* [RS92]*. See also* [Tsu96].
(6) *μ_λ is strongly stochastically stable in the sense of Baladi and Viana (see below)* [BV96].

Properties 3 and 4 of Theorem 20 hold for generic families satisfying H1–H6. For Property 2 one also needs to know that the critical orbit is distributed according to the acip for maps in $\mathcal{A}$. That this is the case for quadratic maps was shown in [BC85].

We comment on the notion of stochastic stability. Let f be an interval map with a natural acip μ and let $\{\Theta_\epsilon\}_{\epsilon>0}$ be a family of probability densities such that Θ_ϵ is supported on $[-\epsilon, \epsilon]$. The idea is to replace the nth iterate f^n of f by

$$(f + t_n) \circ \cdots \circ (f + t_1),$$

where t_i are random numbers chosen independently according to the common distribution Θ_ϵ. One thus obtains a discrete time Markov chain χ^ϵ on the interval. This Markov chain has a unique stationary probability measure m_ϵ.

We say that f is *weakly stochastically stable* under χ^ϵ if $m_\epsilon \to \mu$ in the weak* sense, when $\epsilon \to 0$. In [BY92], Benedicks and Young proved that there exists a set of parameters of positive measure such that the corresponding quadratic maps admit an acip that is weakly stochastically stable under a certain class of perturbations.

f is *strongly stochastically stable* under χ^ϵ if

$$\lim_{\epsilon \to 0} \left\| \frac{d\mu}{dx} - \frac{dm_\epsilon}{dx} \right\|_{L^1(dx)} = 0.$$

Baladi and Viana in [BV96] considered perturbations Θ_ϵ as above with some extra, mild technical assumptions and proved the following theorem.

THEOREM 21. *Let f be an S-unimodal interval map with nondegenerate critical point c. Assume there are constants $h_0 \geq 1$, $\gamma > 0$, and $0 < \alpha < \gamma/4$, such that*

(1) *$|f^n(c) - c| \geq e^{-\alpha n}$ for all $n \geq h_0$;*
(2) *$\left|(f^n)'(f(c))\right| \geq e^\gamma n$ for all $n \geq h_0$;*
(3) *f is topologically mixing on the interval bounded by $f(c)$ and $f^2(c)$.*

Then f has an invariant measure with an integrable density, which is strongly stochastically stable.

The first condition is built into the construction of the set E of Theorem 18, whose existence is implied by conditions H1–H6; see (6.1). The second is of course the property given by Theorem 18, and the third assumption always holds for S-unimodal maps with an acip μ: there is a full measure set of initial points whose orbits are distributed according to μ; in particular these orbits will be dense on the unique interval attractor. Thus any open set will contain a point with a dense orbit, and so the map is topologically mixing. From all this it follows that Theorem 21 is applicable to all maps in the set E of Theorem 18 and Corollary 19 in the S-unimodal case.

Remark. Note that the notions of weak and strong stochastic stability defined above should not be confused with stochastic stability in the sense of Högnäs and Vellekoop [HV97]. They studied generalizations of the Ricker models and the Hassell models where the "environmental" parameter β is a stochastic variable. They called such a Markov chain stochastically stable if it has an attracting, stationary distribution.

Part III. Applications to Population Dynamics

8. Uniqueness of Attractors in Population Dynamics. We now focus on the Ricker and Hassell families, defined in (1.2) and (1.3), and begin with some elementary but useful facts.

LEMMA 22. *Each map in the Ricker family has negative Schwarzian derivative.*

Proof. Calculating the Schwarzian derivative of $R_{\lambda,\beta}$ we find that

$$SR_{\lambda,\beta}(x) = -\frac{1}{2}\frac{\beta^4}{(1-\beta x)^2}\left(x^2 - 4x/\beta + 6\beta^2\right).$$

This is readily shown to be negative for all $x \neq c = 1/\beta$. ∎

LEMMA 23. *Each map in the Hassell family with $\beta \geq 2$ has negative Schwarzian derivative.*

Proof. Straightforward but somewhat tedious calculations give

$$SH_{\lambda,\beta}(x) = \left\{ -\frac{\beta(\beta-1)}{2(1+x)^2\left((\beta-1)x-1\right)^2} \right\} \left\{ (\beta-1)(\beta-2)x^2 - 4(\beta-2)x + 6 \right\}.$$

The first factor is obviously negative for $x \neq c = 1/(1-\beta)$, and the second is positive when $\beta \geq 2$. □

LEMMA 24. *Each map in the Hassell family with $1 < \beta < 2$ has a unique and globally attracting fixed point, regardless of the size of λ.*

Proof. The fixed point at $x = 0$ is globally attracting iff $\lambda \leq 1$. For $\lambda > 1$ there is an interior fixed point $z = \lambda^{1/\beta} - 1$, and calculating the derivative, one finds that

$$1 > H'_{\lambda,\beta}(z) = 1 - \beta(1 - \lambda^{-1/\beta}) > 1 - 2 = -1$$

if $\beta < 2$. z is easily seen to attract all initial conditions $x_0 \neq 0$. □

Let f be a Ricker or a Hassell map. Then the following facts are easily checked:

- if $f(c) \leq c$ or if $c \leq f^2(c) < f(c)$, then f has a globally attracting fixed point or 2-cycle;
- if $f^2(c) < c < f(c)$, then $I = [f^2(c), f(c)]$ absorbs all initial conditions $x_0 \neq 0$, and f restricted to I is unimodal.

Combining these observations with Lemmas 22–24 and Theorem 6, we obtain our first main result on Hassell and Ricker maps.

THEOREM 25. *Each Ricker or Hassell map has a unique metric attractor attracting almost all initial conditions. The attractor is a periodic cycle, an attracting Cantor set, or a finite union of intervals with a dense orbit.*

9. Structure in Parameter Space. Figures 3–6 show one-parameter bifurcation diagrams for our population models. We immediately notice that the λ dependence for each of the families and the β dependence for the Hassell family seem much the same as the parameter-dependence in the logistic family. In this section we will prove some results along these lines.

9.1. Maps with Periodic Attractors. As a corollary to Theorem 17 we immediately obtain the following theorem.

THEOREM 26. *Maps with a stable periodic attractor form a dense subset:*

(1) *in (λ, β)-space, for both the Ricker family $\{R_{\lambda,\beta}\}_{(\lambda,\beta)}$ and the Hassell family, $\{H_{\lambda,\beta}\}_{(\lambda,\beta)}$;*
(2) *in λ-space for each of the families $\{R_{\lambda,\beta_0}\}_\lambda$ and $\{H_{\lambda,\beta_0}\}_\lambda$ obtained when $\beta = \beta_0$ is fixed;*
(3) *in β-space for the one-parameter families $\{H_{\lambda_0,\beta}\}_\beta$, obtained when $\lambda = \lambda_0$ is fixed.*

Proof. All we have to do to apply Theorem 17 is to find an Axiom A map (a map with a stable cycle) and two nonconjugated maps within each of the families. The first statement in the theorem in fact follows from the second, which is easily proved since the interior fixed point starts out as a stable fixed point absorbing all initial conditions $x_0 \neq 0$ for small λ, and loses its stability for some $\hat{\lambda}(\beta)$ when a 2-cycle is born.

For the third statement we argue as follows. For $\lambda_0 \leq 1$, $x = 0$ is a globally attracting fixed point for all β, so this case is trivial. The interior fixed point $z = \lambda_0^{1/\beta} - 1$ exists for $\lambda_0 > 1$ and has multiplier

$$M(\beta) := H'_{\lambda_0,\beta}(z) = 1 - \beta\left(1 - \lambda_0^{-1/\beta}\right).$$

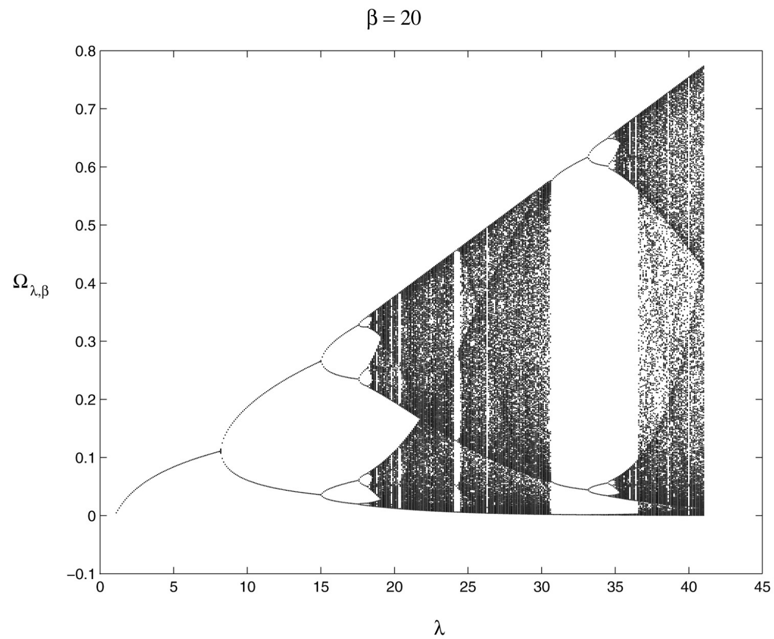

Fig. 3 *Bifurcation diagram $\Omega_{\lambda,\beta}$ versus λ as in Figure 2 but for the Hassell family of* (1.3) *with fixed $\beta = 20$. (We have also performed this experiment with the parameterization and the parameter values used in* [HV97] *and obtained the same type of picture as seen here. We have not been able to reproduce the bifurcation diagram in* [HV97].*)*

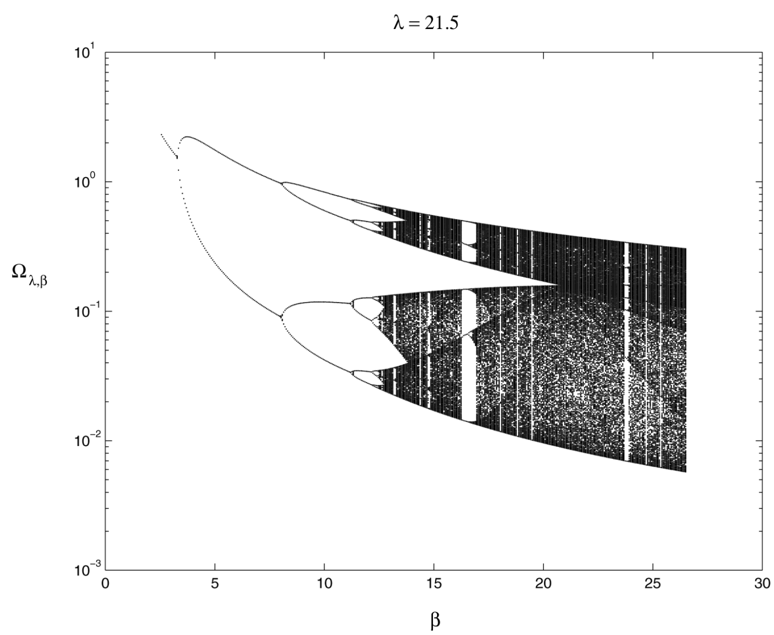

Fig. 4 *Bifurcation diagram $\Omega_{\lambda,\beta}$ versus β for the Hassell family with fixed $\lambda = 21.5$. Note the logarithmic vertical scale.*

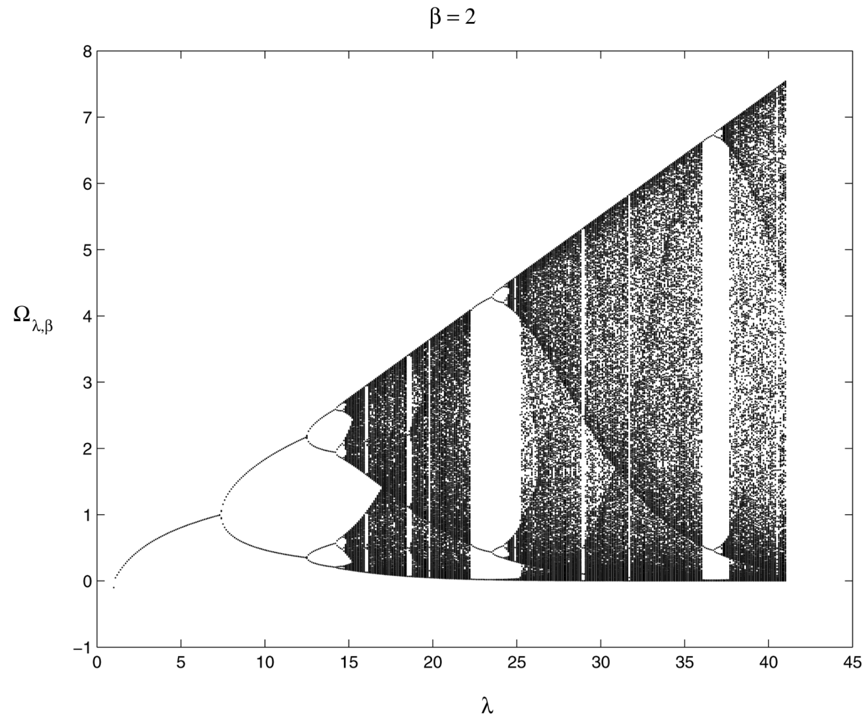

Fig. 5 *Bifurcation diagram $\Omega_{\lambda,\beta}$ versus λ for the Ricker family of* (1.2) *with fixed $\beta = 2$.*

One readily shows that M is decreasing in β and that

$$\lim_{\beta\to\infty} M(\beta) = 1 - \ln \lambda_0.$$

Thus for $1 < \lambda_0 \leq e^2$, z is stable and globally attracting for $H_{\lambda_0,\beta}$ for all β. Once again statement (3) is trivial. For $\lambda_0 > e^2$, z will lose its stability for β large enough. This can only happen when a 2-cycle is born, and a map with a 2-cycle can of course not be conjugated to a map without a 2-cycle, so we may apply Theorem 17. □

Remark. Note that the corresponding statement for one-dimensional sections $\{R_{\lambda_0,\beta}\}_\beta$ of Ricker maps fails. This is so simply because β is a pure scaling parameter. For any $\lambda > 1$ and any $\beta_1, \beta_2 > 0$, R_{λ,β_1} and R_{λ,β_2} are conjugated via a linear change of coordinates, independent of λ. Moving β just moves the map inside a conjugacy class determined by λ. This is illustrated in Figure 6.

This should be an important qualitative difference between Hassell maps and Ricker maps, which in particular suggests that they should behave differently under stochastic perturbations in the β-parameter (sometimes called environmental stochasticity).

9.2. Abundance of Strongly Chaotic Maps. Finally we use Theorem 18 and Corollary 19 to show that genuinely chaotic maps, with positive Lyapunov exponents almost everywhere and an invariant absolutely continuous probability measure, appear with positive frequency in both families. We do this by verifying the conditions H1–H4 and proving the existence of a Misiurewicz parameter $\lambda^*(\beta)$ satisfying H5 and H6 for all β in an open interval. The choice of Misiurewicz parameter is dictated

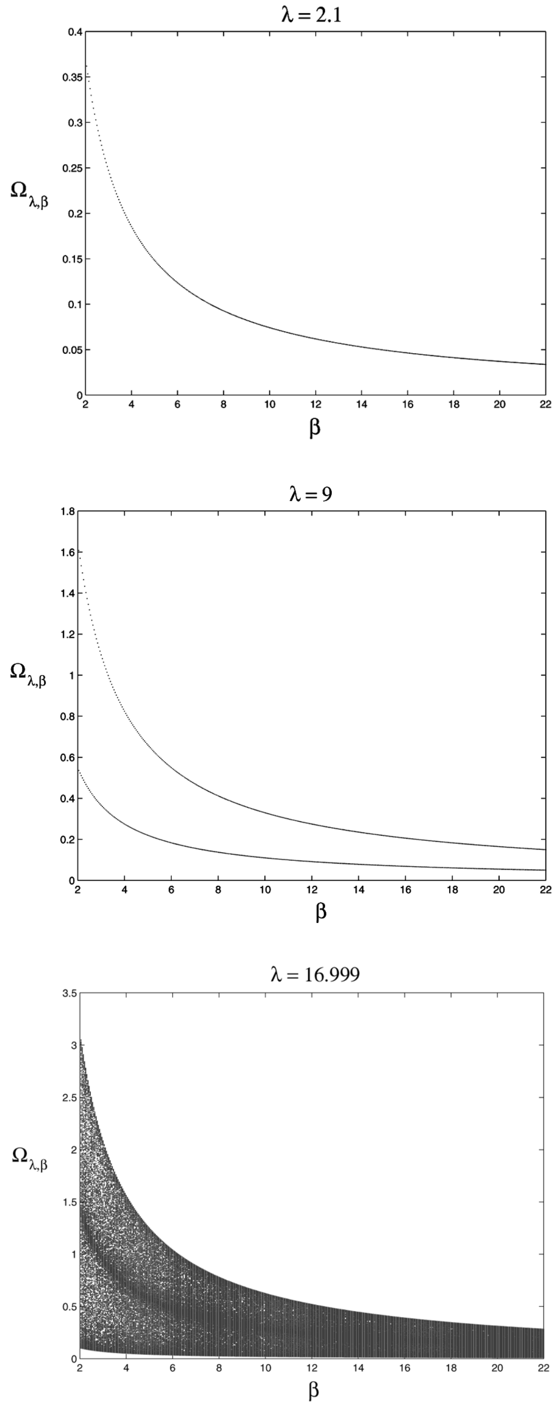

Fig. 6 *For the Ricker family, β is a pure scaling parameter. This figure shows $\Omega_{\lambda,\beta}$ versus β for three fixed values of λ. The qualitative type does not change with β. For $\hat{\lambda} \approx 16.999$ and any $\beta > 0$, we know that $R_{\hat{\lambda},\beta}$ is a Misiurewicz map of type $c \mapsto c_1 \mapsto c_2 \mapsto c_3 \circlearrowleft$, and thus has strong chaotic properties.*

by computing convenience. The reader can, using the same techniques, go hunting for Misiurewicz parameters with the right properties in his or her favorite part of parameter space. We stress the following facts:

- Conditions H1–H4 are easily checked for entire families, and for each fixed $\beta = \beta_0$ they hold for the one-parameter families R_{λ,β_0} and H_{λ,β_0}.
- Misiurewicz maps exists in great abundance in generic families by kneading theory, and condition H6 is of course completely generic.

Proposition 27. *There exists a $\beta_0 \approx 20.914$ such that for each fixed β sufficiently close to β_0, there exists a $\lambda^*(\beta)$, $\lambda^*(\beta_0) = 21.5$, such that $H_\lambda = H_{\lambda,\beta}$ is a one-parameter family with a Misiurewicz map H_{λ^*} fulfilling conditions* H1–H6.

Proof. First H1–H4 are easily checked for any fixed $\beta > 1$ and $\lambda > 1$. Then one verifies that the critical point $c = 1/(\beta - 1)$ is mapped onto the interior (unstable) fixed point $z = \lambda^{1/\beta} - 1$ in exactly three iterates for $\lambda = \lambda_0 = 21.5$ and $\beta = \beta_0 \approx 20.914$:

$$c \mapsto c_1 \mapsto c_2 \mapsto c_3 = z \mapsto z.$$

To do this, one shows numerically that

$$\beta \mapsto \left\{ z(\lambda_0, \beta) - H^3_{\lambda_0,\beta}\left(c\left(\beta\right)\right) \right\}$$

changes sign for $\beta = \beta_0 \approx 20.914$. One also verifies that z is a repeller for these parameter values. Thus the critical orbit does not accumulate on the critical point for H_{λ_0,β_0}, and there are no stable or neutral cycles, since for unimodal maps with negative Schwarzian, these would attract the critical orbit. So we have a Misiurewicz map.

Finally we verify H6 numerically for $\lambda = \lambda_0 = 21.5$ and $\beta = \beta_0 \approx 20.914$. Here we think of $H_\lambda = H(\lambda, \beta_0)$ as a one-parameter family. Define

$$\begin{aligned} F(x;\lambda) &:= H_\lambda(x), \\ F^n(x;\lambda) &:= F\left(F^{n-1}(x;\lambda);\lambda\right) = H^n_\lambda(x) \quad \text{for } n > 1. \end{aligned}$$

For λ close to λ_0 we may define $\zeta(\lambda)$ by

$$\begin{cases} \zeta(\lambda_0) = H_{\lambda_0}(c), \\ F^2\left(\zeta(\lambda);\lambda\right) = z\left(\lambda,\beta_0\right). \end{cases}$$

$\zeta(\lambda)$ will be a point close to $H_\lambda(c)$, with the same type of forward orbit as $H_{\lambda_0}(c)$. Finally, we also write

$$\zeta_1(\lambda) := F\left(\zeta(\lambda);\lambda\right).$$

Condition H6 now reads

$$\frac{d}{d\lambda}\left(\zeta(\lambda) - F(c;\lambda)\right)\bigg|_{\lambda=\lambda_0} \neq 0.$$

From the definition of ζ we get

$$\begin{aligned} z'(\lambda) &= \frac{d}{d\lambda} F\left(F\left(\zeta;\lambda\right);\lambda\right) \\ &= \frac{\partial F}{\partial x}(\zeta_1;\lambda)\frac{d}{d\lambda}F\left(\zeta;\lambda\right) + \frac{\partial F}{\partial \lambda}(\zeta_1;\lambda) \\ &= \frac{\partial F}{\partial x}(\zeta_1;\lambda)\left\{\frac{\partial F}{\partial x}(\zeta;\lambda)\frac{d\zeta}{d\lambda} + \frac{\partial F}{\partial \lambda}(\zeta;\lambda)\right\} + \frac{\partial F}{\partial \lambda}(\zeta_1;\lambda). \end{aligned}$$

Solving for $d\zeta/d\lambda$, we obtain

$$\frac{d\zeta}{d\lambda} = \left(\frac{\partial F}{\partial x}(\zeta;\lambda)\right)^{-1} \left\{ \left(\frac{\partial F}{\partial x}(\zeta_1;\lambda)\right)^{-1} \left(\frac{dz}{d\lambda} - \frac{\partial F}{\partial \lambda}(\zeta_1;\lambda)\right) - \frac{\partial F}{\partial \lambda}(\zeta;\lambda) \right\}.$$

This expression can easily be evaluated at $\lambda = \lambda_0$, since by definition $\zeta(\lambda_0) = H_{\lambda_0}(c)$ and $\zeta_1(\lambda_0) = H^2_{\lambda_0}(c)$, and all other quantities are explicit. H6 can now be verified numerically with rigorous estimates.

Finally, using the implicit function theorem and the fact that condition H6 is open, we conclude that for any β close to β_0, there is a $\lambda^*(\beta)$ with the desired properties. ∎

Combining Theorem 18, Corollary 19, Theorem 21 and the discussion thereafter, Lemma 23, Theorem 25, and Proposition 27, we obtain the following theorem.

THEOREM 28. *Hassell maps are strongly chaotic with positive probability: There exists a nonempty, open set B such that for each $\beta \in B$, there exists a positive measure set Λ_β such that if $\lambda \in \Lambda_\beta$, then*

(1) *$H_{\lambda,\beta}$ has no periodic attractor and the unique metric attractor is a transitive interval attractor;*
(2) *$H_{\lambda,\beta}$ admits an absolutely continuous invariant probability measure $\mu_{\lambda,\beta}$ with the following properties:*
   (a) *$\mu_{\lambda,\beta}$ describes the asymptotic distribution of almost all orbits under $H_{\lambda,\beta}$;*
   (b) *$\mu_{\lambda,\beta}$ has density in L^p, $1 \le p < 2$;*
   (c) *$\mu_{\lambda,\beta}$ is strongly stochastically stable in the sense of Baladi and Viana* [BV96];
(3) *$H_{\lambda,\beta}$ has positive Lyapunov exponent almost everywhere, in particular at the critical value.*

The corresponding result also holds for the Ricker maps, by the same type of arguments. Here we find a map of the type

$$c \mapsto c_1 \mapsto c_2 \mapsto c_3 = z \circlearrowleft$$

for any $\beta > 0$ and $\lambda = \lambda_0 \approx 16.999$ (independent of β). These maps will be Misiurewicz maps, since the fixed point $z = \ln(\lambda_0)/\beta$ has multiplier $R'_{\lambda_0,\beta}(z) = 1 - \ln\lambda_0 < -1$ and thus is unstable. We have checked condition H6 numerically for $0 < \beta \le 200$. We remind the reader that this particular choice of Misiurewicz map is only chosen as a convenient example.

THEOREM 29. *Ricker maps are strongly chaotic with positive frequency in parameter space: for any $\beta > 0$, there exists a positive measure set $\Lambda^{(R)}_\beta$ such that if $\lambda \in \Lambda^{(R)}_\beta$, then*

(1) *$R_{\lambda,\beta}$ has no periodic attractor and the unique attractor is a transitive interval attractor;*
(2) *$R_{\lambda,\beta}$ admits an absolutely continuous invariant probability measure $\mu_{\lambda,\beta}$ with the following properties:*
   (a) *$\mu_{\lambda,\beta}$ describes the asymptotic distribution of almost all orbits of $R_{\lambda,\beta}$;*
   (b) *$\mu_{\lambda,\beta}$ has density in L^p for any $1 \le p < 2$;*
   (c) *$\mu_{\lambda,\beta}$ is strongly stochastically stable in the sense of Baladi and Viana [BV96];*
(3) *$R_{\lambda,\beta}$ has positive Lyapunov exponent almost everywhere, in particular at the critical value.*

Figures 7 and 8 show the Lyapunov exponent and the natural measure for $R_{16.999,2.0}$.

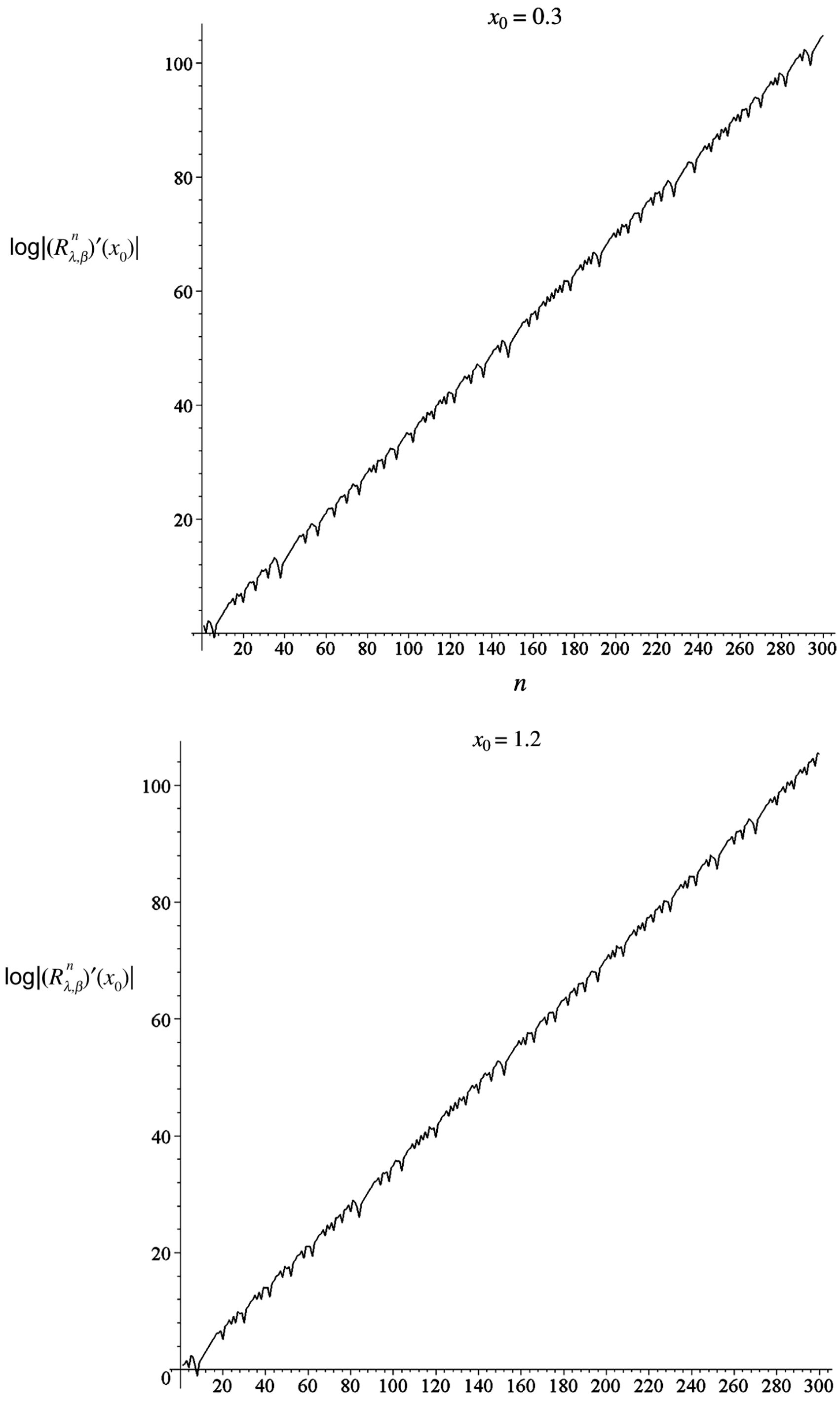

Fig. 7 *Positive Lyapunov exponent. For the Ricker map $R_{\lambda,\beta}$ with $\lambda = 16.999$ and $\beta = 2$, the figure shows $\log|(R^n_{\lambda,\beta})'(x_0)|$ versus n for two different initial conditions $x_0 = 0.3$ and $x_0 = 1.2$.*

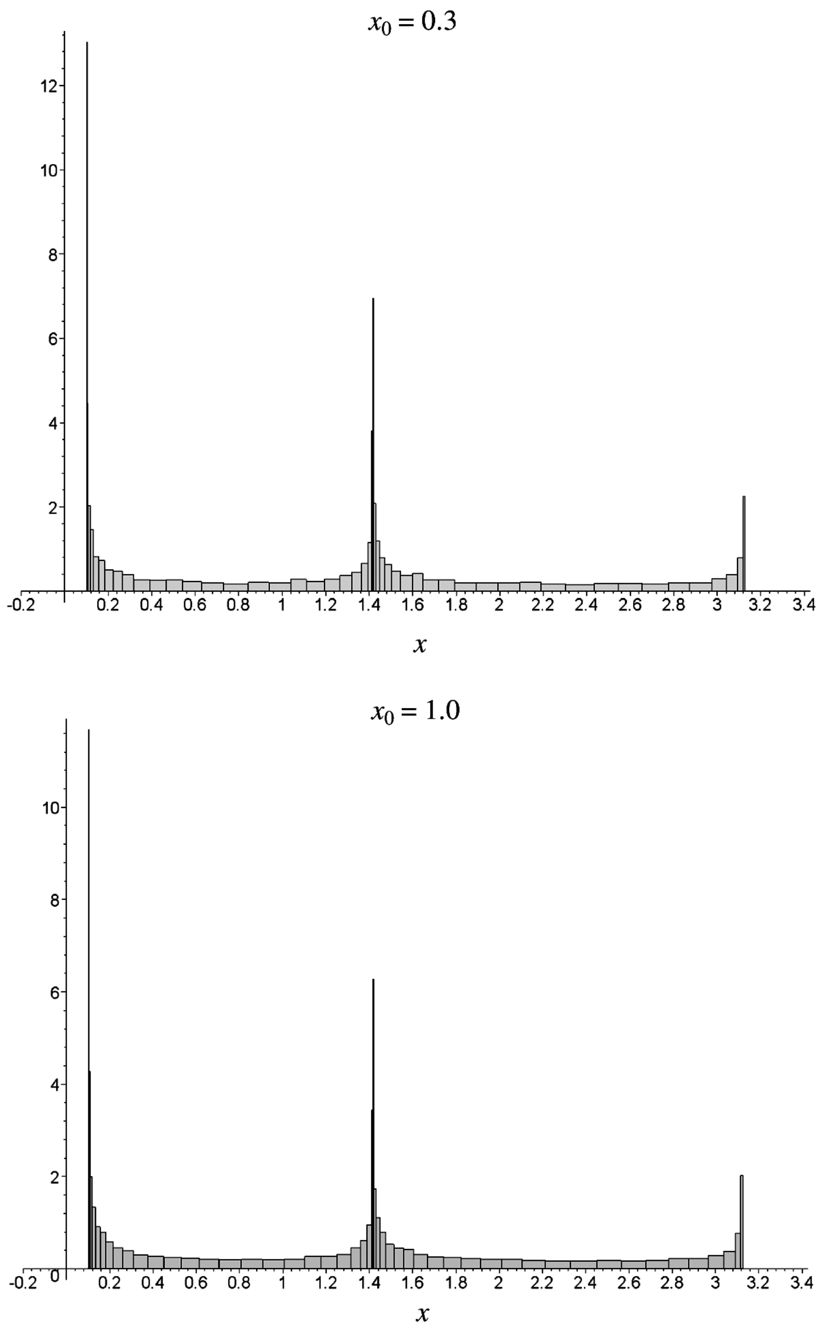

Fig. 8 *Histograms of the distributions of two different orbits for the Ricker map* $R_{\lambda,\beta}$*, again with* $\lambda \approx 16.999$ *and* $\beta = 2$*. These are approximately the same and are approximated by the natural acip of* $R_{\lambda,\beta}$*. This is a Misiurewicz map with finite critical orbit* $c = 0.5 \mapsto c_1 \approx 3.13 \mapsto c_2 \approx 0.10 \mapsto c_3 \approx 1.42 \circlearrowleft$*. Notice the (square-root) singularities building up on the forward images of the critical point. We have computed* 10,000 *points on each orbit and divided the interval into* 50 *subintervals of equal length.*

Remark. One would also expect the natural measures of the chaotic maps in Theorems 28 and 29 to have the sensitive type of λ dependence described in parts (2)–(4) of Theorem 20. Using the methods of [Thu98] we can in fact prove these properties at the special Misiurewicz maps $H_{\lambda^*(\beta),\beta}$ (or $R_{\lambda^*(\beta),\beta}$ defined in the same way) used in the proofs above. To get the full analogue of parts (3) and (4) of Theorem 20, one would need to know that all, or at least most, Misiurewicz maps in the closure of the sets Λ_β satisfy the generic condition H6. This is of course what one expects, but we have no rigorous results to back this up.

10. Concluding Remarks. Using recent results in one-dimensional dynamics, one can harvest a lot of results on basic dynamical properties for explicit families of systems. Here we considered two well-known, and much studied, models in population dynamics, the Ricker family (1.2) and the Hassell family (1.3), and verified the following commonly assumed properties:

- each system has a unique metric attractor, being either a periodic cycle, a Cantor set, or a finite union of intervals;
- periodic attractors appear for a dense subset of parameter space.

We also proved that

- maps with strong chaotic properties, stable under stochastic perturbations, appear with positive frequency in parameter space.

Since we also expect asymptotic distributions (natural measures) as a function of the parameter to behave in a singular way close to chaotic maps, we have a slightly awkward situation when using these families to model real-life observables. If our estimated parameters put us close to the set of strongly chaotic maps, the asymptotic motion will behave in an extremely sensitive way on the parameter, making even statistical predictions of the long-time behavior impossible. This should not be viewed as weakness of the models; it may be an unavoidable difficulty. It is the way these systems, and maybe nature herself, behave.

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