TOPOLOGICAL CHAOS FOR BIRTH–AND–DEATH–TYPE MODELS WITH PROLIFERATION

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Abstract—In this paper we consider a class of infinite systems of linear ODEs. Each system corresponds to a process characterized by two components: the conservative one (birth–and–death process) and the proliferative one. The system of this type can describe the population of neoplastic cells divided into subpopulations characterized by different levels of cellular resistance to antineoplastic drugs. Under suitable assumptions on the "birth" (amplification) coefficients, the "death" (deamplification) coefficients and the averages of the life-spans we prove that this class of models is topologically chaotic.

Key words— topological chaos, birth–and–death process, hypercyclic operators, kinetic model, cancer cells.

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1. Introduction

In the recent decade it has been observed that chaos, typically associated with nonlinear phenomena, can occur also in linear infinite–dimensional systems. To make this statement more precise, let us recall that the (semi–)dynamical system $\{T(t)\}_{t\geq 0}$ on a metric space X is said to be **topologically chaotic** (chaotic in the sense of Devaney [7]) if

it is **topologically transitive**, i.e. for any two non-empty open sets $U, V \subset X$ there exists $t_0 \geq 0$ such that $T(t_0)U \cap V \neq \emptyset$

and

its set of periodic points is dense in X.

Historically speaking, the original Devaney's definition contained also the intuitive notion of sensitive dependence on initial conditions (**sdic**) but it was proved in [4] that in fact the two conditions given above yield sdic (the original proof for discrete in time systems can be repeated with slight modifications for continuous in time systems — see [1]). A link between chaos theory and linear operator theory has been established when it was realized that in a separable Banach space the topological transitivity was equivalent to the notion of hypercyclicity (see e.g. [6], [11]):

> a strongly continuous semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$ on a Banach space X is called **hypercyclic** if for some $x \in X$ the orbit $\{T(t)x\}_{t\geq 0}$ is dense in X.

This path was followed later in [12] (for discrete in time systems in Hilbert spaces) and several other papers (cf. [1]).

The significance of the notion of linear chaos for the applied mathematics (and continuous in time systems) was probably first noted in [17] where the authors showed that a class of linear systems (with constant coefficients) of kinetic type that can model e.g. the death process, scattering with energy loss, and many other phenomena, is chaotic. However, it should be mentioned that in order to be chaotic, the system must contain a "creation" (proliferative) part, as no conservative system (such as e.g. the standard death or birth–and–death process) can be chaotic in the whole Banach space.

The paper [17] was followed by [21] where it was shown that a linear transport equation modelling certain cell population dynamics is chaotic.

The recent paper [6] summarizes and considerably extends many of the existing results. The authors show, among others, that the convection-diffusion on the half-line is chaotic, rendering thus another important example of linear chaotic system. They also provide two important theorems allowing the identification of certain classes of chaotic and non-chaotic systems. As these two theorems will be the main tools in our investigations here, we shall quote them for readers' convenience.

Theorem 1.1 ([6] — Theorem 3.1). *Let X be a separable Banach space and A be the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X. Suppose that the point spectrum $\sigma_P(A)$, of the operator A, is such that there exists a nonempty open set $U \subset \sigma_P(A)$ with $U \cap i\mathbb{R}^1 \neq \emptyset$. Assume that for each $\lambda \in U$ there exists a corresponding nonzero eigenvector x_λ such that for any $0 \neq \Phi \in X^*$ the function $\lambda \to F_\Phi(\lambda) = \langle \Phi, x_\lambda \rangle$ is analytic on U and does not vanish identically on U. Then$\{T(t)\}_{t\geq 0}$ is topologically chaotic.*

Theorem 1.2 ([6] — Theorem 3.3). *Let $\{T(t)\}_{t\geq 0}$ be a hypercyclic semigroup generated by A in a Banach space X. Then the adjoint A^* of A and the dual semigroup $\{T^*(t)\}_{t\geq 0}$ have the following properties:*
(i) *for any $0 \neq \phi \in X^*$ the orbit $\{T^*(t)\phi\}_{t\geq 0}$ is unbounded;*
(ii) *the point spectrum $\sigma_P(A^*)$ is empty.*

A survey of chaotic models coming from the applied sciences together with some discussion of links between discrete and continuous time theory can be found in [1].

The result of [17] was extended in [3] to cover a class of models with variable coefficients showing that the property of being chaotic is in a certain sense stable.

On the other hand, also in [3], it was proved that the "opposite" class of birth models cannot be chaotic.

In the present paper we consider a class of birth–and–death–type models with proliferation, with variable coefficients and under suitable assumptions on the "birth" (amplification) coefficients, the "death" (deamplification) coefficients and the proliferative coefficients (related to the averages of the life-spans) we prove (in Section 3) that the corresponding semigroup is topologically chaotic.

For the death-type model we generalize the result of [3] (in Section 4).

Finally for the "adjoint" ("opposite") processes we prove that the corresponding semigroups cannot be chaotic (in Section 5).

2. Birth–and–death–type model

We consider the following infinite system of ODEs

$$\dot{f}_n = (L\mathbf{f})_n\,, \qquad n \in \mathbb{N}_0\,, \tag{2.1}$$

$$\begin{aligned}(L\mathbf{f})_0 &= -a_0 f_0 + d_1 f_1\,,\\ (L\mathbf{f})_n &= -a_n f_n + b_{n-1} f_{n-1} + d_{n+1} f_{n+1}\,, \qquad n \in \mathbb{N}\,,\end{aligned}$$

where $\mathbf{f} = \{f_n(t)\}_{n\in\mathbb{N}_0}$ is the distribution function, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The coefficients $a_n\,,\ b_n\,,\ d_n\ \geq 0$ are given for $n \in \mathbb{N}_0$; define $c_n = b_n + d_n - a_n$.

Assuming $c_n \geq 0$ for $n \in \mathbb{N}_0$, we can interpret Eq. (2.1) as a kinetic equation corresponding to processes characterized by two components: the conservative one (birth–and–death process, formally for $d_0 = c_0$ and $c_n = 0\ \forall\ n \in \mathbb{N}$) and the proliferative one(formally for $b_n = d_n = 0\ \forall\ n \in \mathbb{N}_0$). The name of birth–and–death process comes from the population theory, where n is interpreted as the size of a population and the possible events are death and birth of individuals. In this situation the functions f_n have the meaning of the probability that the population is of size n at time t. The mathematical theory for the birth–and–death processes can be found in [10], [15] and [18] (see also references therein).

The following very important interpretation of Eq. (2.1) with $a_n = b_n + d_n - c_n$, $c_n \geq 0$, is possible (see [14], [19] and references therein). Equation (2.1) describes the population of neoplastic cells divided into subpopulations characterized by different levels of cellular resistance to antineoplastic drugs. The subpopulations of cells are labeled by numbers $n \in \mathbb{N}_0$. The process considered in [14] and [19] is gene stochastic amplification – deamplification process. The cells of n–subpopulation are identified with ξ_n copies of the drug resistance gene ($\xi_0 = 0$ and ξ_n increases with n). Because the number of gene copies can be very large in cancer cells, it is reasonable to use a model with an infinite number of cells subpopulations. The cells of 0–subpopulation have no copies of the gene and are sensitive to the drug. Cells of n–subpopulation are resistant with the increasing level of resistance for increasing $n \in \mathbb{N}$. The conservative component of the process describes the mutations of cells:

$b_n\Delta t + o(\Delta t)$, for $n \in \mathbb{N}_0$, is the chance of one mutation in the n–subpopulation shifting the mutated cell to the $n+1$–subpopulation and $d_n\Delta t + o(\Delta t)$, for $n \in \mathbb{N}$, is the chance of one mutation in the n–subpopulation shifting the mutated cell to the $n-1$–subpopulation (we assume that $d_0 = 0$). The proliferative component of the process is related to the assumption that the moment of death represents the moment of cell division and that the average life-span is given by the coefficient c_n for the n–subpopulation ($n \in \mathbb{N}_0$).

With this interpretation the proper space for the process defined by Eq. (2.1) is l^1, where the norm

$$\|\mathbf{f}\| = \sum_{n=0}^{\infty} f_n\,, \tag{2.2}$$

of any element $\mathbf{f}$ in the positive cone l^1_+

$$l^1_+ = \left\{\mathbf{f} \in l^1\ ;\quad f_n \geq 0 \quad \forall n \in \mathbb{N}_0\right\}, \tag{2.3}$$

represents the total number of individuals (cells).

Another motivation for studying Eq. (2.1) comes from the theory of kinetic integral equations (generalized Boltzmann models or nonlocal models — see [5] and [16]). In fact, consider the following linear equation

$$\partial_t f = \Gamma * f - \rho f\,, \tag{2.4}$$

in $L^2(\mathbb{T})$, where $\mathbb{T}$ is the 1–dimensional torus, Γ, ρ are given functions on $\mathbb{T}$ and $*$ is the convolution. The bilinear version of this type of equations was studied in [16]. Equation (2.4) in Fourier transform reads

$$\partial_t f_n = \Gamma_n f_n - \sum_{m=-\infty}^{+\infty} \rho_{n-m} f_m\,, \qquad n = 0, \pm 1, \pm 2, \dots\,, \tag{2.5}$$

where f_n is the n-th Fourier coefficient of function f, and Eq. (2.5), under the assumption $\rho_n = 0, \forall\, n \neq -1, 0$ leads to Eq. (2.1) with $a_n = \rho_0 - \Gamma_n$ and $d_n = -\rho_{-1}$ for $n \in \mathbb{N}_0$.

Yet another field of applications of Eq. (2.1) is in the extended kinetic theory (cf. [1], [2] and references therein) where this system describes the isotropic scattering of test particles on the medium composed of field particles at various levels of excitation (here two); the scattering consists in the exchange of a quantum of energy between the test and the field particles.

Assuming that $a_n = a$ ($n \in \mathbb{N}_0$) and $d_n = d$ ($n \in \mathbb{N}$) are independent of n as well as that $b_n = 0$ ($\forall n \in \mathbb{N}$) and $d > a > 0$ we obtain the model considered in [17]

$$\dot{f}_n = (L^{(0)}\mathbf{f})_n = -a f_n + d f_{n+1}\,, \qquad n \in \mathbb{N}_0\,. \tag{2.6}$$

The operator $\mathcal{L}^{(0)}$ determined by $L^{(0)}$ in l^1 is bounded so that there exists a semigroup solving Eq. (2.6). It follows that the point spectrum of $L^{(0)}$ is

$$\sigma_P(L^{(0)}) = \{\lambda = -a + d\mu\ ; \quad |\mu| < 1\}\,,$$

with the eigenvectors

$$\mathbf{f}(\lambda) = (1, \mu, \mu^2, \mu^3, \ldots) \in l^1\,.$$

In [17] it was proved that the corresponding semigroup is chaotic in l^1 (note that the assumption of Theorem 1.1 are satisfied).

In the paper [3] a generalization of the model introduced in [17] was considered. The generalization consisted in allowing variable coefficients a_n, d_n, but keeping the assumptions $b_n = 0$ and $0 < a_n < d_{n+1}$ ($n \in \mathbb{N}_0$). Under the assumption that the sequence $\{a_n\}_{n\in\mathbb{N}_0}$ tends sufficiently fast to some $a > 0$ and the sequence $\{d_n\}_{n\in\mathbb{N}}$ tends to $d > a$ (as $n \to \infty$) it was proved ([3]) that the corresponding semigroup is topologically chaotic in any l^p, $1 \le p < \infty$, and in c_0. It was also proved in [3] that if $d_n = 0$, $0 < a_n < b_{n-1}$ ($n \in \mathbb{N}$), the sequence $\{a_n\}_{n\in\mathbb{N}_0}$ tends sufficiently fast to some $a > 0$ and the sequence $\{b_n\}_{n\in\mathbb{N}_0}$ tends to $b > a$ (as $n \to \infty$), then the corresponding semigroup cannot be even hypercyclic in any of the spaces l^p ($1 \le p < \infty$), c_0.

In the present paper we consider the general equation (2.1) under the assumption that the coefficients a_n, b_n (for $n \in \mathbb{N}_0$), d_n (for $n \in \mathbb{N}$) are nonnegative and

(A1) For some $a \in [0, \infty[$ $\qquad a_n = a + \tilde{\alpha}_n\,, \quad \forall n \in \mathbb{N}_0\,, \qquad$ with $\displaystyle\lim_{n\to\infty} \tilde{\alpha}_n = 0\,;$

(A2) $d_n > 0 \ \ \forall n \in \mathbb{N}\,, \qquad \exists\, d \in]0, \infty[:\ \displaystyle\lim_{n\to\infty} d_n = d\,, \qquad \inf_{n\in\mathbb{N}} \prod_{k=1}^{n} \frac{d_k}{d} > 0\,;$

(A3) For some $b \in [0, \infty[$, $\qquad \displaystyle\overline{\lim_{n\to\infty}}\, b_n = b\,.$

Throughout the paper we use the notation

$$\kappa' = \frac{\kappa}{d} \quad \text{for each quantity } \kappa, \tag{2.7a}$$

and

$$\alpha_n = \tilde{\alpha}'_n = \frac{\tilde{\alpha}_n}{d}\,. \tag{2.7b}$$

In Section 3 we show that under Assumption **(A1)**, **(A2)**, **(A3)**, if additionally

(A4) $\quad 0 < a + b < d\,;$

and the sequences $\{\alpha_n\}_{n\in\mathbb{N}_0}$, $\{b_n d_{n+1}\}_{n\in\mathbb{N}_0}$ satisfy

(A5) $\quad |\alpha_n| \le q^{n+1}\,, \qquad |b'_n d'_{n+1}| \le q^{2n+4}\,, \qquad \forall\, n \in \mathbb{N}_0 \qquad$ for some $q \in]0, q_0[\,,$

where $q_0 \in]0,1[$ is a critical constant, then the semigroup solving Eq. (2.1) is topologically chaotic.

In Section 4 we consider the death–type model, that is Eq. (2.1) with $b_n = 0$ for all $n \in \mathbb{N}_0$. In this case the assumption on the critical constant q_0 can be essentially relaxed. In Section 4 we show that under Assumption **(A1)** and **(A2)**, if additionally

(**A6**) $\quad 0 < a < d\,;$

and the sequence $\{\alpha_n\}_{n\in\mathbb{N}_0}$ satisfies

(**A7**) $\quad |\alpha_n| \le q^{n+1}\,, \qquad \forall\, n \in \mathbb{N}_0\,, \qquad$ for some $q \in]0,1[\,,$

then the semigroup solving Eq. (2.1), with $b_n = 0$ for all $n \in \mathbb{N}_0$, is topologically chaotic. The results of Section 4 are a generalization of paper [3].

Finally, in Section 5, we show that the "adjoint" ("opposite") processes cannot be even hypercyclic.

3. Chaos for the birth–and–death–type model

We denote the infinite matrix of the coefficients of Eq. (2.1) by L. Then by $\mathcal{L}_p$, $p \in \{0\} \cup [1,\infty[$, we denote bounded operators defined by L in, respectively, c_0 and l^p. The operators $\mathcal{L}_p$ are bounded, hence they generate uniformly continuous semigroups of operators in, respectively, c_0 and l^p, denoted hereafter by $\{T_p(t)\}_{t\ge 0}$.

Firstly we shall focus our attention on $\mathcal{L}_1$. The equation for its eigenvectors takes the form

$$\begin{aligned} &\lambda f_0 = -a_0 f_0 + d_1 f_1\,, \\ &\lambda f_n = -a_n f_n + b_{n-1} f_{n-1} + d_{n+1} f_{n+1}\,, \qquad n \in \mathbb{N}\,. \end{aligned} \tag{3.1}$$

We introduce the following notation

$$\mu_n = \frac{\lambda + a_n}{d_{n+1}}\,, \qquad \forall\, n \in \mathbb{N}_0\,. \tag{3.2}$$

Clearly we can take $f_0 = 1$, then $f_1 = \mu_0$ and for $n > 1$ we obtain the recurrence relation

$$f_{n+1} = \mu_n f_n - \frac{b_{n-1}}{d_{n+1}} f_{n-1}\,. \tag{3.3}$$

At this point it is worthwhile to note that f_n are polynomials in λ. Finding the general form of them seems to be rather difficult and, as it turns out, not necessary since we shall need only some estimates. To this end we note the following lemma.

Lemma 3.1. *Assume that there exist constants $n_0 \in \mathbb{N}$, $\beta > 0$ and $\tilde{\mu} > 0$ such that*

$$b_{n-1} \leq \beta d_{n+1}\,, \qquad |\mu_n| \leq \tilde{\mu}\,, \qquad \forall\, n \geq n_0\,. \tag{3.4}$$

Then

$$|f_n| \leq g_n\,, \tag{3.5}$$

for all $n \geq n_0$, where

$$g_{n_0} = |f_{n_0}|\,, \qquad g_{n_0+1} = |f_{n_0+1}|\,, \tag{3.6a}$$

and

$$g_n = \tilde{\mu} g_{n-1} + \beta g_{n-2}\,, \qquad n \geq n_0 + 2\,. \tag{3.6b}$$

Proof. Note that (3.5) holds for $n = n_0$ and $n = n_0 + 1$. Let $m \geq n_0 + 2$ and assume that (3.5) holds for each $n \in \{n_0, \ldots, m\}$. Then

$$|f_{m+1}| \leq \tilde{\mu}|f_m| + \beta|f_{m-1}| \leq \tilde{\mu} g_m + \beta g_{m-1} = g_{m+1}\,, \tag{3.7}$$

and, by induction, (3.5) holds for all $n \geq n_0$.

∎

Note that some generalizations of Lemma 3.1 are possible by using Perron's Theorem (see [8]).

The sequence $\{g_n\}_{n\in\mathbb{N}_0}$ is a Fibonacci sequence and thus its explicit formula can be found by standard calculations. The characteristic equation corresponding to Eq. (3.6b) has the form $\omega^2 - \tilde{\mu}\omega - \beta = 0$ and the characteristic roots are given by

$$\omega_\pm = \frac{\tilde{\mu} \pm \sqrt{\tilde{\mu}^2 + 4\beta}}{2}\,. \tag{3.8}$$

Therefore the general solution to Eq. (3.6b) is

$$g_n = \varrho_+ \omega_+^n + \varrho_- \omega_-^n\,,$$

where ϱ_+ and ϱ_- are constants. Using (3.6a) we obtain the solution to the problem (3.6) as

$$g_n = \frac{1}{\omega_+ - \omega_-}\Big((\omega_+^{n-n_0} - \omega_-^{n-n_0})|f_{n_0+1}| + (\omega_-^{n-n_0}\omega_+ - \omega_-\omega_+^{n-n_0})|f_{n_0}| \Big)\,, \tag{3.9}$$

for all $n \geq n_0$.

Clearly, if $|\omega_\pm| \leq q < 1$, then $|f_n| \leq |g_n| \leq M q^{n-n_0+1}$, $n \geq n_0$, for some M and $\mathbf{f} = (f_0, f_1, \ldots, f_n, \ldots) \in l^1$. We have the following lemma.

Lemma 3.2. *If Assumptions* **(A1)** − **(A4)** *are satisfied, then there exists $r > 0$ such that*

$$\big\{\lambda \in \mathbb{C} \;:\; |\lambda| < r\big\} \;\subset\; \sigma_P(\mathcal{L}_1)\,. \tag{3.10}$$

Proof. By Assumptions **(A1)** − **(A3)** we have

$$\mu = \lim_{n\to\infty} \mu_n = \frac{\lambda + a}{d} \,,$$

and for any $\varepsilon > 0$ there exists n_0 such that

$$|\mu_n| \le |\mu| + \varepsilon \,, \qquad \frac{b_{n-1}}{d_{n+1}} \le b' + \varepsilon \,, \qquad \forall\, n \ge n_0 \,, \tag{3.11}$$

where $b' = \frac{b}{d}$. By Lemma 1 and (3.8) we obtain

$$\omega_\pm(\lambda) \;=\; \frac{|\lambda + a|}{2d} + \varepsilon \pm \left(\left(\frac{|\lambda + a|}{2d} + \varepsilon \right)^2 + b' + \varepsilon \right)^{\frac{1}{2}} . \tag{3.12}$$

Consider the following functions

$$h_\pm(x) = x \pm \left(x^2 + b' + \varepsilon \right)^{\frac{1}{2}} , \qquad x \ge 0 \,. \tag{3.13}$$

We have $|h_\pm(0)| = \sqrt{b' + \varepsilon}$, and by Assumption **(A4)**, ε can be chosen such that $|h_\pm(0)| < 1$. Differentiation gives

$$h'_\pm(x) = 1 \pm \frac{x}{\sqrt{x^2 + b' + \varepsilon}} > 0 \,, \qquad \forall\, x > 0 \,. \tag{3.14}$$

Clearly, we have

$$-1 < -\sqrt{b' + \varepsilon} \le h_-(x) < 0 \,, \qquad \forall\, x > 0 \,. \tag{3.15}$$

On the other hand, we have

$$h_+(x_0) = 1 \,, \qquad \text{for} \quad x_0 = \frac{1 - b' - \varepsilon}{2} \,, \tag{3.16}$$

and therefore

$$\sqrt{b'} \le h_+(x) < h_+(x_0) = 1 \,, \qquad \forall\, 0 < x < x_0 \,. \tag{3.17}$$

Hence, from Assumption **(A4)**, $\varepsilon > 0$ can be chosen small enough so that

$$a + b < d - 3\varepsilon d \,. \tag{3.18}$$

From (3.18) it follows that

$$0 < \frac{a}{2d} + \varepsilon < x_0 \tag{3.19}$$

and therefore

$$|\omega_\pm(0)| < 1 \,. \tag{3.20}$$

Since $\omega_\pm$ are continuous functions of the complex variable λ, there exists a constant $0 < q < 1$ and an open neighbourhood of $\lambda = 0$ where $|\omega_\pm(\lambda)| \le q$, thus by the considerations preceding the lemma the eigenvector $\mathbf{f}$ belongs to l^1.

∎

Some further information on the spectra of operators generated by L with constant coefficients in various weighted l^1–spaces can be found in [14] (see also [19]).

Let us denote by $\mathbf{f}(\lambda)$ the eigenvectors corresponding to the eigenvalue λ,

$$\mathbf{f}(\lambda) = (f_0(\lambda), f_1(\lambda), \ldots)\,.$$

Following Theorem 1.1 we take an arbitrary sequence $\mathbf{\Phi} = \{\Phi_k\}_{k\in\mathbb{N}_0} \in l^\infty$ and consider the function

$$H_{\mathbf{\Phi}}(\lambda) = \sum_{n=0}^{\infty} \Phi_n f_n(\lambda), \tag{3.21}$$

where the series is almost uniformly convergent at least in the circle $|\lambda| < r$, where r was defined in Lemma 3.2.

In order to use the properties of analytic functions we write $H_{\mathbf{\Phi}}$ as the power series in $\mu = \frac{\lambda+a}{d}$. This is possible since, as we noted above, each f_k is a polynomial in μ and the series (3.21) is absolutely convergent for $|\lambda| < r$, thus the terms can be rearranged in an arbitrary way (see e.g. [13]). Therefore we obtain

$$H_{\mathbf{\Phi}}(\lambda) = \sum_{n=0}^{\infty} \mu^n \Big(\sum_{k=n}^{\infty} A_{n,k}\Phi_k \Big)\,, \qquad \mu = \frac{\lambda+a}{d}\,, \tag{3.22}$$

where $A_{n,k}$ is the coefficient at μ^n. It follows immediately from Eq. (3.3) that the matrix $\{A_{n,k}\}_{(n,k)\in\mathbb{N}_0^2}$ is given by the recurrence relation

$$A_{n,k} = \begin{cases} 0 & \text{for} \quad k < n\,, \\ 1 & \text{for} \quad k = n = 0\,, \\ \left(\prod\limits_{j=1}^{n} d'_j\right)^{-1} & \text{for} \quad k = n \ge 1\,, \\ \frac{1}{d'_k}\Big(A_{n-1,k-1} + \alpha_{k-1}A_{n,k-1} - b'_{k-2}A_{n,k-2}\Big) & \text{for} \quad k > n\,, \end{cases}$$

where the convention (2.7) is adhered to and $A_{-1,k} = 0$ for any k.

Since the rearranged series (3.22) is convergent for $\mu = \frac{\hat{r}+a}{d}$, for any $\hat{r} < r$, it is convergent for any μ with $|\mu| < \frac{\hat{r}+a}{d}$. Thus, if $H_{\mathbf{\Phi}}(\lambda)$ vanishes for $|\lambda| < r$, then it vanishes in an open region inside the circle of convergence of the series (3.22) and thus all the coefficients of (3.22) must vanish. Therefore, to show that the last assumption of Theorem 1.1 is satisfied we have to prove that the only bounded solution of the infinite upper–triangular linear system

$$\begin{array}{ccccccccc} \Phi_0 & + & A_{0,1}\Phi_1 & + & A_{0,2}\Phi_2 & + & \ldots & = 0\,, \\ & & A_{1,1}\Phi_1 & + & A_{1,2}\Phi_2 & + & \ldots & = 0\,, \\ & & & & A_{2,2}\Phi_2 & + & \ldots & = 0, \\ & & & & & & \ldots & = 0\,, \end{array} \tag{3.23}$$

is the zero solution.

The direct analysis of the behaviour of the coefficients $A_{n,k}$ is difficult mainly due to the appearance of the factor $\frac{1}{d'_n}$. However, it turns out that we can relate $A_{n,k}$ to the analogous coefficients of system with $d_n = 1$. To show this we introduce the auxiliary system

$$\begin{aligned} \lambda f_0 &= -\alpha_0 f_0 + f_1 \,, \\ \lambda f_n &= -\alpha_n f_n + \gamma_{n-1} f_{n-1} + f_{n+1} \,, \qquad n \in \mathbb{N} \,, \end{aligned} \tag{3.24}$$

where $\gamma_n = b'_n d'_{n+1} = \frac{b_n d_{n+1}}{d^2}$, for $n \in \mathbb{N}_0$.

In the same manner as before we find that the power series (3.22) corresponding to the system (3.24) is given by

$$\mathcal{H}_{\mathbf{\Phi}}(\lambda) = \sum_{n=0}^{\infty} \mu^n \big(\sum_{k=n}^{\infty} B_{n,k} \Phi_k \big) \,, \tag{3.25}$$

where the matrix $\{B_{n,k}\}_{(n,k)\in\mathbb{N}_0}$ is given by the recurrence relation

$$B_{n,k} = \begin{cases} 0 & \text{for} \quad k < n \,, \\ 1 & \text{for} \quad k = n \,, \\ B_{n-1,k-1} + \alpha_{k-1} B_{n,k-1} - \gamma_{k-2} B_{n,k-2} & \text{for} \quad k > n \,, \end{cases}$$

where $B_{-1,k} = 0$ for any k.

Lemma 3.3.

$$A_{n,k} = \Big(\prod_{j=1}^{k} d'_j \Big)^{-1} B_{n,k} \,, \qquad n, k \in \mathbb{N}_0 \,, \tag{3.26}$$

with the understanding that $\prod_{j=1}^{0} = 1$.

Proof. By direct calculation we check the validity of (3.26) for all the pairs (n, k) with $k \le 2$ and arbitrary n (note that due to the form of $B_{n,k}$ this reduces to finite numbers if indices n: $n \le k$). Thus we assume that (3.26) is valid for all pairs $(n, \hat{k})$ where $\hat{k}$ is an arbitrary number satisfying $\hat{k} < k$ and $n \in \mathbb{N}_0$. Let us fix k and take arbitrary n (again there is only the finite range of nontrivial cases: $0 \le n \le k$). We have for $k > n$

$$\begin{aligned} A_{n,k} &= \frac{1}{d'_k} \big(A_{n-1,k-1} + \alpha_{k-1} A_{n,k-1} - b'_{k-2} A_{n,k-2} \big) \\ &= \frac{1}{d'_k} \Bigg(\frac{1}{\prod_{j=1}^{k-1} d'_j} B_{n-1,k-1} + \frac{\alpha_{k-1}}{\prod_{j=1}^{k-1} d'_j} B_{n,k-1} - \frac{b'_{k-2}}{\prod_{j=1}^{k-2} d'_j} B_{n,k-2} \Bigg) \\ &= \Big(\prod_{j=1}^{k} d'_j \Big)^{-1} \Big(B_{n-1,k-1} + \alpha_{k-1} B_{n,k-1} - b'_{k-2} d'_{k-1} B_{n,k-2} \Big) \\ &= \Big(\prod_{j=1}^{k} d'_j \Big)^{-1} B_{n,k} \,, \end{aligned}$$

and Eq. (3.26) for $n \geq k$ follows directly. Thus the lemma is valid for all pairs $(n, \hat{k})$ with arbitrary $\hat{k} \leq k$ and $n \in \mathbb{N}_0$ so that we can continue the induction with respect to k which proves Eq. (3.26) for all $(n,k) \in \mathbb{N}_0^2$.

∎

To estimate $B_{n,k}$ we have the following result.

Lemma 3.4. *If*

$$|\alpha_n| \leq q^{n+1}\,, \qquad \text{and} \qquad |\gamma_n| \leq q^{2n+4}\,, \qquad \text{for some } q \in]0,1[\,, \tag{3.27}$$

then

$$|B_{n,k}| \leq \sum_{1 \leq j_1 \leq \ldots \leq j_{k-n} \leq k} q^{j_1 + \ldots + j_{k-n}}\,, \qquad n < k\,, \quad (n,k) \in \mathbb{N}_0^2\,. \tag{3.28}$$

Proof. We use the same induction argument as in Lemma 3.3. By direct computation we establish the validity of (3.28) for initial values of n and k. Assuming that (3.28) invalid for $n = 0$ and $\hat{k} < k$ we obtain

$$\begin{aligned}
|B_{0,k}| &= |\alpha_{k-1} B_{0,k-1} - \gamma_{k-2} B_{0,k-2}| \\
&\leq q^k \sum_{1 \leq j_1 \leq \ldots \leq j_{k-1} \leq k-1} q^{j_1 + \ldots + j_{k-1}} + q^{2k} \sum_{1 \leq j_1 \leq \ldots \leq j_{k-2} \leq k-2} q^{j_1 + \ldots + j_{k-2}}\,.
\end{aligned}$$

Multiplication in each term increases the length of each chain of indices to k. Moreover, in the first term the index k will appear only once and in the second will appear only twice, so the sequences of indices will be not repeated. Thus

$$|B_{0,k}| \leq \sum_{1 \leq j_1 \leq \ldots \leq j_k \leq k} q^{j_1 + \ldots + j_k}\,.$$

In a similar way we obtain for $n > 0$

$$\begin{aligned}
|B_{n,k}| &= |B_{n-1,k-1} + \alpha_{k-1} B_{n,k-1} - \gamma_{k-2} B_{n,k-2}| \\
&\leq \sum_{1 \leq j_1 \leq \ldots \leq j_{k-n} \leq k-1} q^{j_1 + \ldots + j_{k-n}} + q^k \sum_{1 \leq j_1 \leq \ldots \leq j_{k-n-1} \leq k-1} q^{j_1 + \ldots + j_{k-n-1}} \\
&\quad + q^{2k} \sum_{1 \leq j_1 \leq \ldots \leq j_{k-n-2} \leq k-2} q^{j_1 + \ldots + j_{k-n-2}} \\
&\leq \sum_{1 \leq j_1 \leq \ldots \leq j_{k-n} \leq k} q^{j_1 + \ldots + j_{k-n}}\,.
\end{aligned}$$

∎

Let us denote by $\mathcal{B}_\infty$ the operator generated by the matrix $\{B_{n,k}\}_{(n,k) \in \mathbb{N}_0^2}$ in l^∞.

Lemma 3.5. *There exists a critical constant $q_0 \in]0,1[$ such that if Assumption* **(A5)** *is satisfied, then the operator $\mathcal{B}_\infty$ is an isomorphism in l^∞.*

Proof. We split $\mathcal{B}_\infty$ as

$$\mathcal{B}_\infty = \mathcal{I} + \begin{pmatrix} 0 & B_{0,1} & B_{0,2} & B_{0,3} & \dots \\ 0 & 0 & B_{1,2} & B_{1,3} & \dots \\ 0 & 0 & 0 & B_{2,3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix} = \mathcal{I} + \hat{\mathcal{B}}_\infty \,,$$

where $\mathcal{I}$ is the identity operator in l^∞. We have for any $\mathbf{f} = (f_1, f_2, \dots) \in l^\infty$

$$\|\hat{\mathcal{B}}_\infty \mathbf{f}\|_\infty = \sup_{n\in\mathbb{N}_0} \Big| \sum_{k=n+1}^{\infty} B_{n,k} f_k \Big| \le \|\mathbf{f}\|_\infty \sup_{n\in\mathbb{N}_0} \sum_{k=n+1}^{\infty} |B_{n,k}| \,. \tag{3.29}$$

By Lemma 3.4 we obtain

$$\begin{aligned} \Theta_n &= \sum_{k=n+1}^{\infty} |B_{n,k}| \\ &\le \sum_{1\le j\le n+1} q^j + \sum_{1\le j_1\le j_2\le n+2} q^{j_1+j_2} + \ldots + \sum_{1\le j_1\le\ldots\le j_{k-n}\le k} q^{j_1+\ldots+j_{k-n}} + \ldots \,. \end{aligned}$$

Each power q^j with $j > 2$ can appear several times in the above sum. However, the number of occurrences cannot exceed the number $P(j)$ of all possible integer solutions to $j_1 + \ldots + j_r = j$ (it is clear that any particular sequence $(j_1, \dots, j_r)$ may appear at most once). Thus we can write

$$\Theta_n \le \sum_{j=1}^{\infty} P(j) q^j \,,$$

so the estimate is independent of n. Using the result from combinatorics (see e.g. [20], p. 140)

$$P(j) < \frac{\pi}{\sqrt{6(j-1)}} \exp\left(\pi\sqrt{\frac{2j}{3}}\right) , \qquad \text{for } j > 2 \,, \tag{3.30}$$

we obtain that

$$\overline{\lim_{j\to\infty}} \left(q^j P(j)\right)^{\frac{1}{j}} \le q < 1 \,.$$

Thus the Cauchy criterion yields the convergence of each Θ_n with the common bound independent of n. Defining

$$\theta(q) = \sum_{j=1}^{\infty} P(j) q^j ,$$

we see that θ is a continuous function of q satisfying $\theta(0) = 0$ so that there exists $q_0 \in]0, 1[$ such that $|\theta(q)| < 1$ for $0 < q \le q_0$ and from Eq. (3.29) we get

$$\|\hat{\mathcal{B}}_\infty f\|_\infty \le \theta(q)\|f\|_\infty < \|f\|_\infty\,.$$

Using the Neumann series theorem we obtain that $\mathcal{B}_\infty$ is an isomorphism.

∎

Remark 3.1. *The critical constant* q_0 *in* Lemma 3.5 *can be chosen not smaller than* 0.15.

Proof. If $\varrho = \exp\left(\frac{\pi\sqrt{2}}{3}\right)$, then for $j \ge 3$ we obtain

$$P(j) < \frac{\pi}{\sqrt{6(j-1)}} \exp\left(\pi\sqrt{\frac{2j}{3}}\right) \le \frac{\pi}{\sqrt{12}} \exp\left(\frac{\pi\sqrt{2}j}{3}\right) = \frac{\pi}{\sqrt{12}}\,\varrho^j\,,$$

where we used the fact that the function

$$u(x) = \exp\left(\pi\sqrt{\frac{2}{3}}\left(\sqrt{x} - \frac{x}{\sqrt{3}}\right)\right)$$

is decreasing for $x > 3$ and satisfies $u(3) = 1$. Therefore we obtain

$$\begin{aligned}\theta(q) &\le q + 2q^2 + \frac{\pi}{\sqrt{12}}\sum_{j=3}^{\infty} P(j)q^j \le q + 2q^2 + \frac{\pi}{\sqrt{12}}\sum_{j=3}^{\infty}(\varrho q)^j \\ &= q + 2q^2 + \frac{\pi}{\sqrt{12}}\frac{(\varrho q)^3}{1-\varrho q}.\end{aligned}$$

We need the range of $q > 0$ for which

$$\left|q + 2q^2 + \frac{\pi}{\sqrt{12}}\frac{(\varrho q)^3}{1-\varrho q}\right| < 1\,.$$

Numerically we find that the above inequality is satisfied for $q \in [0\,,\,0.1514\ldots[$.

∎

Theorem 3.1. *Suppose that Assumptions* **(A1)** – **(A5)** *are satisfied with the critical constant* q_0 *as in* Lemma 3.5. *Then the semigroups* $\{T_p(t)\}_{t\ge0}$ *solving the system* (2.1) *in* l^p, $1 \le p < \infty$, *and in* c_0, *are topologically chaotic.*

Proof. From Lemma 3.2 we see that $\sigma_P(\mathcal{L}_1)$ contains an open circle with centre at $\lambda = 0$, that is $\sigma_P(\mathcal{L}_1)$ contains open subsets: $V_+ \subset \mathbb{C}_+ = \{\lambda : \Re\lambda > 0\}$, $V_- \subset \mathbb{C}_- = \{\lambda : \Re\lambda < 0\}$, and an open interval V_0 of the imaginary axis. We have

$$\begin{aligned}H_{\mathbf{\Phi}}(\lambda) &= \sum_{n=0}^{\infty}\mu^n\Big(\sum_{k=n}^{\infty}A_{n,k}\Phi_k\Big) = \sum_{n=0}^{\infty}\mu^n\Big(\sum_{k=n}^{\infty}\frac{\Phi_k}{\prod_{j=1}^{k} d'_j}B_{n,k}\Big) \\ &= \sum_{n=0}^{\infty}\mu^n\Big(\sum_{k=n}^{\infty}\tilde{\Phi}_k B_{n,k}\Big)\,.\end{aligned}$$

Thus if $H_{\mathbf{\Phi}}(\lambda)$ vanishes on either set: $V_{\pm}$ or V_0, then the coefficients of the power series must vanish, which is equivalent to $\mathcal{B}_\infty\tilde{\mathbf{\Phi}} = 0$. Since we proved in Lemma 3.5 that $\mathcal{B}_\infty$ is an isomorphism on l^∞, the only bounded solution is $\tilde{\mathbf{\Phi}} = 0$. Hence, if there is a nontrivial bounded $\mathbf{\Phi}$ for which $H_{\mathbf{\Phi}}$ vanishes identically on $V_{\pm}, V_0$, respectively, then by Assumption (A2), $\tilde{\mathbf{\Phi}}$ would be a nontrivial bounded solution to $\mathcal{B}_\infty\tilde{\mathbf{\Phi}} = 0$ which, as we have seen, is impossible. Thus Theorem 1.1 yields that the semigroup $\{T_1(t)\}_{t\geq 0}$ is topologically chaotic in l^1.

Consider the same problem in c_0 or l^p with $1 < p < +\infty$. In each case the operator $\mathcal{L}_p$, $p \in \{0\} \cup [1, \infty[$ is bounded and thus generates a uniformly continuous semigroup $\{T_p(t)\}_{t\geq 0}$. The eigenvectors belong to each of the above spaces. We have $c_0^* = l^1$ and $(l^p)^* = l^r$, with $1/p + 1/r = 1$. Hence, if the only solution $\mathbf{\Phi} \in l^r$, $1 \leq r \leq +\infty$, of the equation $\mathcal{B}_r\mathbf{\Phi} = 0$ (where $\mathcal{B}_r$ is the operator generated in l^r by the matrix $\{B_{n,k}\}_{(n,k)\in\mathbb{N}_0^2}$) is the zero solution, then the semigroup $\{T_p(t)\}_{t\geq 0}$ is topologically chaotic in each of these spaces. But any element of l^r is necessarily bounded, thus the statement follows from the unique solvability of $\mathcal{B}_\infty\mathbf{\Phi} = 0$ in l^∞.

∎

4. Chaos for the death–type model

In this section we consider the death–type model, that is, Eq. (2.1) with $b_n = 0$ for $n \in \mathbb{N}_0$. In this case the estimates leading to Theorem 3.1 can be substantially improved. This is possible as the transpose to the matrix that corresponds here to the matrix $\{A_{n,k}\}_{(n,k)\in\mathbb{N}_0}$ can be explicitly inverted. We consider the infinite system of ODEs in l^1

$$\dot{f}_n = -a_n f_n + d_{n+1} f_{n+1}\,, \qquad n \in \mathbb{N}_0\,, \tag{4.1}$$

where the coefficients satisfy Assumptions **(A1)**, **(A2)**, **(A6)** and **(A7)**.

Let us denote by K the matrix of coefficients of (4.1) and by $\mathcal{K}_p$, $p \in \{0\} \cup [1, \infty[$, the operator determined by K in the spaces c_0 and l^p, respectively. These operators are bounded and therefore generate semigroups of uniformly bounded operators $\{S_p(t)\}_{t\geq 0}$ in the respective spaces.

As before, firstly we shall focus our attention on $\mathcal{K}_1$. The equation for eigenvectors reads

$$\lambda f_n = -a_n f_n + d_{n+1} f_{n+1}\,, \qquad n \in \mathbb{N}_0\,. \tag{4.2}$$

Formally we obtain its solution as

$$\mathbf{f}(\lambda) = (f_n(\lambda))_{n\in\mathbb{N}_0},$$

where $f_0(\lambda) = 1$, and

$$f_n(\lambda) = \prod_{j=0}^{n-1} \frac{\lambda + a_j}{d_{j+1}}\,, \qquad n \in \mathbb{N}\,.$$

It is clear that the assumptions of Lemma 3.2 are satisfied and the point spectrum of $\mathcal{K}_1$ has the structure required by Theorem 1.1.

Following this theorem, as in Section 3, we take an arbitrary sequence $\tilde{\mathbf{\Phi}} \in l^\infty$ and consider the function

$$F_{\tilde{\mathbf{\Phi}}}(\lambda) = \sum_{n=0}^{\infty} \tilde{\Phi}_n f_n(\lambda)\,, \tag{4.3}$$

where the series is uniformly convergent in some open set of the complex plane containing zero. Due to the simpler structure of the problem we can simplify the above expression at this moment by writing

$$F_{\mathbf{\Phi}}(\mu) = F_{\tilde{\mathbf{\Phi}}}(\lambda) = \sum_{n=0}^{\infty} \frac{\tilde{\Phi}_n}{\prod\limits_{k=1}^{n} d'_k} \prod_{j=0}^{n-1} \Big(\frac{\lambda + a}{d} + \alpha_j \Big) = \sum_{n=0}^{\infty} \Phi_n \prod_{j=0}^{n-1} (\mu + \alpha_j)\,,$$

where $\mu = \frac{\lambda + a}{d}$, and where we used the notation (2.7) as well as the convention that $\prod\limits_{j=0}^{-1} = 1$. The function $F_{\mathbf{\Phi}}$ is analytic in an open neighbourhood of $\mu = 0$. By Assumption **(A2)**, if $\tilde{\mathbf{\Phi}} \in l^\infty$, then also $\mathbf{\Phi} \in l^\infty$. Hence, showing that $F_{\mathbf{\Phi}} \equiv 0$ yields $\mathbf{\Phi} = 0$ for any $\mathbf{\Phi} \in l^\infty$ proves the same statement for $F_{\tilde{\mathbf{\Phi}}}$. Consequently we concentrate on $F_{\mathbf{\Phi}}$.

In order to use the properties of analytic functions we write $F_{\mathbf{\Phi}}(\mu)$ as a power series in μ by multiplying out the brackets and rearranging the terms. We obtain

$$F_{\mathbf{\Phi}}(\mu) = \sum_{n=0}^{\infty} \mu^n \Big(\sum_{k=n}^{\infty} C_{n,k} \Phi_n \Big)\,, \tag{4.4}$$

where

$$C_{n,k} = \begin{cases} 0 & \text{for} \quad k < n\,, \\ 1 & \text{for} \quad k = n\,, \\ \sum\limits_{0 \le j_1 < \ldots < j_{k-n} \le k-1} \alpha_{j_1} \cdot \ldots \cdot \alpha_{j_{k-n}} & \text{for} \quad k > n\,. \end{cases}$$

As in Section 3, in order to use Theorem 1.1 we have to show that the only l^∞–solution of the infinite upper-triangular linear system

$$\begin{array}{ccccccccc} \Phi_0 & + & C_{0,1}\Phi_1 & + & C_{0,2}\Phi_2 & + & \ldots & = 0\,, \\ & & \Phi_1 & + & C_{1,2}\Phi_2 & + & \ldots & = 0\,, \\ & & & & \Phi_2 & + & \ldots & = 0, \\ & & & & & & \ldots & = 0\,, \end{array} \tag{4.5}$$

is the zero solution.

Let $\mathcal{C}_\infty$ be the operator generated by the matrix $\{C_{n,k}\}_{(n,k)\in\mathbb{N}_0}$ in l^∞, let $\mathcal{C}_1^T$ be the operator generated by the transposed matrix in l^1 and finally let $\mathcal{C}_0$ be the restriction of $\mathcal{C}_\infty$ to the space c_0. We have

Lemma 4.1. *Let Assumption* **(A1)**, **(A2)** *and* **(A7)** *be satisfied. Then*

(i) $\mathcal{C}_\infty$ *is a bounded operator in* l^∞*,*
(ii) $\mathcal{C}_1^T$ *is a bounded operator in* l^1*,*
(iii) $(\mathcal{C}_1^T)^* = \mathcal{C}_\infty$*,*
(iv) $\mathcal{C}_0$ *is bounded in* c_0 *and* $\mathcal{C}_0^* = \mathcal{C}_1^T$*.*

Proof. (i) The estimates are similar to that of Lemma 3.5. Let $\mathbf{f} = (f_1, f_2, \ldots) \in l^\infty$, then $\mathcal{C}_\infty \mathbf{f} = \mathbf{g} = (g_1, g_2, \ldots)$, where $g_n = \sum_{k=n}^{\infty} C_{n,k} f_k$ and

$$\|\mathcal{C}_\infty \mathbf{f}\|_\infty = \sup_{n\in\mathbb{N}_0} \Big| \sum_{j=n}^{\infty} C_{n,k} f_k \Big| \le \|f\|_\infty \sup_{n\in\mathbb{N}_0} \sum_{j=n}^{\infty} |C_{n,k}| \,.$$

The sum takes the form

$$\begin{aligned}\Theta_n &= 1 + \Big| \sum_{0\le j\le n} \alpha_j \Big| \\ &\quad + \Big| \sum_{0\le j_1<j_2\le n+1} \alpha_{j_1}\alpha_{j_2} \Big| + \ldots + \Big| \sum_{0\le j_1<\ldots<j_{k-n}\le k-1} \alpha_{j_1} \cdot \ldots \cdot \alpha_{j_{k-n}} \Big| + \ldots \,.\end{aligned}$$

It is clear that we can assume that for $n < n_0$ we have $|\alpha_n| \le M q^{n+1}$ for some constant $M > 0$. With this we get the estimate

$$\begin{aligned}\Theta_n &\le M^{n_0} \Big(1 + \sum_{1\le j\le n+1} q^j \\ &\quad + \sum_{1\le j_1<j_2\le n+2} q^{j_1+j_2} + \ldots \sum_{1\le j_1<\ldots<j_{k-n}\le k} q^{j_1+\ldots+j_{k-n}} + \ldots \Big) \,.\end{aligned}$$

Then, as in Lemma 3.5, we can use the partition function $P = P(j)$ to get

$$\Theta_n \le M^{n_0} \sum_{j=0}^{\infty} P(j) q^j < +\infty \,,$$

provided that $|q| < 1$. These estimates are independent of n which gives the boundedness of $\mathcal{C}_\infty$.

(ii) The argument is similar. Let $\mathbf{f} \in l^1$, then $\mathcal{C}_1^T \mathbf{f} = \mathbf{g}$, where $g_n = \sum_{k=0}^{n} C_{k,n} f_k$ and

$$\|\mathcal{C}_1^T \mathbf{f}\|_1 = \sum_{n=0}^{\infty} \Big| \sum_{k=0}^{n} C_{k,n} f_k \Big| \le \sum_{k=0}^{\infty} f_k \sum_{n=k}^{\infty} |C_{k,n}| = \sum_{k=0}^{\infty} f_k \Theta_k \le \text{const} \|\mathbf{f}\|_1 \,.$$

(iii) Since $\mathcal{C}_1^T$ is bounded and the elements $\mathbf{e}_n = (0, 0, \ldots, 0, 1, 0, \ldots)$ are linearly dense in l^1, it is enough to show that

$$\langle \mathcal{C}_1^T \mathbf{e}_n \,,\, \mathbf{f} \rangle = \langle \mathbf{e}_n \,,\, \mathcal{C}_\infty \mathbf{f} \rangle \,,$$

for any $\mathbf{f} \in l^\infty$. But $\mathcal{C}_1^T \mathbf{e}_n = (0, \ldots, 0, 1, C_{n,n+1}, \ldots)$, and

$$\langle \mathcal{C}_1^T \mathbf{e}_n \,,\, \mathbf{f} \rangle = \sum_{k=n}^{\infty} C_{n,k} f_k \,.$$

On the other hand, $\mathcal{C}_\infty \mathbf{f} = \left\{ \sum_{k=n}^{\infty} C_{n,k} f_k \right\}_{n=0}^{\infty}$, thus

$$\langle \mathbf{e}_n \,,\, \mathcal{C}_\infty \mathbf{f} \rangle = \sum_{k=n}^{\infty} C_{n,k} f_k$$

and (iii) follows.

(iv) To prove that $\mathcal{C}_0$ is a bounded operator in c_0, it is enough to show that c_0 is an invariant subspace for $\mathcal{C}_\infty$. Using more carefully the results of (i) we obtain

$$|g_n| = \Big| \sum_{k=n}^{\infty} C_{n,k} f_k \Big| \le \sup_{k \ge n} |f_k| \sum_{k=n}^{\infty} |C_{n,k}| \le \Theta_n \sup_{k \ge n} |f_k| \,.$$

Since we proved that Θ_n are bounded uniformly with respect to n, we see that if $\mathbf{f} \in c_0$, then also $\mathbf{g} \in c_0$.

The statement $\mathcal{C}_0^* = \mathcal{C}_1^T$ follows from $c_0^* = l^1$ and calculations similar to (iii).

∎

Consider the transpose matrix $C^T = \{C^T_{k,n}\}_{(k,n) \in \mathbb{N}_0^2}$, that is $C^T_{k,n} = C_{n,k}$, $(k,n) \in \mathbb{N}_0^2$. Define the matrix $D = \{D_{k,n}\}_{(k,n) \in \mathbb{N}_0^2}$ such that $D_{k,n} = (-1)^{k-n} C^T_{k,n}$ for all $(k,n) \in \mathbb{N}_0^2$.

Lemma 4.2. *The infinite matrix D is a right-inverse of C.*

Proof. We prove that

$$\sum_{j=n}^{k} C_{j,k} D_{j,n} = \delta_{kn} \,, \qquad \forall\, (k,n) \in \mathbb{N}_0^2 \,, \qquad n \le k \,, \tag{4.6}$$

where δ_{kn} is the *Kronecker delta*. From the definition it follows that (4.6) is satisfied for $k \le n$, thus we have only to prove that

$$\sum_{j=n}^{k} C_{j,k} D_{j,n} = 0 \,, \qquad \forall\, n < k \,. \tag{4.7}$$

As the proof is not particularly intuitive, let us indicate the main ideas behind the considerations that follow below. Namely, it turns out that $C_{n+1,k} D_{n+1,n}$ contains all the terms appearing in $C_{n,k} D_{n,n}$ with the opposite signs so that adding the two first summands of Eq. (4.7) we wipe out all the terms coming from the first summand and this continues as we increase j. Unfortunately, at some levels the terms that had cancelled each other before could appear again. Thus the procedure presented below is aimed at partitioning the set of all the terms that appear in Eq. (4.7) into disjoint classes containing the terms that differ only in sign and proving that they cancel each other within each class.

Fix k and n, $n < k$. Let Υ_m the set of indices over which the summation is carried over in the term $j = n+m$ above. This set is the tensor product of sets of all sequences $J_m : 0 \leq j_1 < j_2 < \ldots < j_{k-(n+m)} \leq k-1$ and $I_m : 0 \leq i_1 \leq \ldots i_m \leq j$. If we count each term according to its multiplicity in this set, then for any m the sequence of indices will have $k-n$ elements. We write any $\mathbf{s} \in \Upsilon_m$ as a nondecreasing sequence and for $\mathbf{s} = (s_1, \ldots, s_{k-n}) \in \Upsilon_m$ we symbolically denote

$$\bar{\alpha}_{\mathbf{s}} = \prod_{i=1}^{k-n} \alpha_{s_i}$$

so that

$$(-1)^{j-n} C_{j,k} D_{j,n} = (-1)^m C_{n+m,k} D_{n+m,n} = \sum_{\mathbf{s} \in \Upsilon_m} \bar{\alpha}_{\mathbf{s}} \,.$$

For further use we also define

$$\Upsilon = \bigcup_{m=0}^{k-n} \Upsilon_m$$

(the sum is not disjoint). Thus we have to prove

$$\sum_{\Upsilon_m \subset \Upsilon} (-1)^m \sum_{\mathbf{s} \in \Upsilon_m} \bar{\alpha}_{\mathbf{s}} = 0 \,. \tag{4.8}$$

Let us first observe that in any sequence J_m taking the tensor product with I_m does not change the "tail" consisting of all the indices which are larger than j, as the indices in I_m can equal at most j. We shall then separate in Υ_m these sequences which contain the longest possible (at m level) subsequence of indices that are larger that j (such a subsequence is necessarily strictly increasing as it can come only from J_m). It follows that such a subsequence is given by $j_1 < \ldots < j_{k-n+m}$, $j_1 > j$ and it is clear that no sequence $\mathbf{s}$ containing such a tail can appear for any $\hat{m} > m$ (the reason for this is that for $\hat{m} > m$ the strictly increasing sequences taken from $J_{\hat{m}}$ are shorter than $k-n+m$ and they are supplemented by elements from $I_{\hat{m}}$, which are at most equal to j).

Hence for each m we select the sequences $\mathbf{s} \in \Upsilon_m$ of the form

$$\mathbf{s}_{I_m} \oplus \mathbf{s}_{J_m} = (i_1, \ldots, i_m, j_1, \ldots, j_{k-(n+m)}) \,, \tag{4.9}$$

where $\mathbf{s}_{I_m} = (i_1, \ldots, i_m) \in I_m$ and $\mathbf{s}_{J_m} = (j_1, \ldots, j_{k-(n+m)}) \in J_m$ so that $i_m \leq j < j_1$. Note that in particular $\mathbf{s}_{I_m}$ is empty for $m = 0$ and $\mathbf{s}_{J_m}$ is empty for $n+m = k$. In such a case we have only either $\mathbf{s}_{J_m}$ or $\mathbf{s}_{I_m}$, respectively.

Let the sequence $\mathbf{s}_{I_m}$ contain a strictly increasing subsequence $j_{s_1} < j_{s_2} < \ldots < j_{s_r}$ for some $r \leq m$. We find all possible sequences $\mathbf{s}_{\hat{m}}$ with $\hat{m} < m$ which give the sequence (4.9). The smallest $\hat{m}$ for which such a sequence can appear is when all the strictly increasing indices are contributed from $J_{\hat{m}}$

$$(j_{s_1}, \ldots, j_{s_r}, j_1, \ldots, j_{k-(n+m)}) \in J_{\hat{m}} \,, \tag{4.10}$$

that is for $\hat{m} = m - r$; then we will have only one sequence of this form. Next, at $\hat{m} = m - r + 1$ the contributions will come from sequences of the form(4.10) with one of the indices s_i missing. Since again for a given missing index s_i there is only one sequence in $\Upsilon_{\hat{m}}$ producing (4.9), we see that (4.9) can be obtained by r sequences in $\Upsilon_{\hat{m}}$. Continuing, for $\hat{m} = m - r + l$ the contribution comes from the sequences of the form (4.10) with any combination of l indices missing, that is (4.9) can be obtained by $\binom{r}{l}$ sequences from Υ_{m-r+l}. Note moreover that the contribution from Υ_{m-r} appeared with factor $(-1)^{m-r}$ and the subsequent ones, for given l, are multiplied by $(-1)^{m-r+l}$.

Let us denote the subset of Υ containing all the sequences (4.9) by $\Upsilon_{I_m J_m, r}$, where r denotes the number of different indices in I_m. By the considerations above, the sets $\Upsilon_{I_m J_m, r}$ constitute a partition of Υ into disjoint sets and moreover for all $\mathbf{s} \in \Upsilon_{I_m J_m, r}$ we have $\bar{\alpha}_{\mathbf{s}} = \alpha_{I_m J_m, r}$ (independent of $\mathbf{s}$). Thus we have

$$\begin{aligned}
\sum_{\Upsilon_m \subset \Upsilon} (-1)^m \sum_{\mathbf{s} \in \Upsilon_m} \bar{\alpha}_{\mathbf{s}} &= \sum_{\{(I_m J_m, r);\; \Upsilon_{I_m J_m, r} \subset \Upsilon\}} (-1)^{m-r} \alpha_{I_m J_m, r} \sum_{l=0}^{r} (-1)^l \binom{r}{l} \\
&= \sum_{\{(I_m J_m, r);\; \Upsilon_{I_m J_m, r} \subset \Upsilon\}} (-1)^{m-r} \alpha_{I_m J_m, r} (1-1)^r = 0 \,,
\end{aligned}$$

so that (4.8) is proved.

∎

Lemma 4.3. *Let Assumptions* **(A1)**, **(A2)** *and* **(A7)** *be satisfied. Then the operator $\mathcal{D}_1$ generated by the matrix D in l^1 is bounded and it is the inverse of $\mathcal{C}_1^T$. Moreover $\mathcal{C}_\infty$ is an isomorphism on l^∞.*

Proof. The norm estimates of $\mathcal{D}_1$ in l^1 are analogous to the estimates of $\mathcal{C}_1^T$ so that $\mathcal{D}$ is bounded. From Lemma 4.2 we see that $\mathcal{D}_1$ is the right-inverse of $\mathcal{C}_1^T$ and since the latter is clearly injective, $\mathcal{D}_1$ is the inverse, thus $\mathcal{C}_1^T$ is an isomorphism in l^1. By Lemma 4.1 (iii) we obtain that $\mathcal{C}_\infty$ is an isomorphism in l^∞.

∎

We now formulate a theorem the proof of which is the same as of Theorem 3.1.

Theorem 4.1. *Suppose that Assumptions* **(A1)**, **(A2)**, **(A6)** *and* **(A7)** *are satisfied. Then the semigroups $\{S_p(t)\}_{t \geq 0}$ solving the system (4.1) in l^p, $1 \leq p < \infty$, and in c_0 are topologically chaotic.*

5. "Adjoint" models

We consider the "adjoint" ("opposite") process to the one considered in Section 3, that is, one with small death rate and possibly large birth-rate. The system of governing equations for such a process can be obtained as the transpose of (2.1)

$$\begin{aligned} \dot{f}_0 &= -a_0 f_0 + b_0 f_1 \,, \\ \dot{f}_n &= -a_n f_n + d_n f_{n-1} + b_n f_{n+1} \,, \qquad n \in \mathbb{N} \,, \end{aligned} \tag{5.1}$$

We adopt the same assumptions as in Section 3, that is, **(A1)** – **(A5)** are supposed to be satisfied. Then it is clear that the matrix L^T defined by the rhs of Eq. (5.1) defines bounded operators $\mathcal{L}_0^T$ in c_0, and $\mathcal{L}_p^T$ in l^p, $1 \leq p < \infty$, respectively. As we noted in Section 3, the transpose matrix L defines a bounded operator $\mathcal{L}_1$ in l^1 and $\mathcal{L}_r$ in l^r, where $\frac{1}{p} + \frac{1}{r} = 1$, with $(\mathcal{L}_0^T)^* = \mathcal{L}_1$ and $(\mathcal{L}_p^T)^* = \mathcal{L}_r$. Denote the semigroup generated by $\mathcal{L}_p^T$, $p \in \{0\} \cup [1, \infty[$, in the respective space, by $\{\mathfrak{T}_p(t)\}_{t\geq 0}$. Let us also recall that in a separable Banach space a semigroup is chaotic if it is hypercyclic (topologically transitive) and has a dense set of periodic points. Then we have

Theorem 5.1. *Suppose that Assumptions* **(A1)** – **(A5)** *are satisfied. Then the semigroups* $\{\mathfrak{T}_p(t)\}_{t\geq 0}$ *solving the system* (5.1) *are not hypercyclic for any* $p \in \{0\} \cup [1, \infty[$.

Proof. Let $p \in \{0\} \cup [1, \infty[$. It follows that the eigenvectors defined by (3.3) are by Lemma 3.2 in any l^r, $1 \leq r < \infty$, so in particular they are eigenvectors of any $(\mathcal{L}_p^T)^*$, which implies that the point spectrum $\sigma_P\Big((\mathcal{L}_p^T)^*\Big)$ of $(\mathcal{L}_p^T)^*$ is not empty. This, by Theorem 1.2 (ii), shows that $\mathcal{L}_p^T$ cannot generate a hypercyclic semigroup. ∎

Similar arguments apply to the "adjoint" process (a birth–type model) corresponding to the one defined by (4.1)

$$\begin{aligned} \dot{f}_0 &= -a_0 f_0 \,, \\ \dot{f}_n &= -a_n f_n + d_n f_{n-1} \,, \qquad n \in \mathbb{N} \,, \end{aligned} \tag{5.2}$$

with the matrix of the rhs obtained as the transpose of K. Let $\{\mathfrak{S}_p(t)\}_{t\geq 0}$, $p \in \{0\} \cup [1, \infty[$ be the corresponding semigroup in c_0 and l^p, respectively.

Theorem 5.2. *Suppose that Assumptions* **(A1)**, **(A2)**, **(A6)** *and* **(A7)** *are satisfied. Then the semigroups* $\{\mathfrak{S}_p(t)\}_{t\geq 0}$ *solving the system* (5.2) *are not hypercyclic for any* $p \in \{0\} \cup [1, \infty[$.

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