Automorphy of non-rigid Calabi–Yau threefolds

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Calabi–Yau Varieties

Definition: A smooth projective variety $X/\mathbb{C}$ of dimension d is said to be *Calabi–Yau* if

(1) $H^i(X, \mathcal{O}_X) = 0$ for every i, $0 < i < d$, and

(2) The canonical bundle $\mathcal{K}_X$ is trivial.

Now introduce Hodge numbers:

$$h^{i,j}(X) := \dim_{\mathbb{C}} H^j(X, \Omega^i_X) \quad \text{for } 0 \le i, j \le d$$

Then

$$h^{i,j}(X) = h^{j,i}(X) \quad \text{by complex conjugation}$$

and

$$h^{i,j}(X) = h^{d-i,d-j}(X) \quad \text{by the Serre duality.}$$

Remark In terms of Hodge numbers, $X/\mathbb{C}$ is Calabi–Yau if

(1) $h^{i,0}(X) = 0 \quad$ for every $i, 0 < i < d$, and

(2) $h^{d,0}(X) = h^{0,d}(X) = \dim_{\mathbb{C}} H^0(X, \Omega_X^d) = \dim_{\mathbb{C}} H^0(X, \mathcal{K}_X) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X) = 1.$

The number $h^{0,d}(X)$ is the *geometric genus* $p_g(X)$ of X.

Numerical characters

- The Betti numbers $B_k(X) := \dim_{\mathbb{C}} H^k(X, \mathbb{C})$.

$$B_k(X) = \sum_{i+j=k} h^{i,j}(X).$$

- The Euler characteristic $E(X) := \sum_{k=0}^{2d} (-1)^k B_k(X)$.

Hodge diamonds

The Hodge numbers of Calabi–Yau varieties are concocted to form the Hodge diamond.

$d = 1$: Elliptic curves

$$h^{1,0} = h^{0,1} = 1$$

$$\begin{array}{ccccc} & 1 & & \qquad & B_0 = 1 \\ 1 & & 1 & \qquad & B_1 = 2 \\ & 1 & & \qquad & B_2 = 1 \\ & & & \qquad & E = 0 \end{array}$$

Dimension one Calabi–Yau varieties are elliptic curves. Elliptic curves over $\mathbb{Q}$ are defined by $y^2 = x^3 + ax + b$ with $4a^3 + 27b^2 \neq 0$.

$d = 2$: **K3 surfaces**

$$h^{1,0} = h^{0,1} = 0, \quad h^{2,0} = h^{0,2} = 1$$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{c} B_0 = 1 \\ B_1 = 0 \\ B_2 = 22 \\ B_3 = 0 \\ B_4 = 1 \\ E = 24 \end{array}$$

Examples: (1) Any quartic surface in $\mathbb{P}^3$. A typical example is the Fermat quartic:

$$X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0 \subset \mathbb{P}^3,$$

or its one-parameter deformation:

$$X_0^4 + X_1^4 + X_2^4 + X_3^4 - 4\lambda X_0 X_1 X_2 X_3 = 0 \subset \mathbb{P}^3 \times \mathbb{P}^1.$$

(2) Double sextic surface, e.g., $w^2 = f_6(x, y, z)$.

(3) Elliptic K3 surfaces, e.g., $Y^2 Z = X^3 + A(t) X Z^2 + B(t) Z^3$ with $4A^3(t) + 27B^2(t) \neq 0$.

(4) Complete intersections.

(5) Toric constructions, reflexive polytopes.

$d = 3$: Calabi–Yau threefolds

$$h^{1,0} = h^{0,1} = 0, \quad h^{2,0} = h^{0,2} = 0, \quad h^{3,0} = h^{0,3} = 1, h^{1,1} > 0$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$$B_0 = 1$$
$$B_1 = 0$$
$$B_2 = h^{1,1}$$
$$B_3 = 2(1 + h^{2,1})$$
$$B_4 = h^{2,2}$$
$$B_5 = 0$$
$$B_6 = 1$$
$$E = 2(h^{1,1} - h^{2,1})$$

Examples: (1) Quintic threefolds in $\mathbb{P}^4$. A typical example is is the Fermat quintic:

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 = 0 \subset \mathbb{P}^4,$$

or its one-parameter deformation:

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 - 5\lambda X_0 X_1 X_2 X_3 X_4 = 0 \subset \mathbb{P}^4 \times \mathbb{P}^1.$$

More generally, hypersurface Calabi–Yau threefolds in projective/weighted projective spaces.

(2) Double octics, fiber products, Calabi–Yau threefolds of product type.

(3) Complete intersection threefolds.

(4) Toric Calabi–Yau threefolds ($\sim$ 600 million).

The largest possible Hodge numbers, or equivalently the Euler characteristic of a Calabi–Yau threefold is not known, but some known examples ($\sim$ 600 million) have $h^{1,1}$(or $h^{2,1}$)$\sim$ 500.

An implication in string theory is that string theory may have as many as 10^{500} vacua that can be described with various choices of branes and fluxes on homology cycles of a CY. Thus string theory has a vast number of different vacua, and only one should describe dynamics of our real world.

Arithmetic Modularity: Langlands Philosophy

(Motivic) L-functions of algebraic varieties over $\mathbb{Q}$ (or a number field) are automorphic L-functions.

Modularity Results in the last two decades

- **Dim 1**: Every elliptic curve E over $\mathbb{Q}$ is modular. There is a modular form f of weight 2 on some $\Gamma_0(N)$ such that $L(E, s) = L(f, s)$.
- **Dim 2**: Every singular K3 surface S over $\mathbb{Q}$ is modular. There is a modular form f of weight 3 on some $\Gamma_0(N) + \chi$ or $\Gamma_1(N)$ such that $L(T(S) \otimes \mathbb{Q}_\ell, s) = L(f, s)$.
- **Dim 3**: Every rigid Calabi–Yau threefold X over $\mathbb{Q}$ is modular. There is a modular form f of weight 4 on some $\Gamma_0(N)$ such that $L(X, s) = L(f, s)$.

Remarks:

(a) The modularity is established for the above varieties over $\mathbb{Q}$. However, we do not know conceptual reasons "why" they are modular. What would be physics implications of modularity?

(b) The above results are obtained by studying 2-dimensional Galois representations associated to Calabi–Yau varieties over $\mathbb{Q}$.

Here 2 coincides with the d-th Betti number of the Calabi–Yau variety of dimension d for $d = 1$ and 3; while 2 is the rank of the transcendental lattice $T(S)$ for $d = 2$.

(c) The modularity of the above varieties over number fields are still open problems.

Automorphy/Modularity of higher dimensional Galois representations

Higher dimensional Galois representations will occur

(a) the rank of $T(S) \geq 3$, for $d = 2$, or

(b) $h^{2,1}(X) \geq 1$ (so that $B_3(X) = 2(1 + h^{2,1}(X)) \geq 4$) for $d = 3$.

The automorphy/modularity question is currently out of reach in the general setting.

We need to have more structures, e.g., for $d = 2$, lattice polarization on $T(S)$, or a Shioda–Inose structure, and for $d = 3$, we require that $H^3(X, \mathbb{Q}_\ell)$ has the splitting property and decomposes into motives of small ranks, or S and X are of CM type.

We will consider some examples. In the case (a), we will look into K3 surfaces of CM type. In the case (b), we will consider Calabi–Yau threefolds X with small $h^{2,1}(X)$, e.g., $h^{2,1}(X) = 4$ or 6.

The simplest case for which the modularity of a Calabi–Yau threefold X can be established is when

$$H^3_{et}(X, \mathbb{Q}_\ell) = U \oplus V$$

where U is 2-dimensional $\mathbb{Q}_\ell$-vector space and V is of the form $\oplus_{i=1}^{h^{2,1}} H^3_{et}(S_i, \mathbb{Q}_\ell)$ where S_i are the ruled surfaces over elliptic curves.

In this case, the modularity of the motives U and V can be established using the 2-dimensional results for $d = 3$ and $d = 1$, respectively.

Theorem (Hulek–Verrill) *Let X be a non-rigid Calabi–Yau threefold over $\mathbb{Q}$. Suppose that X contains birational elliptic ruled surfaces S_i, $i = 1, \cdots, h^{2,1}$ which are defined over $\mathbb{Q}$ and which span $H^{2,1}(X) \oplus H^{1,2}(X)$. Suppose that there is an exact sequence*

$$0 \to U \to H^3_{et}(X, \mathbb{Q}_\ell) \to \oplus_{i=1}^{h^{2,1}} H^3_{et}(S_i, \mathbb{Q}_\ell) \to 0,$$

where U is a $\mathbb{Q}_\ell$-vector space of dimension 2 corresponding to $H^{3,0}(X) \oplus H^{0,3}(X)$.

Then X is modular, and the L-series of X is given by

$$L(X, s) = L(f_4, s) \prod_{i=1}^{h^{2,1}} L(g_2^i, s-1)$$

where f_4 is a weight 4 modular form, the g_2^i are the weight 2 forms associated to the base elliptic curves E_i.

Now we want to construct Calabi–Yau threefolds over $\mathbb{Q}$ for which we can apply the above theorem of Hulek and Verrill.

Examples

(a) Horrocks–Mumford quintics (Schoen, Eddy Lee)

(b) Mirrors of determinantal quintics, or the Reye congruences (Hosono and Takagi)

Remark: Horrocks–Mumford quintics are realized as complete intersections, and are also determinantal quntics. Examples of Hosono and Takagi are also complete intersections.

(c) Orbifolds of rigid Calabi–Yau threefolds (Eddy Lee, Yui, Molnar)

(d) Quadratic twists of rigid Calabi–Yau threefolds (Gouvêa, Kiming and Yui)

(e) Orbifolds of triple products of elliptic curves (Molnar)

(f) Attractive Calabi–Yau threefolds (Livné–Yui)

(g) Calabi–Yau threefolds of Borcea–Voisin type (Goto–Livné–Yui)

(h) Masche's Calabi–Yau threefolds (Schuett, Bini–van Geemen)

Orbifolds of rigid Calabi–Yau threefolds

Let X be a rigid Calabi–Yau threefold which is a resolution of the Schoen quintic

$$Q : x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5x_1x_2x_3x_4x_5 = 0 \subset \mathbb{P}^4$$

Then X is rigid, i.e., $h^{2,1}(X) = 0$ and so $B_3(X) = 2$. The modularity of X was established by Schoen that there is a modular form $f = f_{4,25}$ (expresses in terms of sums of η-products) of weight 4 and level 25 such that

$$L(X, s) = L(f_{4,25}, s).$$

Let $\iota : [x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_1 : x_5 : x_4 : x_3 : x_2]$ is an involution on $\mathbb{P}^4$. Its fixed loci are

$$\mathbb{P}^1_- := \{x_1 = x_2 + x_5 = x_3 + x_4 = 0\}$$

and

$$\mathbb{P}^2_+ := \{x_2 - x_5 = x_3 - x_4 = 0\}.$$

The fixed loci on the Schoen quintic Q consists of the line $\mathbb{P}^1_-$ and the curve

$$E := Q \cap \mathbb{P}^2_+ = \{[x_1 : x_2 : x_3] \,|\, x_1^5 + 2x_2^5 + 2x_3^5 - 5x_1 x_2^2 x_3^2 = 0\}$$

which is of arithmetic genus 6 with 5 nodes, so its normalization $\hat{E}$ is an elliptic curve. The quotient Q/ι still has 60 nodes away from $\mathbb{P}^1_-$ and E. Let Y be a small resolution of Q/ι. Then Hodge numbers are computed to be

$$h^{1,1}(Y) = 85,\ h^{2,1}(Y) = 1$$

so that $B_3(Y) = 4$.

Furthermore, $H^3_{et}(Y, \mathbb{Q}_\ell)$ decomposes as

$$H^3_{et}(Y, \mathbb{Q}_\ell) = H^3_{et}(X, \mathbb{Q}_\ell) \oplus H^1_{et}(\hat{E}, \mathbb{Q}_\ell)(-1)$$

where $H^3_{et}(X, \mathbb{Q}_\ell)$ is the rank 2 motive corresponding to the unique holomorphic 3-form Ω on Q:

$$\Omega = \mathrm{Res} \frac{\sum_j (-1)^j x_j dx_1 \wedge \cdots\cdots \wedge \widehat{dx_j} \cdots \wedge dx_5}{x_1^5 + x_2^5 + \cdots + x_5^5 - 5x_1 x_2 \cdots x_5}$$

which is invariant under ι.

Theorem (E. Lee) *Y is modular, and the L-series of Y is given by*

$$L(Y, s) = L(X, s)L(\hat{E} \times \mathbb{P}^1, s)$$

$$= L(f_{4,5}, s)L(g_{2,50}, s-1).$$

where $f_{4,5}$ is a weight 4 form of level 5 and $g_{2,50}$ is a weight 2 form of level 50.

Attractive Calabi–Yau threefolds

Greg Moore defined an attractive Calabi–Yau threefold X. $H^{3,0}(X) \oplus H^{0,3}(X)$ should be algebraic. Examples of such Calabi–Yau threefolds are orbifolds of triple products of isogenous elliptic curves with CM, or $S \times E$ where S is a singular K3 surface and E is an elliptic curve with CM.

According to Shafarevich, for every integer $n \geq 1$, there are infinitely many singular K3 surfaces defined over number fields K with $[K : \mathbb{Q}] = n$. Schuett has classified singular K3 surfaces defined over $\mathbb{Q}$ that there are (possibly) 101 classes.

Orbifolds of products of these singular K3 surfaces with elliptic curves with CM, both defined over $\mathbb{Q}$ are then attractive Calabi–Yau threefolds.

Theorem: *The attractive Calabi–Yau threefolds over $\mathbb{Q}$ are all motivically modular.*

Example(Livné–Yui): Let Y be a semi-stable elliptic K3 surface with exactly six singular fibers of type I_n, which corresponds to a torsion-free genus zero congruence subgroup of $PSL_2(\mathbb{Z})$ of index 24. Then Y is an elliptic modular singular K3 surface defined over $\mathbb{Q}$, and it corresponds to a weight 3 modular (cusp) form g_Y on $\Gamma_1(M_Y)$ (where M_Y is the level of g_Y). Then the rank 4 submotive of $H^3(Y \times E)$ is associated the tensor product of a cusp form g_Y of weight 3 and a cusp form g_E of weight 2 and

$$L(T(Y \times E) \otimes \mathbb{Q}_\ell, s) = L(g_E \otimes g_Y, s).$$

Calabi–Yau threefolds of Borcea–Voisin type

• Let (E, ι) is an elliptic curve with a non-symplectic involution ι such that the induced map

$$\iota^* : H^{1,0}(E) \to H^{1,0}(E), \quad \iota^*(\omega_E) = -\omega_E$$

and that $E/ < \iota > \simeq \mathbb{P}^1$. Here ω_E is a unique holomorphic 1-form on E.

• Let (S, σ) is a K3 surface with a non-symplectic involution such that the induced map

$$\sigma^* : H^{2,0}(S) \to H^{2,0}(S), \quad \sigma^*(\omega_S) = -\omega_S.$$

Decompose $H^2(S, \mathbb{C})$ into the (+)- and (−)-eigenspaces under the action of $\sigma^* : H^2(S, \mathbb{C}) \to H^2(S, \mathbb{C})$:

$$H^2(S, \mathbb{C}) = H^2(S, \mathbb{C})^+ \oplus H^2(S, \mathbb{C})^-.$$

Set

$$H^2(S,\mathbb{Z})^+ := H^2(S,\mathbb{C})^+ \cap H^2(S,\mathbb{Z})$$

and

$$H^2(S,\mathbb{Z})^- := H^2(S,\mathbb{C})^- \cap H\ 2(S,\mathbb{Z}).$$

Let

$$r := \mathrm{rank}_{\mathbb{Z}} H^2(S,\mathbb{Z})^+.$$

Then $H^2(S,\mathbb{Z})^+$ and $H^2(S,\mathbb{Z})^-$ have signatures $(1, r-1)$ and $(2, 20-r)$ respectively.

Nikulin has classified such pairs (S,σ) of K3 surfaces S with non-symplectic involutions σ, up to deformation.

Theorem (Nikulin, 1979): *There are altogether 75 deformation classes of pairs* (S, σ) *of K3 surfaces* S *with non-symplectic involutions* σ*, and they are completely determined by the triple integers*

$$(r, a, \delta)$$

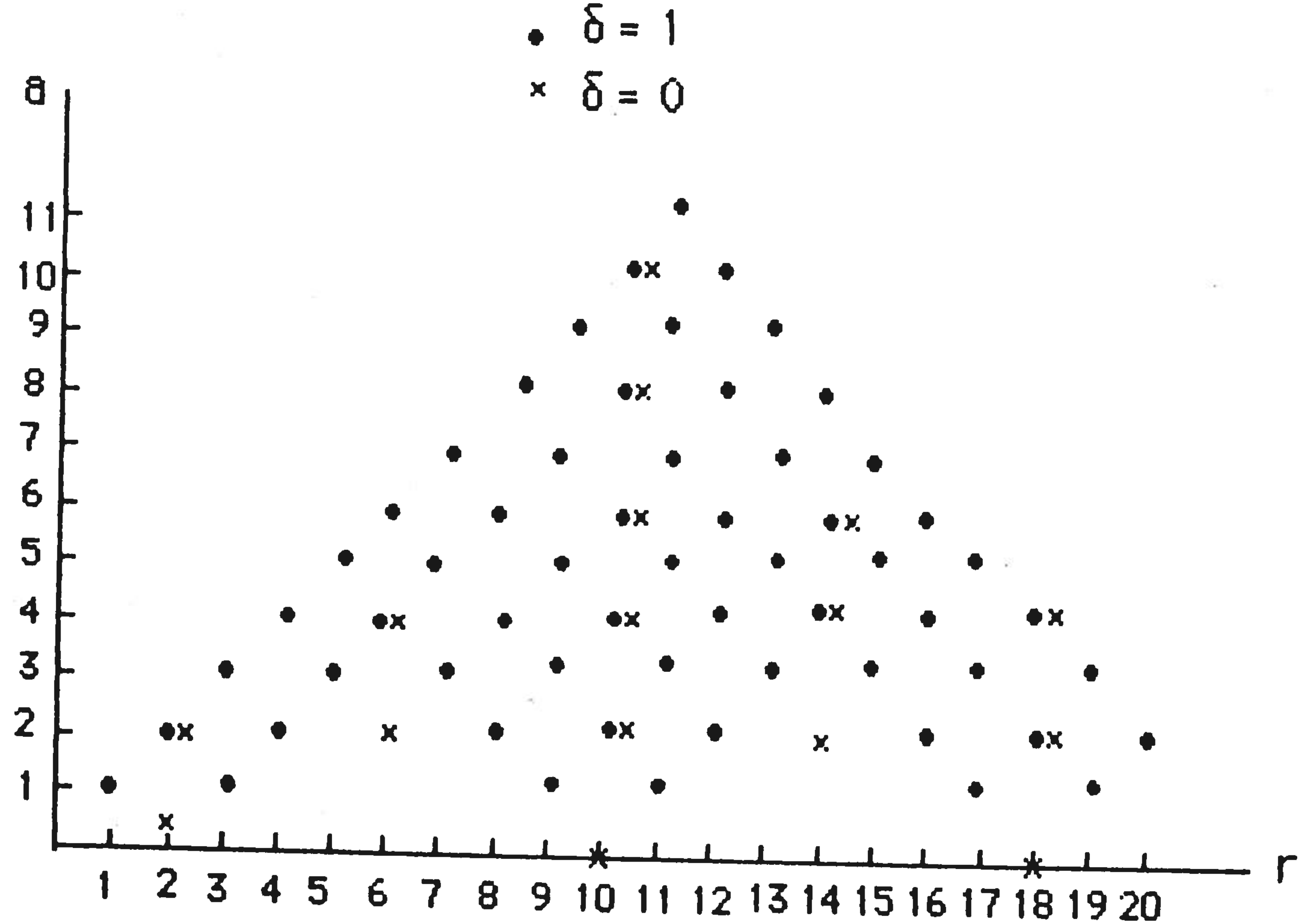
where r *is as above,* a *is the integer determined by*

$$(H^2(S, \mathbb{Z})^+)^\vee / H^2(S, \mathbb{Z})^+ \simeq (\mathbb{Z}/2\mathbb{Z})^a.$$

The intersection pairing on $H^2(S, \mathbb{Z})^+$ *gives rise to a quadratic form* q *with values in* $\mathbb{Q}$*. We define* $\delta = 0$ *if* q *has integer values, and 1 otherwise.*

Nikulin's Pyramid

75 triples (r, a, δ)



Theorem (Nikulin, 1979) : *Let* (S, σ) *be a pair of K3 surface with non-symplectic involution* σ. *Let* S^σ *be the fixed locus of* S *under* σ. *Then*

(1) If $(r, a, \delta) \neq (10, 10, 0), (10, 8, 0)$, *then*

$$S^\sigma = C_g \cup L_1 \cup \ldots \cup L_k \quad \text{(disjoint union)}$$

where C_g *is a genus* $g(\geq 0)$ *curve, and* $L_i(i = 1, \cdots, k)$ *are rational curves.*

(2) If $(r, a, \delta) = (10, 10, 0)$, *then* $S^\sigma = \emptyset$.

(3) If $(r, a, \delta) = (10, 8, 0)$, *then* $S^\sigma = C_1 \cup \bar{C}_1$ *(disjoint union) where* C_1 *and* $\bar{C}_1$ *are elliptic curves.*

Put

$$N := \text{the number of components of } S^\sigma = 1 + k$$

and

$$N' := \text{the sum of genera of components of } S^\sigma = g.$$

Note that

$$g = 11 - \frac{1}{2}(r + a), \ N = 1 + k = 1 + \frac{1}{2}(r - a).$$

Now we will construct Calabi–Yau threefolds of Borcea–Voisin (BV) type. Let (E, ι) and (S, σ) be as above. Take the product $E \times S$. Then the product $\iota \times \sigma$ is an involution on $E \times S$ such that the induced map

$$(\iota \times \sigma)^* : H^{3,0}(E \times S) \rightarrow H^{3,0}(E \times S)$$

is the identity map. Write

$$E^{\iota} = \{P_1, P_2, P_3, P_4\}$$

and

$$S^{\sigma} = \{C_1, C_2, \cdots, C_N\}.$$

Then the fixed point of $\iota \times \sigma$ consists of

$$P_i \times C_j \quad (i = 1, \cdots, 4; j = 1, \cdots, N).$$

The involution $\iota \times \sigma$ lifts naturally to an involution on the blow-up of $E \times S$ along $4N$ curves. The quotient $E \times S/\iota \times \sigma$ and its crepant resolution $E \times \widetilde{S/\iota \times \sigma}$ is our Calabi–Yau threefold of Borcea–Voisin (BV) type, and will be denoted by

$$X = X(r, a, \delta).$$

Note that the exceptional divisors on X are 4 copies of ruled surfaces

$$S^\sigma \times \mathbb{P}^1 := (C_g \times \mathbb{P}^1) \cup (L_1 \times \mathbb{P}^1) \cup \cdots \cup (L_k \times \mathbb{P}^1).$$

Theorem (Borcea, Voisin, 1993, 94): *The Hodge numbers of X are given by*

$$h^{1,1}(X) = 11 + 5N - N'$$

$$h^{2,1}(X) = 11 + 5N' - N$$

and

$$E(X) = 12(N - N').$$

(Note that $B_3(X) = 2(12 + 5N' - N)$.)

Since any elliptic curve E defined over $\mathbb{Q}$ is modular, the modularity of our Calabi–Yau threefolds $X = X(r, a, \delta)$ depends on the modularity of K3 surface component S. We ought to choose appropriate K3 surfaces for S.

Theorem (Reid 1979, Yonemura 1990): *There are 95 admissible weights (w_0, w_1, w_2, w_3) of hypersurface simple K3 singularities defined by non-degenerate polynomials $F(X_0, X_1, X_2, X_3)$ in weighted projective 3-spaces $\mathbb{P}^3(w_0, w_1, w_2.w_3)$ over $\mathbb{Q}$.*

Examples: #1: weight $(1,1,1,1)$ $F = X_0^4 + X_1^4 + X_2^4 + X_3^4$.

#15: weight $(5,4,3,3)$ $F = X_0^3 + X_1^3X_2 + X_1^3X_3 + X_2^5 - X_3^5$.

#95: weight $(7,5,3,2)$

$F = X_0^2X_2 + X_0X_1^2 + X_0X_3^5 + X_1^3X_3 + X_1X_2^4 + X_1X_3^6 + X_2^5X_3 + X_2X_3^7$.

Among these 95 K3 surfaces, we need to find those with the required involutions.

Theorem (Goto–Livné–Yui, 2014) :*Among the* 95 *K3 surfaces,* 92 *have the required non-symplectic involution* σ*. These* 92 *pairs* (S, σ) *realize at least* 40 *triplets* (r, a, δ) *of Nikulin.*

Furthermore, the 86 *out of* 92 *of them are realized as Delsarte surfaces. Consequently these* 86 *pairs* (S, σ) *are of CM type (that is, they are realized as finite Fermat quotients).*

Theorem (Goto–Livné–Yui): *Let (S, σ) be one of the 86 surfaces represented by a Delsarte surface. Let (E, ι) be an elliptic curve over $\mathbb{Q}$. Let X be a Calabi–Yau threefold of Borcea–Voisin (BV) type. Then X has a model defined over $\mathbb{Q}$, and X is automorphic.*

More precisely,

(a) (S, σ) is automorphic, that is,

$$L(S, s) = L(\rho_S, s-1)L(\chi_S, s)$$

is automorphic, where ρ_S and χ_S are Galois representations corresponding to $NS(S)$ and $T(S)$, respectively.

(b) X is automoroic, that is,

$$L(X, s) = L(\rho_E \otimes \rho_S, s)L(\rho_E \otimes \chi_S, s)L(J(C_g), s-1)^4$$

is automorphic, where ρ_E is the Galois representation corresponding to E, $J(C_g)$ is the Jacobian variety of C_g (here C_g is again of CM type).