Moonshine for Finite Groups

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Modular functions

Definition

A meromorphic function $f : \mathbb{H} \mapsto \mathbb{C}$ is a Γ-**modular function** if for every $\gamma \in \Gamma$ we have

$$f(\gamma\tau) = f(\tau).$$

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Example ($\Gamma = \mathrm{SL}_2(\mathbb{Z})$)

The **Hauptmodul** is Klein's j-function ($q := e^{2\pi i\tau}$)

$$J(\tau) := j(\tau) - 744 = \sum_{n=-1}^{\infty} c(n)q^n$$
$$= q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \ldots$$

Glimpses of Monstrous Moonshine

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$$\underbrace{\begin{array}{r} 196884 \\ 21493760 \\ 864299970 \end{array}}_{\text{Coefficients of } j(\tau)} \begin{array}{c} = \\ = \\ = \end{array} \underbrace{\begin{array}{l} 1 + 196883 \\ 1 + 196883 + 21296876 \\ 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \end{array}}_{\text{Dimensions of irreducible representations of the Monster } \mathbb{M}}$$

The Monster characters

The character table for $\mathbb{M}$ (ordered by size) gives dimensions:

$$\begin{aligned} \chi_1(e) &= 1 \\ \chi_2(e) &= 196883 \\ \chi_3(e) &= 21296876 \\ \chi_4(e) &= 842609326 \\ &\vdots \\ \chi_{194}(e) &= 258823477531055064045234375. \end{aligned}$$

Thompson's Monstrous Conjecture

Conjecture (Thompson)

There is a "nice" infinite-dimensional graded module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *for which* $\dim(V_n^\natural) = c(n)$.

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Remark

We can use the trivial representation which has $\dim \chi_1 = 1$. *Using* **too many** *trivial representations is not "nice".*

Web of Numerology?

Definition (Thompson)

Assuming the conjecture, if $g \in \mathbb{M}$, then define the **McKay–Thompson series**

$$T_g(\tau) := \sum_{n=-1}^{\infty} \mathrm{Tr}(g|V_n^\natural)q^n.$$

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Question

Is there a $V^{\natural}$ for which **all** *of the $T_g(\tau)$ are simultaneously nice?*

Monstrous Moonshine Conjecture

Conjecture (Conway and Norton, 1979)

For each $g \in \mathbb{M}$ there is an explicit genus 0 congruence subgroup $\Gamma_g \subset \mathrm{SL}_2(\mathbb{R})$ for which $T_g(\tau)$ is the **Hauptmodul**.

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Theorem (Frenkel–Lepowsky–Meurman (1980s))

If it exists, then the moonshine module $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ *is a specific vertex operator algebra whose automorphism group is* $\mathbb{M}$.

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Theorem (Borcherds (1998 Fields Medal))

The Monstrous Moonshine Conjecture is true.

Aftermath

Inspired by string theory, further moonshines have been found:

- Mathieu (Gannon)
- Umbral (Cheng, Duncan, Harvey, and Duncan, O, Griffin)
- Thompson (Griffin and Mertens)
- Pariah (Duncan, O, Mertens)
- to name a few...

Witten's Problem

Question (Black Hole States)

Consider the monstrous moonshine expressions

$$\begin{aligned} 196884 &= 1 + 196883 \\ 21493760 &= 1 + 196883 + 21296876 \\ 864299970 &= 1 + 1 + 196883 + 196883 + 21296876 + 842609326 \\ &\vdots \\ c(n) &= \sum_{i=1}^{194} \mathbf{m}_i(n)\chi_i(e) \end{aligned}$$

How many '1's, '196883's, etc. show up in these expressions?

Some Proportions

n	$\delta(\mathbf{m}_1(n))$	$\delta(\mathbf{m}_2(n))$	$\cdots$	$\delta(\mathbf{m}_{194}(n))$
1	$1/2$	$1/2$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
40	$4.011\ldots \times 10^{-4}$	$2.514\ldots \times 10^{-3}$	$\cdots$	$0.00891\ldots$
60	$2.699\ldots \times 10^{-9}$	$2.732\ldots \times 10^{-8}$	$\cdots$	$0.04419\ldots$
80	$4.809\ldots \times 10^{-14}$	$7.537\ldots \times 10^{-13}$	$\cdots$	$0.04428\ldots$
100	$4.427\ldots \times 10^{-18}$	$1.077\ldots \times 10^{-16}$	$\cdots$	$0.04428\ldots$
120	$1.377\ldots \times 10^{-21}$	$5.501\ldots \times 10^{-20}$	$\cdots$	$0.04428\ldots$
140	$1.156\ldots \times 10^{-24}$	$1.260\ldots \times 10^{-22}$	$\cdots$	$0.04428\ldots$
160	$2.621\ldots \times 10^{-27}$	$3.443\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
180	$1.877\ldots \times 10^{-28}$	$3.371\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
200	$1.715\ldots \times 10^{-28}$	$3.369\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
220	$1.711\ldots \times 10^{-28}$	$3.368\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$
240	$1.711\ldots \times 10^{-28}$	$3.368\ldots \times 10^{-23}$	$\cdots$	$0.04428\ldots$

Distribution of Monstrous Moonshine

Theorem (Duncan, Griffin, O (2015))

If $1 \leq i \leq 194$, *then as* $n \to +\infty$ *we have*

$$\mathbf{m}_i(n) \sim \frac{\dim(\chi_i)}{\sqrt{2}|n|^{3/4}|\mathbb{M}|} \cdot e^{4\pi\sqrt{|n|}}$$

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Corollary (Duncan, Griffin, O)

The Moonshine module is asymptotically **regular**.
In other words, we have

$$\delta(\mathbf{m}_i) := \lim_{n\to+\infty} \frac{\mathbf{m}_i(n)}{\sum_{i=1}^{194} \mathbf{m}_i(n)} = \frac{\dim(\chi_i)}{\sum_{j=1}^{194} \dim(\chi_j)}.$$

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Definition

A finite group G admits **weak moonshine** if there is an infinite dimensional graded G-module

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$$V_G := \oplus_n V_G(n)$$

such that **for all** $g \in G$ the McKay–Thompson series

$$T_g(\tau) := \sum_n \mathrm{Tr}(g|V_G(n))q^n$$

is a weakly holomorphic modular function.

Finite groups enjoy weak moonshine

Theorem (Dehority, Gonzalez, Vafa, Van Peski (2017))

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Variants

1. *We can require that each $T_g(\tau)$ is modular on $\Gamma_0(\mathrm{ord}_G(g))$.*

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Variants

1. *We can require that each* $T_g(\tau)$ *is modular on* $\Gamma_0(\mathrm{ord}_G(g))$.
2. *In very special cases there are analytic "group compatibility" relations between* $T_g(\tau)$ *and* $T_{g^p}(\tau)$.

Example: Moonshine for D_4 and Q_8

$\mathbf{D_4}$	$\{1\}$	$\{r^2\}$	$\{r, r^3\}$	$\{s, r^2s\}$	$\{rs, r^3s\}$
$\mathbf{Q_8}$	$\{1\}$	$\{-1\}$	$\{i, -i\}$	$\{j, -j\}$	$\{k, -k\}$
	C_1	C_2	C_3	C_4	C_5
χ_1	1	1	1	1	1
χ_2	1	1	−1	1	−1
χ_3	1	1	−1	−1	1
χ_4	1	1	1	−1	−1
χ_5	2	−2	0	0	0

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	C_1	C_2	C_3	C_4	C_5
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- The MT series are Hauptmoduln $J_N(\tau)$ for $\Gamma_0(N)$:

$$T(C_1;\tau) = J_1(\tau)$$
$$T(C_2;\tau) = T(C_4;\tau) = T(C_5;\tau) = J_2(\tau)$$
$$T(C_3;\tau) = J_4(\tau)$$

Example of D_4 and Q_8 continued

- If $1 \leq i \leq 5$ and $n \geq -1$, then let

$$m_i(n) = \#\{\text{mult. of } \rho_i \text{ in } V_G(n)\}.$$

Example of D_4 and Q_8 continued

- If $1 \leq i \leq 5$ and $n \geq -1$, then let

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- The **multiplicity generating functions** are:

$$\mathcal{M}_i(\tau) := \sum_n m_i(n) q^n.$$

$$\mathcal{M}_1(\tau) = q^{-1} + 24788q + 2685440q^2 + 108044482q^3 + O\left(q^4\right),$$
$$\mathcal{M}_2(\tau) = 24640q + 2686464q^2 + 108038912q^3 + O\left(q^4\right),$$
$$\mathcal{M}_3(\tau) = 24640q + 2686464q^2 + 108038912q^3 + O\left(q^4\right),$$
$$\mathcal{M}_4(\tau) = 24512q + 2687488q^2 + 108033280q^3 + O\left(q^4\right),$$
$$\mathcal{M}_5(\tau) = 49152q + 5373952q^2 + 216072192q^3 + O\left(q^4\right).$$

Example of D_4 and Q_8 continued

This weak moonshine is **asymptotically regular.**

n	$\delta_1(n)$	$\delta_2(n) = \delta_3(n)$	$\delta_4(n)$	$\delta_5(n)$
1	$0.16779\dots$	$0.16678\dots$	$0.16592\dots$	$0.33271\dots$
2	$0.16659\dots$	$0.16665\dots$	$0.16671\dots$	$0.33337\dots$
3	$0.16666\dots$	$0.16666\dots$	$0.16665\dots$	$0.33332\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	$1/6$	$1/6$	$1/6$	$1/3$

Natural Problem

Fact

(1) As the D_4 and Q_8 example illustrates, nonisomorphic groups with identical character tables have the same weak moonshine.

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Problem

Can weak moonshine be refined (in a uniformly bounded way) to distinguish groups?

Main Takeaways

Theorem 1 (D–Ono)

If G is a finite group and $s \in \mathbb{Z}^+$, then weak moonshine for G extends to **width** *s weak moonshine.*

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Corollary (D–Ono)

If $s \geq 3$, then **complete** *width s weak moonshine* **determines finite groups up to isomorphism.**

Notation

- G is a finite group
- Let $\rho_1, \rho_2, \ldots, \rho_t$ be the irreducible representations

$$\rho_i \colon G \to \mathrm{GL}_{d_i}(\mathbb{C}).$$

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- Let $V_G = \oplus_n V_G(n)$ be a **weak moonshine** module for G.

Frobenius r-characters

Definition (Frobenius, 1896)

Let χ be a character of G, and for positive integers r we let

$$G^{(r)} := G \times \cdots \times G \quad (r \text{ copies}).$$

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(3) If $r \geq 3$, then $\chi^{(r)}(g_1, g_2, \ldots, g_r)$ is defined by

$$\chi^{(r)}(g_1, \ldots, g_r) := \chi(g_1)\, \chi^{(r-1)}(g_2, \ldots, g_r) \\ - \chi^{(r-1)}(g_1 g_2, \ldots, g_r) - \cdots - \chi^{(r-1)}(g_2, \ldots, g_1 g_r).$$

Basic Facts about r-characters

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Facts (Vanishing)

(1) If $\dim(\chi) = 1$, *then* $\chi^{(2)}(g_1, g_2) = \chi(g_1)\chi(g_2) - \chi(g_1 g_2) = 0$.

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Fact (Expansion as 1-characters)

If $r \geq 2$, *then* $\chi^{(r)}(g_1, \ldots, g_r)$ *is a* **signed** *sum over* S_r *action on*

$$\chi(g_1), \chi(g_2), \ldots, \chi(g_r).$$

Deep Theorem about r-characters

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Remark

Infinitely many nonisomorphic groups share 1 and 2-character tables.

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2. If $1 \leq r \leq s$ and $\underline{g} \in G^{(r)}$, then the *McKay–Thompson series*
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$$T\left(r, \underline{g}; \tau\right) := \sum_{n \gg -\infty} \mathrm{Frob}_r\left(\underline{g}; n\right) q^n$$
is a weakly holomorphic modular function.

COMPUTING $\mathrm{Frob}_r(\overline{g}; n)$

LEMMA

If the $m_i(n)$ are the multiplicities of ρ_i in $V_G(n)$, then

$$\mathrm{Frob}_{r}\left(\underline{g}; n\right) = \mathrm{Tr}\left(\underline{g}|V_G^{(r)}(n)\right) := \sum_{1\leq i\leq t} m_i(n)\chi_i^{(r)}\left(\underline{g}\right).$$

WIDTH s MOONSHINE

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Corollary (D–Ono)

If $s \geq 3$, then **complete** *width s weak moonshine* **determines finite groups up to isomorphism.**

Higher Width MT Series

Question

What information do the higher width MT series

$$\left\{ T(r, \underline{g}; \tau) \ : \ \underline{g} \in G^{(r)} \right\}$$

encode about structure of the "seed" module V_G?

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Answer

The width r MT series know the part of V_G assembled from the characters with $\dim \chi_i \geq r$.

Higher Width MT Series

Theorem 2 (D–Ono)

If width s weak moonshine holds for G, $1 \leq r \leq s$ and $\dim\chi_i \geq r$, then the χ_i multiplicity generating function satisfies

$$\mathcal{M}_i(\tau) := \sum_{n \gg -\infty} m_i(n) q^n$$

$$= \frac{(\dim\chi_i)^{r-1}}{r!|G|^r (\dim\chi_i - 1) \cdots (\dim\chi_i - (r-1))} \sum_{\underline{g} \in G^{(r)}} \overline{\chi_i^{(r)}(\underline{g})} T(r, \underline{g}; \tau).$$

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Remark

Theorem 2 follows from new orthogonality relations for Frobenius r-characters.

Orthogonality of Frobenius r-characters

Theorem (Frobenius, Johnson, D–Ono)

We have that

$$\sum_{\underline{g}\in G^{(r)}} \chi_i^{(r)}\left(\underline{g}\right)\overline{\chi_j^{(r)}\left(\underline{g}\right)} = \frac{r!|G|^r\delta_{ij}}{\left(\dim\chi_i\right)^{r-1}}\left(\dim\chi_i - 1\right)\cdots\left(\dim\chi_i - (r-1)\right).$$

ORTHOGONALITY OF FROBENIUS r-CHARACTERS

THEOREM (FROBENIUS, JOHNSON, D–ONO)

We have that

$$\sum_{\underline{g}\in G^{(r)}} \chi_i^{(r)}(\underline{g})\,\overline{\chi_j^{(r)}(\underline{g})}$$

$$= \frac{r!|G|^r\delta_{ij}}{(\dim\chi_i)^{r-1}}(\dim\chi_i - 1)\cdots(\dim\chi_i - (r-1)).$$

REMARKS

(1) The $r = 1$ case is due to Schur.
(2) The $i \neq j$ case is due to Frobenius and Johnson.
(3) Our contribution is the $i = j$ case which gives the "norms" of Frobenius r-characters.

Example of D_4 and Q_8 revisited

- The MT series for $\left(r^3s, rs\right) \in D_4^{(2)}$ and $(-k, k) \in Q_8^{(2)}$ are

$$T\left(2, \left(r^3s, rs\right); \tau\right) = \sum_{1 \le i \le 5} \chi_i^{(2)}\left(r^3s, rs\right)\mathcal{M}_i(\tau) = \chi_5^{(2)}\left(r^3s, rs\right)\mathcal{M}_5(\tau)$$
$$= 98304q + 10747904q^2 + 432144384q^3 + O\left(q^4\right).$$

$$T\left(2, (-k, k); \tau\right) = \sum_{1 \le i \le 5} \chi_i^{(2)}(-k, k)\mathcal{M}_i(\tau) = \chi_5^{(2)}(-k, k)\mathcal{M}_5(\tau)$$
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- The MT series for $(r^3s, rs) \in D_4^{(2)}$ and $(-k,k) \in Q_8^{(2)}$ are

$$T\left(2, \left(r^3s, rs\right); \tau\right) = \sum_{1 \le i \le 5} \chi_i^{(2)}\left(r^3s, rs\right) \mathcal{M}_i(\tau) = \chi_5^{(2)}\left(r^3s, rs\right) \mathcal{M}_5(\tau)$$
$$= 98304q + 10747904q^2 + 432144384q^3 + O\left(q^4\right).$$

$$T\left(2, (-k,k); \tau\right) = \sum_{1 \le i \le 5} \chi_i^{(2)}(-k,k)\mathcal{M}_i(\tau) = \chi_5^{(2)}(-k,k)\mathcal{M}_5(\tau)$$
$$= -98304q - 10747904q^2 - 432144384q^3 + O\left(q^4\right).$$

- That they are unequal distinguishes D_4 from Q_8.

Summary

Theorem 1 (D–Ono)

If G is a finite group and $s \in \mathbb{Z}^+$, then weak moonshine for G extends to **width** *s weak moonshine. Moreover, G admits* **asymptotically regular** *width s weak moonshine.*

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THEOREM 1 (D–ONO)

If G is a finite group and $s \in \mathbb{Z}^+$, then weak moonshine for G extends to **width** *s weak moonshine. Moreover, G admits* **asymptotically regular** *width s weak moonshine.*

COROLLARY (D–ONO)

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Corollary (D–Ono)

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Theorem 2 (D–Ono)

If $\dim \chi_i \geq r$, then the multiplicity generating functions satisfy

$$\mathcal{M}_i(\tau) := \sum_{n \gg -\infty} m_i(n) q^n =* \sum_{\underline{g} \in G^{(r)}} \overline{\chi_i^{(r)}(\underline{g})}\, T(r, \underline{g}; \tau).$$