SUPPLEMENTARY MATERIALS: APPROXIMATING THE SHAPE OPERATOR WITH THE SURFACE HELLAN–HERRMANN–JOHNSON ELEMENT*

SHAWN W. WALKER†

SM1. Intrinsic Differential Geometry. We review various concepts from differential geometry; see [SM5, SM3, SM2, SM1, SM4], as well as [SM7, Appendix].

Consider a d-dimensional Riemannian manifold $(\Gamma, g_{\mathfrak{ab}})$, where $g_{\mathfrak{ab}}$ is the given metric tensor (discussed below) defined over a (reference) domain $U \subset \mathbb{R}^d$. A point in U is denoted by $(u^1, u^2, ..., u^d)$; in the special case of $d = 2$ that we are mainly concerned with, we may use $(u, v) \in U$. We refer to variables defined on U as *intrinsic* quantities. We keep track of upper and lower indices, where a lower index (subscript) is for *covariant* terms, and an upper index (superscript) is for *contravariant* terms.

The given metric $g_{\mathfrak{ab}}$ is a symmetric, covariant tensor with component functions $g_{\alpha\beta} : U \to \mathbb{R}$, for $1 \leq \alpha, \beta \leq d$, which we assume are at least C^1, and is uniformly positive definite. We write $g := \det g_{\mathfrak{ab}}$ and the inverse metric tensor $g^{\mathfrak{ab}}$ is contravariant with components denoted $g^{\alpha\beta}$, where $g_{\alpha\gamma} g^{\gamma\beta} = \delta^{\beta}_{\alpha}$. Note that $v^{\mathfrak{a}}$ may be converted to $v_{\mathfrak{b}}$ via $v_\beta = g_{\beta\alpha} v^\alpha$; similarly, $w_{\mathfrak{b}}$ may be converted to $w^{\mathfrak{a}}$ by $w^\alpha = g^{\alpha\beta} w_\beta$. When convenient, we write $g_{\mathfrak{ab}} \equiv \boldsymbol{g} = [g_{\alpha\beta}]^2_{\alpha,\beta=1}$ and $g^{\mathfrak{ab}} \equiv \boldsymbol{g}^{-1} = [g^{\alpha\beta}]^2_{\alpha,\beta=1}$ in standard matrix notation for the metric and inverse metric, respectively.

The Christoffel symbols Γ^k_{ij} (of the second kind) are defined by

$$\Gamma^{\gamma}_{\alpha\beta} := \frac{1}{2} g^{\mu\gamma} \left(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\mu\alpha} - \partial_\mu g_{\alpha\beta} \right), \quad 1 \leq \alpha, \beta, \gamma \leq 2, \tag{SM1.1}$$

where $\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\beta\alpha}$, [SM3, SM2]. With this, we recall the definition of covariant (contravariant) derivatives, denoted ∇_α (∇^α), where f is a scalar, $v_{\mathfrak{b}}$ is a covariant vector, and $v^{\mathfrak{c}}$ is a contravariant vector:

$$\begin{aligned} \nabla_\alpha f = \partial_\alpha f, \qquad \nabla_\alpha \nabla_\beta f = \partial_\alpha \partial_\beta f - (\partial_\gamma f) \Gamma^{\gamma}_{\alpha\beta}, \\ \nabla_\alpha v_\beta = \partial_\alpha v_\beta - v_\gamma \Gamma^{\gamma}_{\beta\alpha}, \quad \nabla_\alpha v^\gamma = \partial_\alpha v^\gamma + v^\beta \Gamma^{\gamma}_{\beta\alpha}. \end{aligned} \tag{SM1.2}$$

SM2. Extrinsic Differential Geometry. Suppose that the manifold Γ is embedded in $\mathbb{R}^n$, with $n \geq d$, and that it is represented by a family of charts $\{(U_i, \boldsymbol{\chi}_i)\}$, where a single chart consists of a pair $(U, \boldsymbol{\chi})$, with $U \subset \mathbb{R}^d$ (reference domain) and $\boldsymbol{\chi} : U \to \mathbb{R}^n$, [SM3]. For simplicity of exposition, assume there is only one chart $(U, \boldsymbol{\chi})$, where $\Gamma = \boldsymbol{\chi}(U)$. We refer to variables in $\mathbb{R}^n$ as *extrinsic* quantities.

For example, $\boldsymbol{\chi} = (\chi^1, ..., \chi^n)^T \in \mathbb{R}^n$, and $\chi^i : U \to \mathbb{R}$ for each $i \in \{1, 2, ..., n\}$. A point $\mathbf{x} \in \mathbb{R}^n$ has its j-th coordinate denoted by x^j. Moreover, ∂_k is the partial derivative with respect to coordinate x^k. Repeated indices are summed over. We typically bold-face extrinsic vectors and tensors, e.g. let $\boldsymbol{w}$ be a 2-tensor in $\mathbb{R}^n$ with components w_{ij} for $i, j \in \{1, 2, ..., n\}$. The canonical (orthonormal) basis in $\mathbb{R}^n$, is denoted by $\{\mathbf{a}_k\}^n_{k=1}$, where $\mathbf{a}_1 = (1, 0, ..., 0)^T$ (column vector), etc. With the Kronecker delta δ^j_i, we have the dual basis $\{\mathbf{a}^k\}$ of $\{\mathbf{a}_k\}$ by the formula $\mathbf{a}_i \cdot \mathbf{a}^j = \delta^j_i$.

*Supplementary material for SISC MS#M153196.
https://doi.org/10.1137/22M1531968
†Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803 USA (walker@math.lsu.edu).

We now specialize to the case of a surface in $\mathbb{R}^3$, i.e. $d = 2$, $n = 3$. The tangent space $T_{\mathbf{x}}(\Gamma)$, at a point $\mathbf{x} \in \Gamma$, is a subspace of $\mathbb{R}^3$ spanned by $\{\mathbf{e}_1, \mathbf{e}_2\}$ (the covariant basis) where

$$\mathbf{e}_\alpha = \partial_\alpha \boldsymbol{\chi}(u^1, u^2), \quad 1 \leq \alpha \leq 2, \quad \text{where } (u^1, u^2) = \boldsymbol{\chi}^{-1}(\mathbf{x}).$$

In this case, the metric tensor $g_{\mathfrak{ab}}$ is given by $g_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$, for $1 \leq \alpha, \beta \leq 2$. The contravariant tangent basis is given by $\{\mathbf{e}^1, \mathbf{e}^2\}$, where $\mathbf{e}^\beta = \mathbf{e}_\alpha g^{\alpha\beta} = (\partial_\alpha \boldsymbol{\chi}) g^{\alpha\beta}$, [SM1]. Sometimes, we express $g_{\mathfrak{ab}} \equiv \boldsymbol{g} = \boldsymbol{J}^T \boldsymbol{J}$, where $\boldsymbol{J} = [\mathbf{e}_1, \mathbf{e}_2]$ is an 3×2 matrix.

An alternative view of the tangent space is the following. Let $\boldsymbol{\nu} : \Gamma \to \mathbb{R}^3$ be the surface unit normal vector of Γ, which satisfies $\boldsymbol{\nu} = \mathbf{e}_1 \times \mathbf{e}_2$. The tangent space projection $\boldsymbol{P} : \mathbb{R}^3 \to \mathbb{R}^3$, defined on Γ, is given by

$$\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{\nu} \otimes \boldsymbol{\nu} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2, \tag{SM2.1}$$

and note that (in local coordinates) $\boldsymbol{J}\boldsymbol{g}^{-1}\boldsymbol{J}^T = \boldsymbol{P} \circ \boldsymbol{\chi}$, [SM6]. Thus, given a vector $\mathbf{v} \in \mathbb{R}^3$, it is in the tangent space $T_{\mathbf{x}}(\Gamma)$ if there exists a (contravariant) vector $v^{\mathfrak{a}}$ such that $\mathbf{v}(\mathbf{x}) = v^\alpha \mathbf{e}_\alpha \circ \boldsymbol{\chi}^{-1}(\mathbf{x})$.

We define the tangent bundle:

$$T(\Gamma) = \{(\mathbf{x}, \mathbf{v}) \mid \mathbf{x} \in \Gamma, \mathbf{v}(\mathbf{x}) \in T_{\mathbf{x}}(\Gamma)\},$$

thus, we say $\mathbf{v} \in T(\Gamma)$ if $\mathbf{v}(\mathbf{x}) \in T_{\mathbf{x}}(\Gamma)$ for every $\mathbf{x} \in \Gamma$; in this case, we write $\mathbf{v} : \Gamma \to T(\Gamma)$. We introduce extrinsic differential operators via their intrinsic counterpart, starting with the surface gradient $\nabla_\Gamma f : \Gamma \to T(\Gamma)$ defined in local coordinates by

$$(\nabla_\Gamma f) \circ \boldsymbol{\chi} = (\nabla_\alpha f) g^{\alpha\beta} \mathbf{e}_\beta^T = \partial_\alpha (f \circ \boldsymbol{\chi}) g^{\alpha\beta} (\partial_\beta \boldsymbol{\chi})^T \equiv \nabla (f \circ \boldsymbol{\chi}) \boldsymbol{g}^{-1} \boldsymbol{J}^T, \tag{SM2.2}$$

for any differentiable function $f : \Gamma \to \mathbb{R}$. Furthermore, let $\mathrm{id}_\Gamma : \Gamma \to \Gamma$ be the identity map, i.e. $\mathrm{id}_\Gamma = \boldsymbol{\chi} \circ \boldsymbol{\chi}^{-1}$, or $\mathbf{x} = \mathrm{id}_\Gamma(\mathbf{x})$ for all $\mathbf{x} \in \Gamma$. Then, defining

$$\nabla_\Gamma \mathrm{id}_\Gamma := [\nabla_\Gamma \mathrm{id}_\Gamma^1; \nabla_\Gamma \mathrm{id}_\Gamma^2; \nabla_\Gamma \mathrm{id}_\Gamma^3] \in \mathbb{R}^{3\times 3},$$

where $\mathrm{id}_\Gamma = \left(\mathrm{id}_\Gamma^1, \mathrm{id}_\Gamma^2, \mathrm{id}_\Gamma^3\right)$ (i.e. id_Γ^k is the kth component of id_Γ), it holds that $\nabla_\Gamma \mathrm{id}_\Gamma = \boldsymbol{P}$.

The (extrinsic) surface gradient, and surface divergence, of a tangential vector field $\mathbf{v} \in T(\Gamma)$ is

$$\begin{aligned} \nabla_\Gamma \mathbf{v} \circ \boldsymbol{\chi} &:= \mathbf{e}_\gamma g^{\gamma\alpha} (\nabla_\beta v_\alpha) g^{\beta\mu} \mathbf{e}_\mu^T = \mathbf{e}_\gamma g^{\gamma\alpha} (\partial_\beta v_\alpha - v_\omega \Gamma_{\alpha\beta}^\omega) g^{\beta\mu} \mathbf{e}_\mu^T, \\ (\nabla_\Gamma \cdot \mathbf{v}) \circ \boldsymbol{\chi} &:= \mathrm{tr}(\nabla_\Gamma \mathbf{v} \circ \boldsymbol{\chi}) = g^{\gamma\alpha} (\nabla_\beta v_\alpha) g^{\beta\mu} \mathbf{e}_\mu \cdot \mathbf{e}_\gamma \\ &= \nabla_\beta (g^{\gamma\alpha} v_\alpha) g^{\beta\mu} g_{\mu\gamma} = \delta_\gamma^\beta (\nabla_\beta v^\gamma) = \nabla_\gamma v^\gamma. \end{aligned}$$

The (extrinsic) surface Hessian of f is given by

$$\begin{aligned} (\nabla_\Gamma \nabla_\Gamma f) \circ \boldsymbol{\chi} &:= \mathbf{e}_\mu g^{\mu\alpha} [\nabla_\alpha \nabla_\beta f] g^{\beta\rho} \mathbf{e}_\rho^T \\ &= \mathbf{e}_\mu g^{\mu\alpha} [\partial_\alpha \partial_\beta (f \circ \boldsymbol{\chi}) - \partial_\gamma (f \circ \boldsymbol{\chi}) \Gamma_{\alpha\beta}^\gamma] g^{\beta\rho} \mathbf{e}_\rho^T. \end{aligned} \tag{SM2.3}$$

Lastly, setting $\nabla_\Gamma \boldsymbol{\nu} := [\nabla_\Gamma \nu_1; \nabla_\Gamma \nu_2; \nabla_\Gamma \nu_3]$, it can be shown that [SM6]:

$$\nabla_\Gamma \boldsymbol{\nu} = \kappa_1 \boldsymbol{d}_1 \otimes \boldsymbol{d}_1 + \kappa_2 \boldsymbol{d}_2 \otimes \boldsymbol{d}_2, \tag{SM2.4}$$

where κ_1, κ_2 are the *principle curvatures* of Γ and $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ are the *principle directions* (which are tangent to Γ). Thus, $\nabla_\Gamma \boldsymbol{\nu}$ is the (extrinsic) *shape operator*.

SM3. Parametrization Via Curved Element Map. Recall $\boldsymbol{F}_T^l : T^1 \to T^l$ from subsection 3.1. It is useful to consider this map as a parametrization of T^l in the following sense. Apply a rigid rotation of coordinates $\mathbf{x}$ to $\mathbf{x}'$ so that $T^s \to T^{s\prime}$ (for any s) and $T^{1\prime} \subset \mathbb{R}^2$. In the rotated coordinates, we view $\boldsymbol{F}_T^{l\,\prime}$ as a function of two variables, so that $(T^{1\prime}, \boldsymbol{F}_T^{l\,\prime})$ is a local chart for $T^{l\prime}$. Next, let $\boldsymbol{J}' = [\partial_1 \boldsymbol{F}_T^{l\,\prime}, \partial_2 \boldsymbol{F}_T^{l\,\prime}]$ be the 3×2 Jacobian matrix with induced metric $\boldsymbol{g}' = (\boldsymbol{J}')^T \boldsymbol{J}'$. In addition, define the 3×2 matrix $\bar{\boldsymbol{P}}_\star{}' = [\mathbf{a}_1, \mathbf{a}_2]$, where $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ are the canonical basis vectors of $\mathbb{R}^3$, $(\bar{\boldsymbol{P}}_\star{}')^T \bar{\boldsymbol{P}}_\star{}' = \boldsymbol{I}_2$, and $\bar{\boldsymbol{P}}_\star{}'(\bar{\boldsymbol{P}}_\star{}')^T = \bar{\boldsymbol{P}}' := \boldsymbol{I}_3 - \bar{\boldsymbol{\nu}}' \otimes \bar{\boldsymbol{\nu}}'$, where $\bar{\boldsymbol{\nu}}' \equiv \mathbf{a}_3$ is the unit normal of $T^{1\prime}$.

All results derived in the rotated coordinates can be mapped back to the original coordinates. For example, let $\bar{\boldsymbol{P}}_\star = [\mathbf{b}_1, \mathbf{b}_2]$, where $\mathbf{b}_1$, $\mathbf{b}_2$ are any two orthogonal unit vectors in $\mathbb{R}^3$ pointing in the plane of T^1, and note that $\bar{\boldsymbol{P}}_\star{}^T \bar{\boldsymbol{P}}_\star = \boldsymbol{I}_2$, and $\bar{\boldsymbol{P}}_\star \bar{\boldsymbol{P}}_\star{}^T = \bar{\boldsymbol{P}} := \boldsymbol{I}_3 - \bar{\boldsymbol{\nu}} \otimes \bar{\boldsymbol{\nu}}$ (see (SM2.1)), where $\bar{\boldsymbol{\nu}} = \mathbf{b}_1 \times \mathbf{b}_2$ is the unit normal of T^1. Then, $\boldsymbol{J} = (\nabla_{T^1} \boldsymbol{F}_T^l) \bar{\boldsymbol{P}}_\star$, $\boldsymbol{g} = \boldsymbol{J}^T \boldsymbol{J}$, and by (3.2),

$$|\boldsymbol{J} - \bar{\boldsymbol{P}}_\star| = O(h), \quad \boldsymbol{g} = \bar{\boldsymbol{P}}_\star{}^T \bar{\boldsymbol{P}}^T \bar{\boldsymbol{P}} \bar{\boldsymbol{P}}_\star + O(h) = \boldsymbol{I}_2 + O(h),$$

so $\boldsymbol{g}$ is invertible for h sufficiently small. Note that, in terms of $\boldsymbol{F}_T^l$, the surface gradient (SM2.2) of $f : T^l \to \mathbb{R}$ can be written as $(\nabla_{T^l} f) \circ \boldsymbol{F}_T^l = (\nabla_{T^1} \bar{f}) \bar{\boldsymbol{P}}_\star \boldsymbol{g}^{-1} \boldsymbol{J}^T$, where $\bar{f} := f \circ \boldsymbol{F}_T^l$.

REFERENCES

[SM1] P. G. Ciarlet, *Linear and Nonlinear Functional Analysis with Applications*, SIAM, 1st ed., 2013.

[SM2] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, Upper Saddle River, New Jersey, 1976.

[SM3] M. P. do Carmo, *Riemannian Geometry*, Mathematics: Theory and Applications, Birkhäuser, Boston, 2nd (english) ed., 1992.

[SM4] E. Hebey, *Sobolev Spaces on Riemannian Manifolds*, Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 1996, https://doi.org/10.1007/BFb0092907.

[SM5] E. Kreyszig, *Differential Geometry*, Dover, 1991.

[SM6] S. W. Walker, *The Shapes of Things: A Practical Guide to Differential Geometry and the Shape Derivative*, vol. 28 of Advances in Design and Control, SIAM, 1st ed., 2015.

[SM7] S. W. Walker, *The Kirchhoff plate equation on surfaces: The surface Hellan–Herrmann–Johnson method*, IMA Journal of Numerical Analysis, 42 (2021), pp. 3094–3134, https://doi.org/10.1093/imanum/drab062, https://arxiv.org/abs/https://academic.oup.com/imajna/article-pdf/42/4/3094/46323774/drab062.pdf.