Existence of eta-quotients of squarefree levels

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April 13, 2019
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Dedekind's η function and η-quotients

The Dedekind η-function $\eta : \mathbb{H} \to \mathbb{C}$ is defined by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q := e^{2\pi i \tau}$. We have $\eta(\tau)^{24} = \Delta(\tau) \in M_{12}(\mathrm{SL}_2(\mathbb{Z}))$.
An eta-quotient is a function of the form

$$f(\tau) = \prod_{0<\delta|N} \eta(\delta\tau)^{r_\delta}$$

where each $r_\delta \in \mathbb{Z}$ and $N \in \mathbb{N}$.

Modular form spaces generated by η-quotients

Ono - Every modular form on $SL_2(\mathbb{Z})$ may be expressed as a rational function in $\eta(\tau)$, $\eta(2\tau)$, and $\eta(4\tau)$.

Rouse and Webb - There are precisely 121 positive integers $N \leqslant 500$ such that the graded ring of modular forms for $\Gamma_0(N)$ is generated by eta-quotients.

Rouse and Webb's computations make use of the following bound, originally obtained by Bhattacharya:

Theorem (Bhattacharya, Rouse and Webb)

Suppose that $f(\tau) = \prod \eta(\delta\tau)^{r_\delta}$ is modular of level N and weight . Then

$$\sum_{\delta | N} |r_\delta| \leqslant 2k \prod_{p|N} \left(\frac{p+1}{p-1}\right)^{\min\{2, ord_p(N)\}}.$$

Informally, this says that in order for $M(\Gamma_0(N))$ to be generated by η-quotients, it is necessary that N be "sufficiently composite".

Necessary Tools

Theorem (Newman)
Let $f(\tau) = \prod_{\delta | N} \eta^{r_\delta}(\delta\tau)$. *If* f *satisfies*

$$\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24}$$

$$\sum_{\delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then for $\mathrm{k} = \frac{1}{2} \sum_{\delta | N} r_\delta$ *and* $\chi(d) = \left(\frac{(-1)^{\mathrm{k}} s}{d}\right)$ *where* $s = \prod_{\delta | N} \delta^{r_\delta}$*,*
$f \in M^!_{\mathrm{k}}(\Gamma_0(N), \chi)$.

In the case $\gcd(N, 6) = 1$, the two congruence conditions are equivalent as every element of $(\mathbb{Z}/24\mathbb{Z})^x$ is its own inverse. Additionally, when N is coprime to 6 the converse of this theorem holds.

Theorem (Ligozat)

Let c, d, and N be positive integers with $d \mid N$ and $\gcd(c, d) = 1$. If $f(\tau)$ is an η-quotient satisfying the conditions given in the prior theorem, then the order of vanishing for $f(\tau)$ at the cusp c/d is

$$\frac{N}{24} \sum_{\delta \mid N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, {}^N\!/_d) d\delta}.$$

Note: If N is squarefree, the set $\{1/d : d \mid N\}$ is a complete set of representatives of the cusps of $\Gamma_0(N)$.

Weakly holomorphic η-quotients of squarefree level

Suppose N is coprime to 6. Newman's theorem states that existence of a weakly holomorphic modular η-quotient $f(\tau) = \prod_{\delta | N} \eta(\delta\tau)^{r_\delta}$ of level N and weight k is equivalent to existence of a solution in r_δ to

$$24m = \sum_{\delta | N} \delta r_\delta$$

for some $m \in \mathbb{Z}$. As $2\mathrm{k} = \sum_{\delta | N} r_\delta$,

$$2\mathrm{k} = 24m - \sum_{\delta | N} (\delta - 1) r_\delta.$$

Thinking of this as a linear Diophantine equation in the variables m and r_δ, we obtain the following proposition:

> *Let $N = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$ be coprime to 6, and let $h_N = \frac{1}{2} gcd(p_1 - 1, p_2 - 1, \ldots, p_\ell - 1, 24)$. Then there exist eta-quotients in $M_{\mathrm{k}}^{!}(\Gamma_0(N))$ if and only if $h_N \mid \mathrm{k}$.*

Theorem (A., Anderson, Hamakiotes, Oltsik, Swisher)
Let $p \geqslant 5$ be prime, set $h_p = \frac{1}{2}gcd(p-1, 24)$, and let k be an even integer. Then there exists a modular η-quotient $f = \eta(\tau)^{r_1}\eta(p\tau)^{r_p}$ of level N and weight k if and only if

(1) $h_p \mid \mathrm{k}$
(2) It is not the case that $p \neq 5, p \equiv 5 \pmod{24}$, *and* $\mathrm{k} = 2$.

The forwards direction is by exhaustion on the possible residues of p modulo 24. We will focus on the exceptional case laid out in (2).

We want to show that there are no η-quotients $\eta(\tau)^{r_1}\eta(p\tau)^{r_p}$ in $M_2(\Gamma_0(p))$ if $p \neq 5$, but $p \equiv 5 \pmod{24}$.
Recall the bound of Bhattacharya,

$$|r_1| + |r_p| \leqslant 4\left(\frac{p+1}{p-1}\right) < 5.$$

By Ligozat's theorem,

$$24v_1 = pr_1 + r_p$$
$$24v_{1/p} = r_1 + pr_p.$$

As $|r_1| + |r_p| \leqslant 4$, for these to be non-negative, we must have r_1 and r_p both non-negative. But by Newman's theorem we must also have

$$r_1 + 5r_p \equiv 0 \pmod{24}.$$

No such r_1, r_p exist.

Semiprime levels

Theorem (AAHOS)
Let $p, q \geqslant 5$ be distinct primes, $N = pq$ and k be an even integer. Let $h_N = \frac{1}{2}\gcd(p-1, q-1, 24)$. Then there exists $f(\tau) = \prod_{\delta|N} \eta(\delta\tau)^{r_\delta} \in M_{\mathrm{k}}(\Gamma_0(N))$ if and only if

(1) $h_N \mid \mathrm{k}$

(2) It is not the case that $(p, q) \pmod{24} \in \{(1,5), (5,1), (5,5)\}$, $p, q \neq 5$, *and* $\mathrm{k} = 2$.

Remark: When N is composite, for η-quotients to generate the graded ring of modular forms on $\Gamma_0(N)$ it is necessary that η-quotients span $M_2(\Gamma_0(N))$. So, for all exceptional p, q described in (2) as well as all p, q such that $h_{pq} > 2$, $M(\Gamma_0(N))$ is not generated by η-quotients.

The exceptional case

Again we use Bhattacharya's bound to obtain

$$|r_1| + |r_p| + |r_q| + |r_N| \leqslant 4\left(\frac{p+1}{p-1}\right)\left(\frac{q+1}{q-1}\right) < 5.$$

By Ligozat's Theorem,

$$\begin{aligned} 24v_1 &= Nr_1 + qr_p + pr_q + r_N \\ 24v_{1/p} &= qr_1 + Nr_p + r_q + pr_N \\ 24v_{1/q} &= pr_1 + r_p + Nr_q + qr_N \\ 24v_{1/N} &= r_1 + pr_p + qr_q + Nr_N. \end{aligned}$$

Again, in order for these all to be non-negative we must have each $r_\delta \geqslant 0$. But, for such r_δ, there exist no solutions to either of the equations

$$\begin{aligned} r_1 + r_p + 5r_q + 5r_N &\equiv 0 \pmod{24} \\ r_1 + 5r_p + 5r_q + r_N &\equiv 0 \pmod{24}. \end{aligned}$$

Squarefree levels

Theorem

Let $N = p_1 p_2 \cdots p_\ell$ be squarefree and coprime to 6 and let $\mathrm{k} \in \mathbb{N}$ be even. Define $h_N = \frac{1}{2}\gcd(p_1 - 1, p_2 - 1, \ldots, p_\ell - 1, 24.)$ Suppose p_1 is the smallest prime dividing N and that

$$4 \prod_{p \mid N} \frac{p+1}{p-1} < p_1 + 1.$$

Then there exist η-quotients in $M_k(\Gamma_0(N))$ if and only if

(1) $h_N \mid \mathrm{k}$

(2) It is not the case that $\mathrm{k} = 2$, each $p_i \equiv 1$ or $5 \pmod{24}$, at least one p_i is congruent to $5 \pmod{24}$, and no $p_i = 5$.

That is, we gain no "new" cases where there are weakly holomorphic η-quotients by no fully holomorphic η-quotients when we go from semiprime to squarefree.

The exceptional case

For the exceptional case, we use the fact that for each $\delta \mid N$

$$24 v_{1/\delta} = N r_\delta + \dots .$$

So, if any $r_\delta < 0$, then the largest that $24 v_{1/\delta}$ can be is if $r_\delta = -1$ and $r_{\delta'}$ is as large as possible (which is $p_1 - 1$ by hypothesis), where δ' is picked so that $r_{\delta'}$ appears in the equation for $v_{1/\delta}$ with coefficient $p_2 \cdots p_\ell$. But even in this scenario,

$$24 v_{1/\delta} = -N + (p_1 - 1)\frac{N}{p_1} < 0.$$

Thus, every r_δ must be non-negative. By Newman's theorem,

$$\sum_{\substack{\delta \mid N \\ \delta \equiv 1}} r_\delta + 5 \sum_{\substack{\delta \mid N \\ \delta \equiv 5}} r_\delta \equiv 0 \pmod{24},$$

and as $\mathrm{k} = 2$,

$$\sum_{\delta \mid N} r_\delta = 4.$$

This is impossible to achieve.

The inequality hypothesis for squarefree levels

We could only extend our techniques to squarefree levels when

$$4\prod_{p|N}\frac{p+1}{p-1} < p_1 + 1.$$

A very reasonable question to then ask is how easily this inequality fails.
The smallest integer of the desired form for which this fails would be obtained by taking the product of every prime congruent to either 1 or 5 modulo 24 starting from 29 until the product on the left exceeds 30.
If we look at the product taking every such prime from 29 up to 10^7, the product is still only approximately 8.434.

Further directions and obstacles

Extending past squarefree:
If N is not squarefree, the "sudoku" property for the orders of vanishing no longer holds. For any r_δ, there is still a cusp whose order of vanishing involves the expression Nr_δ, but there could be other N's on other r_δ's.

Dropping the assumption that $4\prod_{p|N}\frac{p+1}{p-1} < p_1 + 1$:
We lose the fact that all r_δ must be nonnegative, which makes $\sum r_\delta = 4$ a significantly looser restriction.

Including 2 and 3:
In this case, Newman's theorem no longer gives a necessary condition, so it would need a completely different approach.