Integrality properties of the Weil representation of a finite quadratic module

Luca Candelori

Wayne State University, Detroit, MI

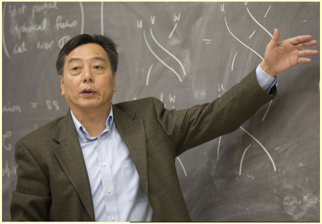
Southern Regional Number Theory, April 13th, 2019

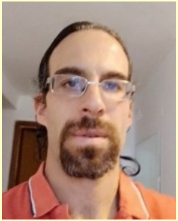
Acknowledgments

This talk is based on two ongoing projects:

- *Integrality for the modular representation of a modular tensor category*, with Siu-Hung "Richard" Ng (LSU) and Yilong Wang (LSU)

- *Explicit integral bases for the Weil representation*, with Yilong Wang (LSU) and Shaul Zemel (Hebrew University).





Finite quadratic modules

Definition

A finite quadratic module is a pair (D, q) of a finite abelian group D and a (non-degenerate) quadratic form $q : D \to \mathbb{Q}/\mathbb{Z}$, whose associated bilinear form we denote by

$$b(x, y) := q(x + y) - q(x) - q(y).$$

Finite quadratic modules

Definition

A finite quadratic module is a pair (D, q) of a finite abelian group D and a (non-degenerate) quadratic form $q : D \to \mathbb{Q}/\mathbb{Z}$, whose associated bilinear form we denote by

$$b(x, y) := q(x + y) - q(x) - q(y).$$

Examples

$p > 2$ prime:

$$A_p = (\mathbb{Z}/p\mathbb{Z}, x \mapsto x^2/p)$$

$$A_{p^r} = (\mathbb{Z}/p^r\mathbb{Z}, x \mapsto x^2/p^r)$$

$$A_{p^r}^t = (\mathbb{Z}/p^r\mathbb{Z}, x \mapsto t\,x^2/p^r), (t, p) = 1$$

The Weil Representation

Let (D, q) be a finite quadratic module. Let $\mathbb{C}(D)$ be the $\mathbb{C}$-vector space of functions $f : D \to \mathbb{C}$. This space has a canonical basis $\{\delta_x\}_{x \in D}$ of delta functions, i.e. $\delta_x(y) = \delta_{x,y}$.

The Weil Representation

Let (D, q) be a finite quadratic module. Let $\mathbb{C}(D)$ be the $\mathbb{C}$-vector space of functions $f : D \to \mathbb{C}$. This space has a canonical basis $\{\delta_x\}_{x\in D}$ of delta functions, i.e. $\delta_x(y) = \delta_{x,y}$.

Definition

The Weil representation $\rho_D : \mathrm{Mp}_2(\mathbb{Z}) \longrightarrow \mathrm{GL}(\mathbb{C}(D))$ is defined with respect to the basis $\{\delta_x\}_{x\in D}$ by

$$\rho_D(T)(\delta_x) = e^{2\pi i\, q(x)}\delta_x$$

$$\rho_D(S)(\delta_x) = \frac{p^-(D)}{|D|}\sum_{y\in D} e^{-2\pi i\, b(x,y)}\delta_y,$$

where $p^-(D) = \sum_{x\in D} e^{-2\pi i q(x)}$.

Metaplectic covers of $\mathsf{SL}_2(\mathbb{R})$

- $\pi_1(\mathsf{SL}_2(\mathbb{R})) \simeq \pi_1(\mathrm{SO}(2)) \simeq \mathbb{Z}$

Metaplectic covers of $\mathsf{SL}_2(\mathbb{R})$

- $\pi_1(\mathsf{SL}_2(\mathbb{R})) \simeq \pi_1(\mathrm{SO}(2)) \simeq \mathbb{Z}$
- For each $n \in \mathbb{Z} \geq 1$, there is a unique normal (topological) cover $\mathsf{Mp}_n(\mathbb{R}) \to \mathsf{SL}_2(\mathbb{R})$.

Metaplectic covers of $\mathsf{SL}_2(\mathbb{R})$

- $\pi_1(\mathsf{SL}_2(\mathbb{R})) \simeq \pi_1(\mathrm{SO}(2)) \simeq \mathbb{Z}$
- For each $n \in \mathbb{Z} \geq 1$, there is a unique normal (topological) cover $\mathsf{Mp}_n(\mathbb{R}) \to \mathsf{SL}_2(\mathbb{R})$.
- $\mathsf{Mp}_n(\mathbb{R})$ is the group of pairs $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{R}), \phi(\tau) \in \mathcal{O}_{\mathbb{H}}^*)$ such that $\phi(\tau)^n = c\tau + d$.

Metaplectic covers of $\mathsf{SL}_2(\mathbb{R})$

- $\pi_1(\mathsf{SL}_2(\mathbb{R})) \simeq \pi_1(\mathrm{SO}(2)) \simeq \mathbb{Z}$
- For each $n \in \mathbb{Z} \geq 1$, there is a unique normal (topological) cover $\mathsf{Mp}_n(\mathbb{R}) \to \mathsf{SL}_2(\mathbb{R})$.
- $\mathsf{Mp}_n(\mathbb{R})$ is the group of pairs $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{R}), \phi(\tau) \in \mathcal{O}_{\mathbb{H}}^*)$ such that $\phi(\tau)^n = c\tau + d$.
- Group law: $(\gamma_1, \phi_1)(\gamma_2, \phi_2) = (\gamma_1\gamma_2, \phi_2(\tau)\phi_1(\gamma_2\tau))$.

Metaplectic covers of $SL_2(\mathbb{Z})$

Definition

The metaplectic group of degree n $\mathrm{Mp}_n(\mathbb{Z})$ is the inverse image of $SL_2(\mathbb{Z})$ in $\mathrm{Mp}_n(\mathbb{R})$.

Metaplectic covers of $\mathrm{SL}_2(\mathbb{Z})$

Definition

The metaplectic group of degree n $\mathrm{Mp}_n(\mathbb{Z})$ is the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{Mp}_n(\mathbb{R})$.

- $\mathrm{Mp}_n(\mathbb{Z})$ is a non-trivial central extension:

$$1 \to \mu_n \to \mathrm{Mp}_n(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}) \to 1$$

Metaplectic covers of $\mathrm{SL}_2(\mathbb{Z})$

Definition

The metaplectic group of degree n $\mathrm{Mp}_n(\mathbb{Z})$ is the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{Mp}_n(\mathbb{R})$.

- $\mathrm{Mp}_n(\mathbb{Z})$ is a non-trivial central extension:

$$1 \to \mu_n \to \mathrm{Mp}_n(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}) \to 1$$

- $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $\mathrm{SL}_2(\mathbb{Z})$. The pairs

$$S_n = (S, \sqrt[n]{\tau}), \quad T_n = (T, 1)$$

generate $\mathrm{Mp}_n(\mathbb{Z})$, $\quad S_n^4 = (I_2, e^{2\pi i/n}), \quad (S_n T_n)^3 = (S_n)^2.$

Examples of Weil representations

- $A_2 = (\mathbb{Z}/2\mathbb{Z}, x \mapsto x^2/4)$, $\rho_{A_2} : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{C})$

Examples of Weil representations

- $A_2 = (\mathbb{Z}/2\mathbb{Z}, x \mapsto x^2/4)$, $\rho_{A_2} : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{C})$

$$\rho_{A_2}(T,1) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad \rho_{A_2}(S,\sqrt{\tau}) = \frac{1-i}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Examples of Weil representations

- $A_2 = (\mathbb{Z}/2\mathbb{Z}, x \mapsto x^2/4)$, $\rho_{A_2} : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{C})$

$$\rho_{A_2}(T,1) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad \rho_{A_2}(S,\sqrt{\tau}) = \frac{1-i}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- $A_3 = (\mathbb{Z}/3\mathbb{Z}, x \mapsto x^2/3)$, $\rho_{A_3} : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_3(\mathbb{C})$

$$\rho_{A_3}(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix} \quad \rho_{A_3}(S) = \frac{1-2\zeta_3^2}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 \end{pmatrix}$$

$$\zeta_3 = e^{2\pi i/3}$$

Properties of the Weil representations

- $N = \mathrm{ord}(\rho_D(T)) =$ level of the finite quadratic module.

Properties of the Weil representations

- $N = \mathrm{ord}(\rho_D(T))$ = level of the finite quadratic module.
- ρ_D factors through finite group, a central extension

$$1 \to \{\pm 1\} \to G \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \to 1$$

Properties of the Weil representations

- $N = \mathrm{ord}(\rho_D(T))$ = level of the finite quadratic module.
- ρ_D factors through finite group, a central extension

$$1 \to \{\pm 1\} \to G \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \to 1$$

- Easy to decompose $\rho_D \simeq \bigoplus_i \rho_i$ into irreducibles.

Properties of the Weil representations

- $N = \mathrm{ord}(\rho_D(T))$ = level of the finite quadratic module.
- ρ_D factors through finite group, a central extension

$$1 \to \{\pm 1\} \to G \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \to 1$$

- Easy to decompose $\rho_D \simeq \bigoplus_i \rho_i$ into irreducibles.

Theorem (Nobs, 1970s)

Every complex irreducible representation of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ appears as a factor of a suitable Weil representation ρ_D, except for 18 exceptional ones when $N = 2^r$.

Integrality for the Weil representation

- The matrix entries of ρ_D are in $\mathbb{Z}[1/N, \zeta_N]$, $\zeta_N = e^{2\pi i/N}$.

Integrality for the Weil representation

- The matrix entries of ρ_D are in $\mathbb{Z}[1/N, \zeta_N]$, $\zeta_N = e^{2\pi i/N}$.

Question: "clear the denominator"?

Is there a choice of basis for $\mathbb{C}(D)$ such that the matrix entries of ρ_D are in $\mathbb{Z}[\zeta_N]$?

Integrality for the Weil representation

- The matrix entries of ρ_D are in $\mathbb{Z}[1/N, \zeta_N]$, $\zeta_N = e^{2\pi i/N}$.

Question: "clear the denominator"?

Is there a choice of basis for $\mathbb{C}(D)$ such that the matrix entries of ρ_D are in $\mathbb{Z}[\zeta_N]$?

Concretely: find a matrix M such that

$$M\rho_D(T)M^{-1} \in \mathsf{GL}_{|D|}(\mathbb{Z}[\zeta_N])$$

$$M\rho_D(S)M^{-1} \in \mathsf{GL}_{|D|}(\mathbb{Z}[\zeta_N])$$

Example: A_3

$$M = \begin{pmatrix} 0 & -\frac{2}{3}\zeta_3 + \frac{2}{3} & 0 \\ 1 & \frac{1}{3}\zeta_3 + \frac{2}{3} & 1 \\ 1 & \frac{1}{3}\zeta_3 + \frac{2}{3} & -1 \end{pmatrix}$$

Example: A_3

$$M = \begin{pmatrix} 0 & -\frac{2}{3}\zeta_3 + \frac{2}{3} & 0 \\ 1 & \frac{1}{3}\zeta_3 + \frac{2}{3} & 1 \\ 1 & \frac{1}{3}\zeta_3 + \frac{2}{3} & -1 \end{pmatrix}$$

$$\rho_{A_3}(T): \quad M \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix} M^{-1} = \begin{pmatrix} \zeta_3 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}$$

Example: A_3

$$M = \begin{pmatrix} 0 & -\frac{2}{3}\zeta_3 + \frac{2}{3} & 0 \\ 1 & \frac{1}{3}\zeta_3 + \frac{2}{3} & 1 \\ 1 & \frac{1}{3}\zeta_3 + \frac{2}{3} & -1 \end{pmatrix}$$

$$\rho_{A_3}(T): \quad M \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix} M^{-1} = \begin{pmatrix} \zeta_3 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}$$

$$\rho_{A_3}(S): \quad M \left(\frac{1 - 2\zeta_3^2}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 \end{pmatrix} \right) M^{-1} = \begin{pmatrix} \zeta_3 & -1 & 0 \\ -\zeta_3 & -\zeta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example: A_3

$$M = \begin{pmatrix} 0 & -\frac{2}{3}\zeta_3 + \frac{2}{3} & 0 \\ 1 & \frac{1}{3}\zeta_3 + \frac{2}{3} & 1 \\ 1 & \frac{1}{3}\zeta_3 + \frac{2}{3} & -1 \end{pmatrix}$$

$$\rho_{A_3}(T): \quad M \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix} M^{-1} = \begin{pmatrix} \zeta_3 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}$$

$$\rho_{A_3}(S): \quad M \left(\frac{1-2\zeta_3^2}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3^2 & \zeta_3 \\ 1 & \zeta_3 & \zeta_3^2 \end{pmatrix} \right) M^{-1} = \begin{pmatrix} \zeta_3 & -1 & 0 \\ -\zeta_3 & -\zeta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\rho_{A_p} \simeq \rho_{A_p}^{+} \oplus \rho_{A_p}^{-} \Rightarrow$ block decompositions

Example: A_5

In [76]:
```
M
```

Out[76]:

$$\begin{pmatrix}
0 & \frac{2}{5}\zeta_5^3 - \frac{2}{5}\zeta_5^2 - \frac{2}{5}\zeta_5 + \frac{2}{5} & -\frac{4}{5}\zeta_5^3 - \frac{2}{5}\zeta_5^2 - \frac{4}{5}\zeta_5 & 0 & 0 \\
1 & -\frac{2}{5}\zeta_5^3 + \frac{1}{5}\zeta_5 + \frac{1}{5} & \frac{3}{5}\zeta_5^3 + \frac{2}{5}\zeta_5^2 + \frac{2}{5}\zeta_5 + \frac{3}{5} & 1 & -\frac{2}{5}\zeta_5^3 - \frac{4}{5}\zeta_5^2 - \frac{1}{5}\zeta_5 - \frac{3}{5} \\
0 & \frac{1}{5}\zeta_5^3 + \frac{1}{5}\zeta_5^2 + \frac{3}{5} & -\frac{1}{5}\zeta_5^3 - \frac{1}{5}\zeta_5^2 + \frac{2}{5} & 0 & \frac{1}{5}\zeta_5^3 - \frac{3}{5}\zeta_5^2 - \frac{2}{5}\zeta_5 - \frac{1}{5} \\
0 & \frac{1}{5}\zeta_5^3 + \frac{1}{5}\zeta_5^2 + \frac{3}{5} & -\frac{1}{5}\zeta_5^3 - \frac{1}{5}\zeta_5^2 + \frac{2}{5} & 0 & -\frac{1}{5}\zeta_5^3 + \frac{3}{5}\zeta_5^2 + \frac{2}{5}\zeta_5 + \frac{1}{5} \\
1 & -\frac{2}{5}\zeta_5^3 + \frac{1}{5}\zeta_5 + \frac{1}{5} & \frac{3}{5}\zeta_5^3 + \frac{2}{5}\zeta_5^2 + \frac{2}{5}\zeta_5 + \frac{3}{5} & -1 & \frac{2}{5}\zeta_5^3 + \frac{4}{5}\zeta_5^2 + \frac{1}{5}\zeta_5 + \frac{3}{5}
\end{pmatrix}$$

In [65]:
```
M^(-1)*T*M
```

Out[65]:

$$\begin{pmatrix}
\zeta_5 & \zeta_5^3 + \zeta_5^2 + 1 & -\zeta_5^3 - \zeta_5^2 - \zeta_5 - 1 & 0 & 0 \\
0 & -\zeta_5^3 & \zeta_5^2 + \zeta_5 & 0 & 0 \\
0 & -\zeta_5^2 - 1 & -\zeta_5^2 - \zeta_5 & 0 & 0 \\
0 & 0 & 0 & \zeta_5 & -\zeta_5^3 \\
0 & 0 & 0 & 0 & -\zeta_5^3 - \zeta_5^2 - \zeta_5 - 1
\end{pmatrix}$$

In [66]:
```
M^(-1)*S*M
```

Out[66]:

$$\begin{pmatrix}
-\zeta_5^2 - 1 & -\zeta_5^2 - \zeta_5 & -\zeta_5^3 - 2\zeta_5^2 - \zeta_5 - 1 & 0 & 0 \\
\zeta_5^3 + \zeta_5^2 + \zeta_5 + 1 & \zeta_5^2 + \zeta_5 & \zeta_5^3 + \zeta_5^2 + 2\zeta_5 + 1 & 0 & 0 \\
-\zeta_5^3 - \zeta_5 & -\zeta_5^3 - \zeta_5^2 - \zeta_5 - 1 & -\zeta_5 & 0 & 0 \\
0 & 0 & 0 & -\zeta_5^3 - \zeta_5^2 - \zeta_5 - 1 & \zeta_5^3 + 1 \\
0 & 0 & 0 & -1 & \zeta_5^3 + \zeta_5^2 + \zeta_5 + 1
\end{pmatrix}$$

Main Theorem

Theorem

Let (D, q) be a finite quadratic module of level N. Then the Weil representation ρ_D is integral over $\mathbb{Z}[\zeta_N]$, i.e. there exists a basis for $\mathbb{C}(D)$ such that the matrix entries of ρ_D with respect to this basis are in $\mathbb{Z}[\zeta_N]$.

Main Theorem

Theorem

Let (D, q) be a finite quadratic module of level N. Then the Weil representation ρ_D is integral over $\mathbb{Z}[\zeta_N]$, i.e. there exists a basis for $\mathbb{C}(D)$ such that the matrix entries of ρ_D with respect to this basis are in $\mathbb{Z}[\zeta_N]$.

Note: by general representation theory of finite groups, ρ_D is defined over $\mathcal{O}_K$, for $K \supseteq \mathbb{Q}(\zeta_N)$.

Main Theorem

Theorem

Let (D, q) be a finite quadratic module of level N. Then the Weil representation ρ_D is integral over $\mathbb{Z}[\zeta_N]$, i.e. there exists a basis for $\mathbb{C}(D)$ such that the matrix entries of ρ_D with respect to this basis are in $\mathbb{Z}[\zeta_N]$.

Note: by general representation theory of finite groups, ρ_D is defined over $\mathcal{O}_K$, for $K \supseteq \mathbb{Q}(\zeta_N)$. Not good enough!

Decomposition of finite quadratic modules

- Decompose (D, q):

$$(D, q) \simeq \oplus(D_i, q_i)$$

Decomposition of finite quadratic modules

- Decompose (D, q):

$$(D, q) \simeq \oplus (D_i, q_i)$$

and each (D_i, q_i) is one of:

$$A^t_{p^k} := \left(\mathbb{Z}/p^k\mathbb{Z},\ \frac{t\,x^2}{p^k}\right), \quad p > 2 \text{ prime}, \quad (t, p) = 1,$$

$$A^t_{2^k} := \left(\mathbb{Z}/2^k\mathbb{Z},\ \frac{t\,x^2}{2^{k+1}}\right), \qquad (t, 2) = 1,$$

$$B_{2^k} := \left(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \frac{x^2 + 2xy + y^2}{2^k}\right),$$

$$C_{2^k} := \left(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \frac{xy}{2^k}\right).$$

Decomposition of finite quadratic modules

- Decompose (D, q):

$$(D, q) \simeq \oplus (D_i, q_i)$$

and each (D_i, q_i) is one of:

$$A^t_{p^k} := \left(\mathbb{Z}/p^k\mathbb{Z},\ \frac{t\,x^2}{p^k}\right), \quad p > 2 \text{ prime}, \quad (t, p) = 1,$$

$$A^t_{2^k} := \left(\mathbb{Z}/2^k\mathbb{Z},\ \frac{t\,x^2}{2^{k+1}}\right), \qquad (t, 2) = 1,$$

$$B_{2^k} := \left(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \frac{x^2 + 2xy + y^2}{2^k}\right),$$

$$C_{2^k} := \left(\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}, \frac{xy}{2^k}\right).$$

- If $(D, q) \simeq \bigoplus_i (D_i, q_i)$, then $\rho_D \simeq \bigotimes_i \rho_{D_i}$

$A^t_{p^k}$ case p odd

Let $(D, q) = A^t_{p^k}$, p odd.

$$R := \mathbb{Z}[\zeta_{p^n}], R' = \mathbb{Z}[1/p^n, \zeta_{p^n}]$$

ρ_D defines an $R'[G]$-module $W(D, q)$, free of rank $|D|$ as an R'-module.

$A^t_{p^k}$ case p odd

Let $(D, q) = A^t_{p^k}$, p odd.

$$R := \mathbb{Z}[\zeta_{p^n}], R' = \mathbb{Z}[1/p^n, \zeta_{p^n}]$$

ρ_D defines an $R'[G]$-module $W(D, q)$, free of rank $|D|$ as an R'-module.

Lemma (Curtis-Reiner, Thm. 75.2)

There exists an $R[G]$-module $U(D, q)$ such that

$$W(D, q) \simeq U(D, q) \otimes_R R'$$

as $R'[G]$-modules.

$A^t_{p^k}$ case p odd

Let $(D,q) = A^t_{p^k}$, p odd.

$$R := \mathbb{Z}[\zeta_{p^n}], R' = \mathbb{Z}[1/p^n, \zeta_{p^n}]$$

ρ_D defines an $R'[G]$-module $W(D,q)$, free of rank $|D|$ as an R'-module.

Lemma (Curtis-Reiner, Thm. 75.2)

There exists an $R[G]$-module $U(D,q)$ such that

$$W(D,q) \simeq U(D,q) \otimes_R R'$$

as $R'[G]$-modules.

We need to prove that $U(D,q)$ is free over R.

How to prove a finitely generated module is free?

How to prove a finitely generated module is free?

- Structure theorem of finitely generated modules over a Dedekind domain:

$$U(D,q) \simeq \bigoplus_{i=1}^{p} J_i$$

where each J_i is a fractional ideal of R.

How to prove a finitely generated module is free?

- Structure theorem of finitely generated modules over a Dedekind domain:

$$U(D,q) \simeq \bigoplus_{i=1}^{p} J_i$$

  where each J_i is a fractional ideal of R.
- The isomorphism class of $U(D,q)$ is determined by its *Steinitz class*

$$[U(D,q)] := [\prod J_i] = \prod [J_i] \in \mathrm{Cl}(R).$$

How to prove a finitely generated module is free?

- Structure theorem of finitely generated modules over a Dedekind domain:

$$U(D,q) \simeq \bigoplus_{i=1}^{p} J_i$$

where each J_i is a fractional ideal of R.

- The isomorphism class of $U(D,q)$ is determined by its *Steinitz class*

$$[U(D,q)] := [\prod J_i] = \prod [J_i] \in \mathrm{Cl}(R).$$

Theorem

The Steinitz class $[U(D,q)]$ is trivial in $\mathrm{Cl}(R)$.

This implies $U(D,q)$ is free.

Finding explicit integral bases

Finding explicit integral bases

- For $(\mathbb{Z}/p\mathbb{Z}, x \mapsto x^2/p)$ p prime: Pat Gilmer (LSU), Yilong Wang (LSU).

Finding explicit integral bases

- For $(\mathbb{Z}/p\mathbb{Z}, x \mapsto x^2/p)$ p prime: Pat Gilmer (LSU), Yilong Wang (LSU).

- Take

$$\epsilon_+ = \delta_1 + \delta_{p-1}, \quad \epsilon_- = \delta_1 - \delta_{p-1}$$

and let the basis be

$$\epsilon_+, \rho_D(U)(\epsilon_+), \ldots, \rho_D(U)(\epsilon_+)^{(p+1)/2},$$

$$\epsilon_-, \rho_D(U)(\epsilon_-), \ldots, \rho_D(U)(\epsilon_-)^{(p-1)/2}$$

where $U = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

More integral bases: $\mathbb{Z}/9\mathbb{Z}$

$$\left(\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & \zeta_9 & -\zeta_9^4 & 0 & -\zeta_9^4-\zeta_9 & \zeta_9^4+\zeta_9 & 0 & \zeta_9^4 & -\zeta_9 \\ 0 & \zeta_9^2 & \zeta_9^5+\zeta_9^2 & 0 & \zeta_9^5 & -\zeta_9^5 & 0 & -\zeta_9^5-\zeta_9^2 & -\zeta_9^2 \\ 0 & 0 & 0 & -\zeta_9^3-2 & 0 & 0 & \zeta_9^3+2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & \zeta_9 & \zeta_9^4 & 0 & -\zeta_9^4-\zeta_9 & -\zeta_9^4-\zeta_9 & 0 & \zeta_9^4 & \zeta_9 \\ 0 & \zeta_9^2 & -\zeta_9^5-\zeta_9^2 & 0 & \zeta_9^5 & \zeta_9^5 & 0 & -\zeta_9^5-\zeta_9^2 & \zeta_9^2 \\ 2\zeta_9^3 & 0 & 0 & -\zeta_9^3 & 0 & 0 & -\zeta_9^3 & 0 & 0 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\zeta_9^3-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\zeta_9^3-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\zeta_9^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\zeta_9^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\zeta_9^3-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\zeta_9^3-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \zeta_9^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_9^3 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_9^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_9^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right)$$

Applications

- Modular representations

$$\mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z}) \longrightarrow \mathrm{GL}_*(\mathbb{F}_p)$$

(Gilmer-Massbaum, 2015 for $n = 1$). What's known for $n > 1$?

Applications

- Modular representations

$$\mathsf{SL}_2(\mathbb{Z}/p^n\mathbb{Z}) \longrightarrow \mathsf{GL}_*(\mathbb{F}_p)$$

(Gilmer-Massbaum, 2015 for $n = 1$). What's known for $n > 1$?

- Integral Weil representations are toy models for Integral Topological Quantum Field Theories of Gilmer and Massbaum.

Applications

- Modular representations

$$\mathsf{SL}_2(\mathbb{Z}/p^n\mathbb{Z}) \longrightarrow \mathsf{GL}_*(\mathbb{F}_p)$$

(Gilmer-Massbaum, 2015 for $n = 1$). What's known for $n > 1$?

- Integral Weil representations are toy models for Integral Topological Quantum Field Theories of Gilmer and Massbaum.

- (2+1) TQFTs = Modular Tensor Categories

Modular tensor categories

Sort of a Definition

A modular tensor category $\mathcal{C}$ is a braided spherical fusion category satisfying a certain non-degeneracy condition...

Modular tensor categories

Sort of a Definition

A modular tensor category $\mathcal{C}$ is a braided spherical fusion category satisfying a certain non-degeneracy condition...

- Simple objects $\Pi(\mathcal{C}) = \{x_1, \ldots, x_r\}$, $r =$ rank of $\mathcal{C}$.

Modular tensor categories

Sort of a Definition

A modular tensor category $\mathcal{C}$ is a braided spherical fusion category satisfying a certain non-degeneracy condition...

- Simple objects $\Pi(\mathcal{C}) = \{x_1, \ldots, x_r\}$, $r =$ rank of $\mathcal{C}$.
- $\mathcal{C}$ is a 'machine' producing projective representations of mapping class groups

$$\mathcal{M}(X) \longrightarrow \mathrm{PGL}_*(\mathbb{C})$$

where X is an orientable surface with boundary components labeled by $\Pi(\mathcal{C})$.

The modular representation

$\mathcal{M}(T^2) = \mathsf{SL}_2(\mathbb{Z}) \to \mathrm{PGL}_r(\mathbb{C})$ is the modular representation of $\mathcal{C}$.

The modular representation

$\mathcal{M}(T^2) = \mathsf{SL}_2(\mathbb{Z}) \to \mathrm{PGL}_r(\mathbb{C})$ is the modular representation of $\mathcal{C}$.

Theorem

The modular representation lifts canonically to a linear representation

$$\rho_{\mathcal{C}} : \mathsf{Mp}_c(\mathbb{Z}) \longrightarrow \mathsf{GL}_r(\mathbb{C})$$

where c is the order of the square of the global anomaly of $\mathcal{C}$.

The modular representation

$\mathcal{M}(T^2) = \mathsf{SL}_2(\mathbb{Z}) \to \mathrm{PGL}_r(\mathbb{C})$ is the modular representation of $\mathcal{C}$.

Theorem

The modular representation lifts canonically to a linear representation

$$\rho_{\mathcal{C}} : \mathsf{Mp}_c(\mathbb{Z}) \longrightarrow \mathsf{GL}_r(\mathbb{C})$$

where c is the order of the square of the global anomaly of $\mathcal{C}$.

$\rho_{\mathcal{C}}$ factors through $\mathsf{SL}_2(\mathbb{Z}/N\mathbb{Z})$ where $N = \mathrm{ord}(\rho_{\mathcal{C}}(T_c))$ is the level of $\mathcal{C}$ (Ng-Schauenberg, 2010)

Example: Fibonacci modular category

- $\mathcal{C}$ is a fusion category with two simple objects (rank 2) $\{1, \sigma\}$, $\sigma \otimes \sigma \simeq 1 \oplus \sigma$.

Example: Fibonacci modular category

- $\mathcal{C}$ is a fusion category with two simple objects (rank 2) $\{1, \sigma\}$, $\sigma \otimes \sigma \simeq 1 \oplus \sigma$.

- Modular representation $\rho : \mathrm{Mp}_5(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{C})$:

$$\rho(T, 1) = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_5^2 \end{pmatrix}$$

$$\rho(S, \sqrt[5]{\tau}) = \frac{1 + \varphi^2 \zeta_5^3}{2 + \varphi} \begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}$$

$\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio

Integrality for modular tensor categories

Theorem (C., Ng, Wang 2019)

Let $\mathcal{C}$ be a modular tensor category of level $N = p^n$, for prime p. Then the modular representation $\rho_{\mathcal{C}}$ of the metaplectic cover $\mathrm{Mp}_c(\mathbb{Z})$ is integral over $\mathbb{Z}[\zeta_N]$.

Integrality for modular tensor categories

Theorem (C., Ng, Wang 2019)

Let $\mathcal{C}$ be a modular tensor category of level $N = p^n$, for prime p. Then the modular representation $\rho_{\mathcal{C}}$ of the metaplectic cover $\mathrm{Mp}_c(\mathbb{Z})$ is integral over $\mathbb{Z}[\zeta_N]$.

- Uses the Cauchy Theorem of modular tensor categories (Bruillard, Ng, Rowell, Zhang, 2015).

Integrality for modular tensor categories

Theorem (C., Ng, Wang 2019)

Let $\mathcal{C}$ be a modular tensor category of level $N = p^n$, for prime p. Then the modular representation $\rho_{\mathcal{C}}$ of the metaplectic cover $\mathrm{Mp}_c(\mathbb{Z})$ is integral over $\mathbb{Z}[\zeta_N]$.

- Uses the Cauchy Theorem of modular tensor categories (Bruillard, Ng, Rowell, Zhang, 2015).
- $\rho_{\mathcal{C}} = \rho_D$ is the Weil rep., when $\mathcal{C}$ is pointed!

Integrality for modular tensor categories

Theorem (C., Ng, Wang 2019)

Let $\mathcal{C}$ be a modular tensor category of level $N = p^n$, for prime p. Then the modular representation $\rho_{\mathcal{C}}$ of the metaplectic cover $\mathrm{Mp}_c(\mathbb{Z})$ is integral over $\mathbb{Z}[\zeta_N]$.

- Uses the Cauchy Theorem of modular tensor categories (Bruillard, Ng, Rowell, Zhang, 2015).
- $\rho_{\mathcal{C}} = \rho_D$ is the Weil rep., when $\mathcal{C}$ is pointed!
- Find explicit integral bases?

Fibonacci example

Let $p^{\pm} = 1 + \varphi^2 \zeta_5^{\pm 2}, \quad \varphi = \frac{1+\sqrt{5}}{2}$

$$\rho(T, 1) = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_5^2 \end{pmatrix}$$

$$\rho(S, \sqrt[5]{\tau}) = \frac{p^-}{2+\varphi} \begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}$$

Fibonacci example

Let $p^{\pm} = 1 + \varphi^2 \zeta_5^{\pm 2}, \quad \varphi = \frac{1+\sqrt{5}}{2}$

$$\rho(T, 1) = \begin{pmatrix} 1 & 0 \\ 0 & \zeta_5^2 \end{pmatrix}$$
$$\rho(S, \sqrt[5]{\tau}) = \frac{p^-}{2+\varphi} \begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 1/p^+ \\ 0 & \varphi/p^+ \end{pmatrix}$$

$$M^{-1}TM = \begin{pmatrix} 1 & \zeta_5^3 \\ 0 & \zeta_5^2 \end{pmatrix}, \quad M^{-1}SM = \begin{pmatrix} 0 & -\zeta_5^4 \\ 1 & 0 \end{pmatrix}$$

Thank you!