OPEN PROBLEMS ON CALABI–YAU VARIETIES ARITHMETIC, GEOMETRY AND PHYSICS

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• **Problem 1**: The modularity of elliptic curves, singular K3 surfaces and rigid Calabi–Yau threefolds defined over $\mathbf{Q}$ have been established. All these cases are associated to two-dimensional Galois representations.

Now suppose that these Calabi–Yau varieties are defined over number fields. Can one establish the modularity of the associated two-dimensional Galois representations (the potential modularity)? One ought to determine the L-series and interpret them in terms of some modular forms.

• **Problem 2**: Even though we have been able to establish the modularity of Calabi–Yau varieties over $\mathbf{Q}$ in Problem 1, we do not know the conceptual reason why the modularity holds. We need to have some necessary and sufficient condition for the modularity from geometry or physics, e.g., the Arakelov–Yau bound. For instance, the six rigid Calabi–Yau threefolds corresponding to the index 12 subgroups of $\mathrm{PSL}_2(\mathbf{Z})$ attain the Arakelov–Yau bound. Also the eight non-rigid Calabi–Yau threefolds corresponding to the index 24 subgroups of $\mathrm{PSL}_2(\mathbf{Z})$ reach the Arakelov–Yau bound. All these Calabi–Yau threefolds are defined over $\mathbf{Q}$ and their modularity/automorphy have been established. Is the Arakelov–Yau bound a necessary and sufficient condition for the modularity/automorphy?

• **Problem 3**: This is the so-called *geometric realization problem*.

For the dimension 2 case, every singular K3 surface X over $\mathbf{Q}$ is motivically modular in the sense that the group of transcendental cycles $T(X)$ corresponds to a newform of weight 3. Conversely, every Hecke eigenform of weight 3 with integral Fourier coefficients corresponds to $T(X)$ of a singular K3 surface X defined over $\mathbf{Q}$ (Elkies and Schuett).

For dimension 3, rigid Calabi–Yau threefolds over $\mathbf{Q}$ are modular in the sense that there is a weight 4 modular form on some $\Gamma_0(N)$ that determines the L-series. The converse problem (raised by Mazur, and van Straten) is: *Which newforms of weight* 4 *on some* $\Gamma_0(N)$ *with integral Fourier coefficients would arise from rigid Calabi–Yau threefolds* X *over* $\mathbf{Q}$*?*

• **Problem 4**: This is about singular K3 surfaces. They are in one-to-one correspondence with the $\mathrm{SL}_2(\mathbf{Z})$-equivalence classes of positive definite even integral quadratic forms. By Shafarevich, singular K3 surfaces are all of CM type, and they are defined over number fields. Shafarevich proved that for every positive integer n, there are a finite number of singular K3 surfaces defined over a field of degree n over $\mathbf{Q}$. For a fixed n, is there a bound on the number of singular K3 surfaces defined over a field of degree n? What is the minimal field of definition of a singular K3 surface? How many are defined over $\mathbf{Q}$? (According to Schütt, there should be 101 plus possibly one more.) Find defining equations over $\mathbf{Q}$ for singular K3 surfaces.

• **Problem 5**: Consider K3 surfaces with Picard number ≤ 19. For instance, let X be a K3 surface with Picard number 19 defined over $\mathbf{Q}$. Then there should be a cusp form g of weight 2 on GL(2) such that the symmetric square $\mathrm{Sym}^2(g)$ should realize $T(X)$. Is $L(T(X)\otimes\mathbf{Q}_\ell, s) = L(\mathrm{Sym}^2(g), s)$? Suppose that a Kummer surface is given by the product $E\times E$ of some non-CM elliptic curve E over $\mathbf{Q}$. Then the Shioda–Inose structure induces an isomorphism of integral Hodge structures on the transcendental lattices, so X and $\mathrm{Km}(E\times E)$ have the same $\mathbf{Q}$-Hodge structure. In this case, the two-dimensional Galois representation ρ_E associated to E induces the three-dimensional Galois representation $\mathrm{Sym}^2(\rho_E)$ over some number field, say K. Therefore, $T(X)$ is potentially modular, that is, $L(T(X)\otimes\mathbf{Q}_\ell, s)$ is determined over K by a modular form g associated to E. What is a necessary and sufficient condition for the $\mathrm{Sym}^2(\rho_E)$ to be defined over $\mathbf{Q}$?

• **Problem 6**: This is about rigid Calabi–Yau threefolds over $\mathbf{Q}$. These Calabi–Yau threefolds are all modular. Hence they correspond to weight 4 modular forms for some congruence subgroups of $\mathrm{PSL}_2(\mathbf{Z})$. It happens in examples that two rigid Calabi–Yau threefolds with the same set of Hodge numbers correspond to the same weight 4 modular form, so that their L-series coincide. The Tate Conjecture asserts that there should be an algebraic correspondence between the two rigid Calabi–Yau threefolds defined over $\mathbf{Q}$ which induces the identification of the L-series. Construct such an algebraic correspondence.

• **Problem 7**: This is about Calabi–Yau threefolds of CM type. Construct Calabi–Yau threefolds over $\mathbf{Q}$ of CM type and classify them. Start with rigid Calabi–Yau threefolds over $\mathbf{Q}$ of CM type. There are toric constructions of Calabi–Yau threefolds, about 600 million, though rigid Calabi–Yau threefolds arise very rarely via this contruction. It is not clear from these combinatorial data which Calabi–Yau threefolds are of CM type.

• **Problem 8**: Let X be a Calabi–Yau threefold. The intermediate Jacobian $J(X)$ of X is a complex torus of dimension $B_3(X)/2$ where $B_3(X)$ is the third Betti number of X. By definition, $J(X)$ is Hodge theoretic, and hence of transcendental nature. One would like to detect some algebro-geometric, or arithmetic properties of $J(X)$ from some specific transcendental features of Hodge structures of X.

For a rigid Calabi–Yau threefold X, $J(X)$ is a simply a complex torus of dimension 1. If X has a model defined over $\mathbf{Q}$, it is expected that $J(X)$ would have a model defined over $\mathbf{Q}$. Are there any relations between the L-series of X and $J(X)$? Molnar has some results along this line.

For non-rigid Calabi–Yau threefolds, one can ask the same question. Start with the case when $J(X)$ is a complex torus of dimension 2. Or consider special types of Calabi–Yau threefolds, e.g., K3-fibered or elliptic fibered Calabi–Yau threefolds.

• **Problem 9**: Now we consider families of Calabi–Yau varieties. The modularity of the mirror map ought to be investigated. To construct mirror maps, one need to look at the Picard–Fuchs differential equations of Calabi–Yau families. This problem may be tractable if the Picard–Fuchs differential equations are of hypergeometric type, or GKZ hypergeometric systems. Then one needs to determine the monodromy groups for the Picard–Fuchs differential equations. This is a very difficult problem. The only result known so far is that the Zariski closure of the monodromy groups of the Dwork families in $\mathbf{P}^n$ is $\mathrm{Sp}_n(\mathbf{Z})$.

For the dimension 2 case, consider one-or-two parameter families of K3 surfaces. Compute mirror maps, and discuss their modularity. Start with K3 surfaces with Picard numbers $19, 18$ or 17 equipped with lattice polarizations.

For dimension 3, start with one-parameter families of Calabi–Yau threefolds. For the 14 one-parameter Calabi–Yau threefold families of hypergeometric type, the Picard–Fuchs differential equations have been computed, and the monodromy groups have shown to be arithmetic for 7 families, but for the remaining 7 families, the monodromy groups have been shown to be thin. One of the implications of the monodromy group being thin is that the mirror map of the family is most unlikely to have a modular property. Thin monodromy groups might be non-congruence subgroups of $\mathrm{Sp}_4(\mathbf{Z})$ and mirror maps have unbounded denominators.

For one-parameter families of Calabi–Yau threefolds whose Picard–Fuchs differential equations are of non-hypergeometric type (which are considered by Yang and myself), the first thing is to look into their monodromy groups (arithmetic or thin). We know that the Zariski closures of the monodromy groups are $\mathrm{Sp}_4(\mathbf{Z})$. Then compute their mirror maps by the Frobenius method. Discuss the modularity (or non-modularity) of mirror maps.

• **Problem 10**: This is the so-called a *geometric modularity problem*. This is about the quasi-modularity of the generating functions of the Gromov–Witten invariants and some related invariants (e.g., the Gopakummar–Vafa invariants, the Donaldson–Thomas invariants) for Calabi–Yau manifolds. (However, the Gromov–Witten invariants are trivial on K3 surfaces.)

There are many examples on calculation of Gromov–Witten invariants and related invariants and their generating functions. For instance, consider an elliptic Calabi–Yau threefold defined over a del Pezzo surface of degree 8. The generating function of Gromov–Witten invariants and Gopakummer–Vafa invariants can be expressed in terms of Eisenstein series and quasi-modular forms.

Again we do not know a conceptual reason why quasi-modular forms appear in this landscape. Perhaps, we need string theoretic reason(s), e.g., properties like, the S-duality or elliptic genera, etc., satisfied by these generating functions?

• **Problem 11**: Mirror symmetry (in String theory). One formulation of "Mirror symmetry (conjecture)" from String theory is: *There exists a pair of Calabi–Yau threefolds* $(X, X^\vee)$ *and an isomorphism* $H^{1,1}(X) \to H^{2,1}(X^\vee)$ *such that this isomorphism exchanges the prepotential of the A-model Yukawa coupling defined on the complexified Kähler cone*

$\kappa_X \subset H^{1,1}(X)$ with the prepotential of the B-model Yukawa coupling defined on the complex structure moduli space whose tangent space is $H^{2,1}(X^\vee)$.

For dimension 1, the generating function of simply ramified covers of an elliptic curve with $2g-2$ marked points has been shown to be a quasi-modular form of weight $6g-6$ on $\mathrm{PSL}_2(\mathbf{Z})$. This is the A-model (Fermion) calculation. On the B-model (Bosonic) calculation, one ought to compute Feynman integrals on graphs with $2g-2$ vertices and $3g-3$ edges. Conjecturally, the A-model and the B-model calculations should give the same result. A conceptual reason why quasi-modular forms enter the scene may be deduced from the B-model calculation.

For dimension 2, i.e, K3 surfaces, there is the beautiful formula due to Yau and Zaslow that expresses the generating function of the numbers of nodal curves in terms of the Dedekind eta-function. Let $n(g)$ (with convention $n(0)=0$) be the number of rational (highly singular) curves on a K3 surface X that represent a homology class $A\in H_2(X,\mathbf{Z})$ with $A^2=2g-2$. The Yau–Zaslow formula is $\sum_{g\geq 0} n(g)q^g = q/\Delta(q)$ where $\Delta(q)$ is the weight 12 cups form for $\mathrm{PSL}_2(\mathbf{Z})$. There are many generalizations of the Yau–Zaslow formula, all of which relate generating functions to quasi-modular forms. As far as I am aware, there seems no calculations done on the B-model size.

For dimension 3, there are a couple of examples in support of the mirror symmetry conjecture. For instance, let X be the quintic Calabi–Yau threefold, and let $X^\vee$ be it its mirror, which is an orbifold of a one-parameter deformation of X quotiented out by a finite discrete group of symmetries. Then the A-model Yukawa coupling is $\Phi_{A,X} = 5+\sum_{d=1}^\infty n_d d^3 \frac{q^d}{1-q^d}$ where n_d denotes the number of rational curves of degree d in X. The B-model Yukawa coupling is $\Phi_{B,X^\vee} = 5+2,875q+609,250q^2+317,206,375q^3+O(q^4)$. The mirror symmetry asserts that $\Phi_{A,X}=\Phi_{B,X^\vee}$ for all d.

Establish the mirror symmetry conjecture for many mirror pairs of Calabi–Yau threefolds. For this problem, you need a good physics background.

• **Problem 12**: Arithmetic mirror symmetry. One can consider mirror symmetry from an arithmetic point of view. There are several variants of mirror symmetry, e.g., Berglund–Hübsch mirror symmetry, Landau–Ginzburg mirror symmetry, toric mirror symmetry, and many more.

For K3 surfaces also, there are several variants of mirror symmetry, e.g., Arnold's strange duality, toric duality, etc.

Assume that a Calabi–Yau variety X has a model, e.g., a hypersurface, or a complete intersection, defined over $\mathbf{Q}$. Its mirror partner $X^\vee$ arises in a family. Again assume that $X^\vee$ has a model, e.g., a hypersurface defining equation or a complete intersection defined over $\mathbf{Q}$ with one deformation parameter.

Count the number of rational points modulo a good prime p and construct the congruence zeta-function of X. There are several methods available for point counting. On the mirror side, do the same at some special point assigning a value to the deformation parameter.

Detect the mirror symmetry phenomenon at the level of zeta-functions and L-series.

Also compute the formal group of X and that of a mirror partner. Compute the unit root zeta-functions. Is the height bounded above by some geometric invariant? Is there any relation between heights of mirror pairs?

• **Problem 13**: For a rigid Calabi–Yau threefold, its mirror partner is not a Calabi–Yau threefold. It is expected that mirrors of rigid Calabi–Yau threefolds are Fano varieties. Construct Fano varieties as mirrors of rigid Calabi–Yau threefolds.

When a rigid Calabi–Yau threefold and a Fano variety have models defined over $\mathbf{Q}$, compute their zeta-functions and L-series, and establish arithmetic mirror symmetry.

• **Problem 14**: Once one can compute the zeta-functions and L-series of Calabi–Yau manifolds defined over $\mathbf{Q}$, discuss their special values at some special points. This problem leads to the Birch–Swinnerton-Dyer Conjecture for elliptic curves, the Bloch–Beilinson Conjecture for K3 surfaces. For Calabi–Yau threefolds, we have yet to even formulate a conjectural formula for special values. Here is a conjecture.

Conecture: *Let X be a Calabi–Yau threefold defined over a number field K. Then*

(a) Define, for a prime ℓ,

$$CH^2(X_K)_{\mathrm{hom}} := Ker[CH^2(X_K) \to H^4(X_K, \mathbf{Z}_\ell(2))].$$

Then $CH^2(X_K)_{\mathrm{hom}}$ is a finitely generated abelian group,

(b) the rank of $CH^2(X_K)_{\mathrm{hom}}$ is equal to the order of $L(X_K, s)$ at $s = 2$.

(c) Suppose that X is a rigid Calabi–Yau threefold over $\mathbf{Q}$. Suppose that the intermediate Jacobian $J^2(X)$ has a model defined over $\mathbf{Q}$. Assume that the Abel–Jacobi map $CH^2(X) \otimes \mathbf{Q} \to J^2(X) \otimes \mathbf{Q}$ is injective. Then

$$\mathrm{rank}_{\mathbf{Z}}\, CH^2(X)_{\mathrm{hom}} \leq \mathrm{rank}_{\mathbf{Z}} J^2(X)(\mathbf{Q})$$

or equivalently,

$$\mathrm{ord}_{s=2} L(X, s) \leq \mathrm{ord}_{s=1} L(J^2(X), s).$$

The equality of the right hand side on $J^2(X)$ and $\mathrm{ord}_{s=1} J^2(X)(\mathbf{Q})$ is the conjecture of Birch and Swinnerton-Dyer. The equality on the left hand side on the rank of $CH^2(X)_{\mathrm{hom}}$ and $\mathrm{ord}_{s=2} L(X, s)$ is the conjecture of Beilinson–Bloch.

References

If you are interested in getting into the subject of Calabi–Yau varieties from all fronts, I recommend the following lecture notes/articles.

[1] *Introduction to Calabi–Yau Varieties: Arithmetic, Geometry and Physics: Lecture Notes on Concentrated Graduate Courses*, Fields Institute Monograph, Vol.??, 2015. Edited by R. Laza, M. Schütt and N. Yui. Springer. (To appear in the spring of 2015.)

This is the must-read book if you are interested in getting to the subject of Calabi–Yau Varieties.

[2] *Arithmetic and Geometry of K3 surfaces and Calabi–Yau threefolds*, Fields Institute Communications Vo1. **67**, 2013. Edited by R. Laza, M. Schütt and N. Yui. Springer. 2013.

[3] *Arithmetic of certain Calabi–Yau varieties and mirror symmetry*, by N. Yui, in "Arithmetic algebraic geometry (Park City UT, 1999), 5-7–569. IAS/Park City Math. Ser. **9**, 2001. AMS.

[4] *Update on the modularity of Calabi–Yau varieties. With an appendix by Helena Verrill*, by N. Yui, Fields Institute Commun. **38**, in "Calabi–Yau varieties and mirror symmetry" (Toronto, ON 2001), 307–362. 2003. AMS.

[5] *Modularity of Calabi–Yau varieties*, by K. Hulek, R. Kloosterman and M. Schütt, in "Global aspects of complex geometry", 271–309. Springer Berlin, 2006.

For physics (String theory) background, I recommend

[6] *Mirror Symmetry*, edited by V. Vafa and E. Zaslow, Clay Math. Monograph Vol.**1**, AMS/CMI, 2003.