Generalized Legendre Curves and Quaternionic Multiplication

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joint with
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a Women in Numbers 3 project

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Mini-workshop on Algebraic Varieties, Hypergeometric series, and Modular Forms

${}_2F_1$-hypergeometric Function

Let $a, b, c \in \mathbb{R}$. The hypergeometric function ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right]$ is defined by

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, $$

where $(a)_n = a(a+1)\dots(a+n-1)$ is the Pochhammer symbol.

Facts. Assume $a, b, c \in \mathbb{Q}$.

- ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right]$ satisfies a hypergeometric differential equation, whose monodromy group is a triangle group.
- ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right]$ can be viewed as a quotient of periods on some abelian varieties defined over $\overline{\mathbb{Q}}$.

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Hypergeometriic Differential Equation

$_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}; z\right]$ satisfies the differential equation $HDE(a, b, c; z)$:

$$z(1-z)F'' + [(a+b+1)z - c]F' + abF = 0.$$

Theorem (Schwarz)

Let f, g be two independent solutions to $HDE(a, b; c; \lambda)$ at a point $z \in \mathfrak{H}$, and let $p = |1-c|$, $q = |c-a-b|$, and $r = |a-b|$. Then the Schwarz map $D = f/g$ gives a bijection from $\mathfrak{H} \cup \mathbb{R}$ onto a curvilinear triangle with vertices $D(0), D(1), D(\infty)$, and corresponding angles $p\pi, q\pi, r\pi$.

When p, q, r are rational numbers in the lowest form with $0 = \frac{1}{\infty}$, let e_i be the denominators of p, q, r arranged in the non-decreasing order, the monodromy group is isomorphic to the triangle group (e_1, e_2, e_3).

Arithmetic triangle groups

- A *triangle group* (e_1, e_2, e_3) with $2 \leq e_1, e_2, e_3 \leq \infty$ is

$$\langle x, y \mid x^{e_1} = y^{e_2} = (xy)^{e_3} = id \rangle.$$

- A triangle group Γ is called *arithmetic* if it has a unique embedding to $SL_2(\mathbb{R})$ with image commensurable with norm 1 group of an order of an indefinite quaternion algebra.
- Γ acts on the upper half plane. The fundamental half domain $\Gamma \backslash \mathfrak{h}$ gives a tessellation of $\mathfrak{h}$ by congruent triangles with internal angles $\pi/e_1, \pi/e_2, \pi/e_3$. $(1/e_1 + 1/e_2 + 1/e_3 < 1)$
- The quotient space is a modular curve when at least one of e_i is ∞; otherwise, it is a Shimura curve.
- Arithmetic triangle groups Γ have been classified by Takeuchi.

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Examples

- The triangle group corresponding to

$$ {}_2F_1\left[\begin{matrix}\frac{1}{12} & \frac{5}{12} \\ & 1\end{matrix};z\right], \quad {}_2F_1\left[\begin{matrix}\frac{7}{12} & \frac{11}{12} \\ & \frac{3}{2}\end{matrix};z\right] $$

is $(2,3,\infty) \simeq \mathrm{SL}(2,\mathbb{Z})$.

- The triangle group corresponding to

$$ {}_2F_1\left[\begin{matrix}\frac{1}{5} & \frac{2}{5} \\ & \frac{4}{5}\end{matrix};z\right], \quad {}_2F_1\left[\begin{matrix}\frac{1}{84} & \frac{43}{84} \\ & \frac{2}{3}\end{matrix};z\right] $$

is $(2,3,7)$.

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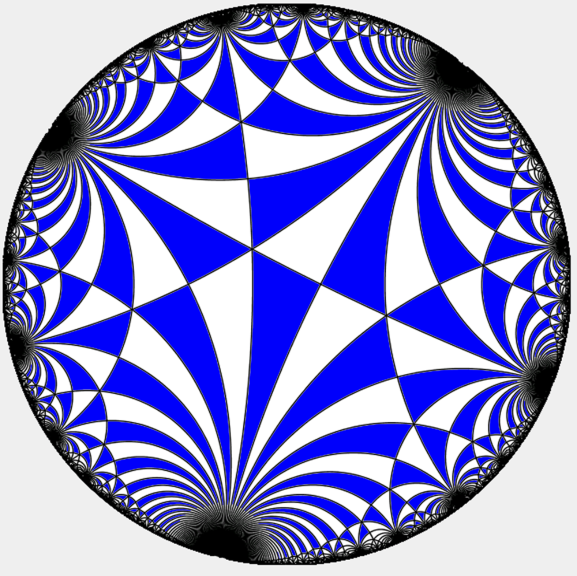
$(2,3,\infty)$-tessellation of the hyperbolic plane



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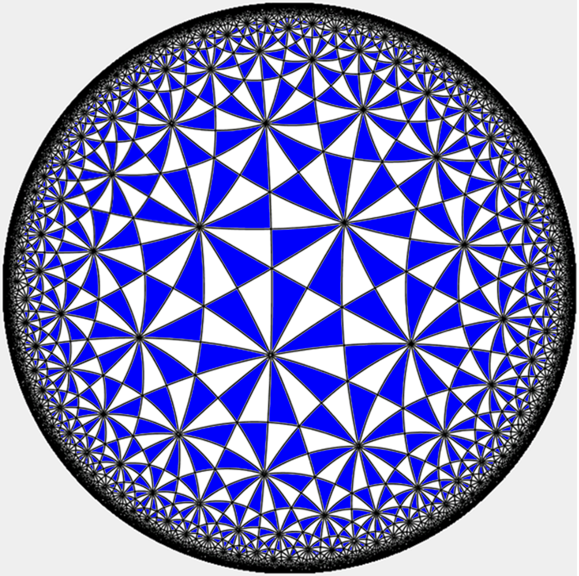

Legendre Family

For $\lambda \neq 0, 1$, let

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

be the elliptic curve in Legendre normal form.

- The periods of the Legendre family of elliptic curves are

$$\Omega(E_\lambda) = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

- If $0 < \lambda < 1$, then

$${}_2F_1\left[\begin{matrix}\frac{1}{2} & \frac{1}{2} \\ & 1\end{matrix}; \lambda\right] = \frac{\Omega(E_\lambda)}{\pi}.$$

The triangle group $\Gamma = (\infty, \infty, \infty) \simeq \Gamma(2)$.

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Generalized Legendre Curves

- Euler's integral representation of the ${}_2F_1$ with $c > b > 0$

$$\mathcal{P}(\lambda) = \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-\lambda x)^{-a}dx$$
$$= {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; \lambda\right] B(b, c-b),$$

where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is the so-called Beta function.

- Following Wolfart , $\mathcal{P}(\lambda)$ can be realized as a *period* of

$$C_\lambda^{[N;i,j,k]} : y^N = x^i(1-x)^j(1-\lambda x)^k,$$

where $N = lcd(a,b,c)$, $i = N(1-b)$, $j = N(1+b-c)$, $k = Na$.

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Let $N \geq 2$. For the curve

$$C_\lambda^{[N;i,j,k]} : \; y^N = x^i(1-x)^j(1-\lambda x)^k,$$

- a period can be chosen as

$$\mathcal{P}(\lambda) = B\left(1 - \frac{i}{N}, 1 - \frac{j}{N}\right) {}_2F_1\left[\begin{matrix} \frac{k}{N}, \frac{N-i}{N} \\ \frac{2N-i-j}{N} \end{matrix} ; \lambda\right],$$

- the corresponding Schwarz triangle is a triangle with angles

$$\left|\frac{N-i-j}{N}\right|\pi, \quad \left|\frac{N-k-j}{N}\right|\pi, \quad \left|\frac{N-i-k}{N}\right|\pi.$$

Example. For the curve $C_\lambda^{[6;4,3,1]} : \; y^6 = x^4(1-x)^3(1-\lambda x)$,

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Petkoff-Shiga's result

Fact. The triangle group $\Gamma = (3,6,6)$ can be realized as the norm 1 group of the maximal order $\mathcal{O}_6$ of the quaternion algebra B_6 over $\mathbb{Q}$ of discriminant 6.

Petkoff-Shiga. The Jacobians of these genus 3 Picard curves

$$C(\lambda):\ w^3 = (z^2 - 1/4)\left(z^2 - \lambda/4\right)$$

decompose into $E'(\lambda) \oplus A'(\lambda)$ where

- $E'(\lambda):\ w^3 = (z - 1/4)(z - \lambda/4)$ is a CM elliptic curve
- $A'(\lambda)$ is an abelian surface with QM by $\mathcal{O}_6$.

Definition. For a simple abelian surface A, we say that A is with quaternionic multiplication (QM) by an order $\mathcal{O}$ if $\mathrm{End}(A) \simeq \mathcal{O}$.

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$C_\lambda^{[6;4,3,1]}$ with $\Gamma = (3,6,6)$

Question. Can we construct abelian surfaces with QM by $\mathcal{O}_6$ from the family

$$C_\lambda^{[6;4,3,1]} : y^6 = x^4(1-x)^3(1-\lambda x)?$$

For $\lambda \neq 0, 1 \in \overline{\mathbb{Q}}$, the Jacobian variety of the smooth model $X_\lambda^{[6;4,3,1]}$ of $C_\lambda^{[6;4,3,1]}$ is decomposed as

$$\mathrm{Jac}(X_\lambda^{[6;4,3,1]}) = E(\lambda) \oplus A(\lambda),$$

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Motivation

Question.

- Can we construct abelian surfaces with QM from the generalized Legendre family $C^{[N;,i,j,k]}$.
- Can we construct abelian surface A from $C^{[N;,i,j,k]}$ with $\mathrm{End}_0(A)$ contains a quaternion algebra?

Assume $N \geq 2$, $1 \leq i, j, k < N$, $\lambda \neq 0, 1 \in \overline{\mathbb{Q}}$. Let $J_\lambda = J_\lambda^{[N;i,j,k]}$ be the Jacobian variety of the smooth model $X_\lambda^{[N;i,j,k]}$ of $C_\lambda^{[N;i,j,k]}$.

Facts.

- For each $n \mid N$, $J_\lambda^{[n;i,j,k]}$ is a natural quotient of $J_\lambda^{[N;i,j,k]}$.
- Let J_λ^{new} be the primitive part of J_λ so that its intersection with any abelian subvariety isomorphic to $J_\lambda^{[n;i,j,k]}$ for each $n \mid N$ is zero.

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Question:

- Given a hypergeometric differential equation, when does J_λ^{new} contain a subvariety A such that of $\mathrm{End}_0(A)$ contains a quaternion algebra?
- If the monodromy group of the hypergeometric differential equation is an arithmetic triangle group Γ, when does $\mathrm{End}_0(A)$ contains the corresponding quaternion algebra H_Γ?

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Assumption. Assume $N \geq 2$, $1 \leq i, j, k < N$, $\gcd(i, j, k, N) = 1$, $\lambda \neq 0, 1 \in \overline{\mathbb{Q}}$. Furthermore, suppose $N \nmid i + j + k$.

Theorem (Deines, Long, Fuselier, Swisher, T.)

Let $N = 3, 4, 6$. Then for each $\lambda \in \overline{\mathbb{Q}}$, the endomorphism algebra of J_λ^{new} contains a quaternion algebra H over $\mathbb{Q}$ if and only if

$$B\left(\frac{N-i}{N}, \frac{N-j}{N}\right) \Big/ B\left(\frac{k}{N}, \frac{2N-i-j-k}{N}\right) \in \overline{\mathbb{Q}},$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Beta function, and $\Gamma(\cdot)$ is the Gamma function.

Remark.

- $H = H_\Gamma$.
- Our methods apply more generally. For general N, $H = H_\Gamma$?

Assumption. Assume $N \geq 2$, $1 \leq i, j, k < N$, $\gcd(i, j, k, N) = 1$, $\lambda \neq 0, 1 \in \overline{\mathbb{Q}}$. Furthermore, suppose $N \nmid i + j + k$.

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Holomorphic Differential 1-forms on $X_\lambda^{[N;i,j,k]}$

Let $X_\lambda = X_\lambda^{[N;i,j,k]}$ be the smooth model of $C_\lambda^{[N;i,j,k]}$. A basis of $H^0(X_\lambda, \Omega^1)$ is given by

$$\omega = \frac{x^{b_0}(1-x)^{b_1}(1-\lambda x)^{b_2}dx}{y^n}, \quad 0 \le n \le N-1,\ b_i \in \mathbb{Z},$$

satisfying the following conditions

$$b_0 \ge \frac{ni+\gcd(N,i)}{N} - 1,$$
$$b_1 \ge \frac{nj+\gcd(N,j)}{N} - 1,$$
$$b_2 \ge \frac{nk+\gcd(N,k)}{N} - 1,$$

$$b_0 + b_1 + b_2 \le \frac{n(i+j+k)-\gcd(N,i+j+k)}{N} - 1.$$

Examples

- For $C_\lambda^{[3;1,2,1]}$ ($\Gamma = (3, \infty, \infty)$), a basis for the space of holomorphic 1-forms is
$$\frac{dx}{y}, \quad \frac{dx}{y^2}.$$
- For $C_\lambda^{[4;1,1,1]}$ ($\Gamma = (2, 2, 2)$), the space of holomorphic 1-forms are spanned by
$$\frac{dx}{y^2}, \frac{dx}{y^3}, \frac{xdx}{y^3},$$
and
$$\frac{(1-x)dx}{y^3}, \frac{(1-\lambda x)dx}{y^3}.$$

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Let $\zeta_N = e^{2\pi i/N}$. For each $0 \le n < N$, we let V_n denote the isotypical component of $H^0(X_\lambda, \Omega^1)$ associated to the character $\chi_n : \zeta_N \mapsto \zeta_N^n$. Then

$$H^0(X(\lambda), \Omega^1) = \bigoplus_{n=0}^{N-1} V_n.$$

If $\gcd(n, N) = 1$,

- $\dim V_n = \left\{\frac{ni}{N}\right\} + \left\{\frac{nj}{N}\right\} + \left\{\frac{nk}{N}\right\} - \left\{\frac{n(i+j+k)}{N}\right\}$, where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.
- $\dim V_n + \dim V_{N-n} = 2$.

The subspace

$$H^0(X_\lambda, \Omega^1)^{\text{new}} = \bigoplus_{\gcd(n,N)=1} V_n$$

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The abelian variety J_λ^{new}

Assume $N < i + j + k < 2N$. For $gcd(N, n) = 1$, we have

$$V_n = \mathbb{C}\langle dx/y^n \rangle.$$

Wolfart. The primitive Jacobian subvariety J_λ^{new} is isogenious to $\mathbb{C}^{\phi(N)}/\Lambda(\lambda)$, where $\Lambda(\lambda)$ can be identified with the $\mathbb{Z}$-module generated by the $2\phi(N)$ columns

$$\left(\sigma_n(\zeta_N^i) \int_0^1 \omega_n\right)_i, \quad \left(\sigma_n(\zeta_N^i) \int_{1/\lambda}^\infty \omega_n\right)_i, \quad (n, N) = 1,\ i = 0..\phi(N) - 1$$

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Remark. These periods are all of first kind. When $N = 3, 4, 6$, the abelian variety J_λ^{new} is 2-dimensional.

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$\phi(N) = 2$

All the periods are:

$$\int_0^1 \omega_1 = B\left(\frac{N-i}{N}, \frac{N-j}{N}\right) {}_2F_1\left[\begin{matrix} \frac{k}{N} & \frac{N-i}{N} \\ & \frac{2N-i-j}{N} \end{matrix}; \lambda\right],$$

$$\int_{\frac{1}{\lambda}}^\infty \omega_1 = (-1)^{\frac{k+j}{N}} \lambda^{\frac{i+j-N}{N}} B\left(\frac{i+j+k-N}{N}, \frac{N-k}{N}\right) {}_2F_1\left[\begin{matrix} \frac{j}{N} & \frac{i+j+k-N}{N} \\ & \frac{i+j}{N} \end{matrix}; \lambda\right]$$

$$= \alpha(\lambda) B\left(\frac{i+j+k-N}{N}, \frac{N-k}{N}\right) {}_2F_1\left[\begin{matrix} \frac{N-k}{N} & \frac{i}{N} \\ & \frac{i+j}{N} \end{matrix}; \lambda\right],$$

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$$\tau_1 = \int_0^1 \omega_1 = B\left(\frac{N-i}{N}, \frac{N-j}{N}\right) {}_2F_1\left[\begin{matrix} \frac{k}{N} & \frac{N-i}{N} \\ & \frac{2N-i-j}{N} \end{matrix}; \lambda\right],$$

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$$\tau_1' = \int_{\frac{1}{\lambda}}^{\infty} \omega_1 = \tau_{N-1}\alpha(\lambda) B\left(\frac{i+j+k-N}{N}, \frac{N-k}{N}\right) / B\left(\frac{i}{N}, \frac{j}{N}\right),$$

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$$\gamma = \frac{\tau_1' \tau_{N-1}'}{\tau_1 \tau_{N-1}} = \frac{\left(\sin \frac{i}{N}\pi\right)\left(\sin \frac{j}{N}\pi\right)}{\left(\sin \frac{k}{N}\pi\right)\left(\sin \frac{2N-i-j-k}{N}\pi\right)} \in \mathbb{Q}(\zeta_N + \zeta_N^{-1}).$$

Example: $X_\lambda^{[6;4,3,1]}$

For the curve $[6;4,3,1]$, the lattice Λ is generated by

$$\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}, \begin{pmatrix}\zeta_6\tau_1\\ \zeta_6^{-1}\tau_2\end{pmatrix}, \begin{pmatrix}\beta_1\tau_2\\ \beta_2\tau_1\end{pmatrix}, \begin{pmatrix}\zeta_6\beta_1\tau_2\\ \zeta_6^{-1}\beta_2\tau_1\end{pmatrix},$$

where

$$\tau_1 = B(1/3,1/2)\,{}_2F_1\begin{bmatrix}\frac{1}{6} & \frac{1}{3}\\ & \frac{5}{6}\end{bmatrix};\lambda\Bigg], \quad \tau_2 = B(2/3,1/2)\,{}_2F_1\begin{bmatrix}\frac{5}{6} & \frac{2}{3}\\ & \frac{7}{6}\end{bmatrix};\lambda\Bigg],$$

$$\beta_1 = -\left(\lambda^{1/6}(1-\lambda)^{1/3}\sqrt[3]{2}\right), \quad \beta_2 = 2/\beta_1.$$

The endomorphism algebra $\mathrm{End}(J_\lambda^{new})$ contains

$$E = \begin{pmatrix}\zeta_6 & 0\\ 0 & \zeta_6^{-1}\end{pmatrix}, \quad J = \begin{pmatrix}0 & \beta_1\\ \beta_2 & 0\end{pmatrix},$$

$$I = 2E - (\zeta_6 + \zeta_6^{-1}) = \begin{pmatrix}\sqrt{-3} & 0\\ 0 & -\sqrt{-3}\end{pmatrix}.$$

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Example: $X_\lambda^{[6;4,3,1]}$

Note that $I^2 = -3$, $J^2 = 2$, and $IJ = -JI$. Thus $\mathrm{End}(J_\lambda^{new})$ contains the quaternion algebra

$$\left(\frac{-3,2}{\mathbb{Q}}\right) = \mathbb{Q} + \mathbb{Q}I + \mathbb{Q}J + \mathbb{Q}EJ, \quad I^2 = -3,\ J^2 = 2,\ IJ = -JI,$$

which is isomorphic to $H_{(3,6,6)}$.

End(J_λ^{new}) with $\phi(N) = 2$

When $N = 3, 4, 6$, a period matrix of J_λ^{new} is

$$\begin{pmatrix} \tau_1 & \zeta_N \tau_1 & \alpha(\lambda)\beta\tau_{N-1} & \zeta_N \alpha(\lambda)\beta\tau_{N-1} \\ \tau_{N-1} & \zeta_N^{-1}\tau_{N-1} & \gamma\tau_1/\beta\alpha(\lambda) & \zeta_N^{-1}\gamma\tau_1/\beta\alpha(\lambda) \end{pmatrix},$$

where

$$\beta = B\left(\frac{i+j+k-N}{N}, \frac{N-k}{N}\right) / B\left(\frac{i}{N}, \frac{j}{N}\right),$$

and

$$\gamma/\beta = B\left(\frac{N-i}{N}, \frac{N-j}{N}\right) / B\left(\frac{k}{N}, \frac{2N-i-j-k}{N}\right).$$

If $\beta \in \overline{\mathbb{Q}}$ ($\gamma/\beta \in \overline{\mathbb{Q}}$), then End$_0$($J_\lambda^{new}$) contains the endomorphisms

$$E = \begin{pmatrix} \zeta_N & 0 \\ 0 & \zeta_N^{-1} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \alpha(\lambda)\beta \\ \frac{\gamma}{\alpha(\lambda)\beta} & 0 \end{pmatrix}.$$

End(J_λ^{new}) with $\phi(N) = 2$

When $N = 3, 4, 6$, if

$$\beta = B\left(\frac{N-i}{N}, \frac{N-j}{N}\right) / B\left(\frac{k}{N}, \frac{2N-i-j-k}{N}\right) \in \overline{\mathbb{Q}},$$

the algebra $\mathrm{End}_0(J_\lambda^{new})$ contains the quaternion algebra defined over $\mathbb{Q}$ generated by

$$I = 2E - (\zeta_N + \zeta_N^{-1}) = \begin{pmatrix} \zeta_N - \zeta_N^{-1} & 0 \\ 0 & \zeta_N^{-1} - \zeta_N \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \alpha(\lambda)\beta \\ \frac{\gamma}{\alpha(\lambda)\beta} & 0 \end{pmatrix}$$

which satisfy

$$I^2 = \left(\zeta_N - \zeta_N^{-1}\right)^2, J^2 = \gamma \in \mathbb{Q}, \quad IJ + JI = 0.$$

End(J_λ^{new}) with $\phi(N) = 2$

Claim. When $N = 3, 4, 6$, if $\mathrm{End}_0(J_\lambda^{new})$ contains a quaternion algebra over $\mathbb{Q}$, then

$$\beta = B\left(\frac{i+j+k-N}{N}, \frac{N-k}{N}\right) / B\left(\frac{i}{N}, \frac{j}{N}\right) \in \overline{\mathbb{Q}}.$$

Idea.

$$\begin{array}{c} {}_2F_1 - \text{Gaussian hypergeometric function} \\ \downarrow \\ L_p(J_\lambda^{new}, s) \\ \uparrow \\ \text{Galois representations} \end{array}$$

"Computing" the Galois representation of $C_\lambda^{[N;,i,j,k]}$ via Gaussian hypergeometric functions.

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Hypergeometric functions over $\mathbb{F}_q$

Let p be a prime, and $q = p^s$.

Definition.

- Let $\widehat{\mathbb{F}_q^\times}$ denote the group of multiplicative characters on $\mathbb{F}_q^\times$.
- Extend $\chi \in \widehat{\mathbb{F}_q^\times}$ to $\mathbb{F}_q$ by setting $\chi(0) = 0$.
- (Greene, 1984) Let $\lambda \in \mathbb{F}_q$, and $A, B, C \in \widehat{\mathbb{F}_q^\times}$. Define

$$ {}_2F_1\begin{pmatrix} A & B \\ & C \end{pmatrix}; \lambda\Big)_q = \varepsilon(\lambda)\frac{BC(-1)}{q}\sum_{x\in\mathbb{F}_q} B(x)\overline{B}C(1-x)\overline{A}(1-\lambda x), $$

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Jacobi sums and Beta functions

If $\chi \in \widehat{\mathbb{F}_q^\times}$ is of order N, we have the following analogy

$$\begin{array}{ccc} \frac{i}{N} & \Longleftrightarrow & \chi^i \\ \Gamma(\frac{i}{N}) & \Longleftrightarrow & g(\chi^i) \\ B(\frac{i}{N}, \frac{j}{N}) & \Longleftrightarrow & J(\chi^i, \chi^j) \end{array}$$

$$C_\lambda^{[N;i,j,k]} \Longleftrightarrow \widetilde{C}_\lambda^{[N;i,j,k]}/\mathbb{F}_q$$

$$ {}_2F_1\left[\begin{matrix} \frac{k}{N} & \frac{N-i}{N} \\ & \frac{2N-i-j}{N} \end{matrix}; \lambda\right] \qquad {}_2F_1\left(\begin{matrix} \chi^{-k} & \chi^i \\ & \chi^{i+j} \end{matrix}; \lambda\right)_q$$

$$\Longleftrightarrow$$

$$ {}_2F_1\left[\begin{matrix} \frac{N-k}{N} & \frac{i}{N} \\ & \frac{i+j}{N} \end{matrix}; \lambda\right] \qquad {}_2F_1\left(\begin{matrix} \chi^{k} & \overline{\chi}^i \\ & \overline{\chi}^{i+j} \end{matrix}; \lambda\right)_q$$

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$$

Counting points on generalized Legendre curves

Theorem.
Let $p > 3$ be prime and $q = p^s \equiv 1 \pmod{N}$, and let i, j, k be natural numbers with $1 \leq i, j, k < N$. Further, let $\xi \in \widehat{\mathbb{F}_q^\times}$ be a character of order N. Then for $\lambda \in \mathbb{F}_q \setminus \{0, 1\}$,

$$\#X_\lambda^{[N;i,j,k]}(\mathbb{F}_q) = 1 + q + q\sum_{m=1}^{N-1} \xi^{mj}(-1)\,{}_2F_1\left(\begin{matrix}\xi^{-km} & \xi^{im} \\ & \xi^{m(i+j)}\end{matrix}\,;\lambda\right)_q + n_0 + n_1 + n_{\frac{1}{\lambda}} + n_\infty - 4,$$

where $n_0, n_1, n_{\frac{1}{\lambda}}, n_\infty$ are the numbers of points on $X_\lambda^{[N;i,j,k]}$ from resolving the singularities $0, 1, \frac{1}{\lambda}, \infty$ respectively of $C_\lambda^{[N;i,j,k]}$

Galois Representations

Suppose $C_\lambda^{[N;i,j,k]}$ has genus g. One can construct a compatible family of degree-$2g$ representations

$$\rho_\ell(\lambda) : G_{\mathbb{Q}} := Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_{2g}(\overline{\mathbb{Q}}_\ell)$$

via the Tate module of the Jacobian $J_\lambda^{[N;i,j,k]}$ of $X_\lambda^{[N;i,j,k]}$.

Let $\zeta \in \mu_N$, the multiplicative group of Nth roots of unity. The map $A_\zeta : (x,y) \mapsto (x, \zeta^{-1} y)$ induces an action on the ρ_ℓ. Consequently,

$$\rho_\ell(\lambda)|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))} = \bigoplus_{n=1}^{N-1} \sigma_n(\lambda)$$

where $\sigma_n(\lambda)$ is 2-dimensional when $(n, N) = 1$.
Let ρ^{new} be the subrepresentation of ρ that corresponds to J_λ^{new}.

4-dimensional Galois representations with QM

Proposition.

$$-\mathrm{Tr}\sigma_m(\mathrm{Frob}_q) \quad \text{and} \quad {}_2F_1\left(\begin{matrix}\xi^{-km} & \xi^{im} \\ & \xi^{m(i+j)}\end{matrix}; \lambda\right)_q \cdot \xi^{mj}(-1)q$$

agree up to different embeddings of $\mathbb{Q}(\zeta_N)$ in $\mathbb{C}$.

Theorem

Let $\varphi(N) = 2$. If $End_0(J_\lambda^{new})$ contains a quaternion algebra, then the corresponding representations σ_1 and σ_{N-1} of $G_{\mathbb{Q}(\zeta_N)}$, which are assumed to be absolutely irreducible, differ by a character.

Criterion

Proposition. If $A, B, C \in \widehat{\mathbb{F}_q^\times}$ $A, B \neq \varepsilon$, $A, B \neq C, \varepsilon$, and $\lambda \in \mathbb{F}_q \setminus \{0, 1\}$,

$$J(A, \overline{A}C)\,{}_2F_1\begin{pmatrix} A & B \\ & C \end{pmatrix}; \lambda\Big)_q =$$

$$AB(-1)\overline{C}(-\lambda)C\overline{AB}(1-\lambda)J(B, \overline{B}C)\,{}_2F_1\begin{pmatrix} \overline{A} & \overline{B} \\ & \overline{C} \end{pmatrix}; \lambda\Big)_q.$$

Theorem. For the curve $C^{[N;i,j,k]}$ with $\phi(N) = 2$, if $\mathrm{End}(J^{new})$ contains a quaternion algebra, then, as $A = \eta_N^{-k}$, $B = \eta_N^i$, $C = \zeta_n^{(i+j)}$,

$${}_2F_1\left(\begin{matrix} \eta_N^{-k} & \eta_N^{i} \\ & \eta_N^{(i+j)} \end{matrix}; \lambda\right)_q, \quad {}_2F_1\left(\begin{matrix} \eta_N^{k} & \eta_N^{-i} \\ & \eta_N^{-(i+j)} \end{matrix}; \lambda\right)_q$$

differ by a character. Equivalently,

$$F(\eta_N) := J(\eta_N^i, \eta_N^j)/J(\eta_N^{-k}, \eta_N^{i+j+k})$$

has to be a character of N ($2N$ when N is odd).

$$g(\chi)\overline{g(\chi)} = p,$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(z\pi)}.$$

Hasse-Davenport Relation.

$$g(\chi^{\ell a}) = (-1)^{\ell}\chi(\ell^{\ell a - N/2})\chi(2^{N/2})^{1-\ell} g(\chi^{N/2})^{1-\ell} \prod_{j=0}^{\ell-1} g(\chi^{a+(N/\ell)j})$$

$$\Gamma(\ell z) = \ell^{(\ell z - \frac{1}{2})} 2^{\frac{(1-\ell)}{2}} \Gamma\left(\frac{1}{2}\right)^{1-\ell} \prod_{j=0}^{\ell-1} \Gamma\left(z + \frac{j}{\ell}\right).$$

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Hasse-Davenport Relation.

$$g(\chi^{\ell a}) = (-1)^{\ell}\chi(\ell^{\ell a - N/2})\chi(2^{N/2})^{1-\ell} g(\chi^{N/2})^{1-\ell} \prod_{j=0}^{\ell-1} g(\chi^{a+(N/\ell)j})$$

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Proposition. Let $N \geq 4$ be an even integer such that N divides $p-1$ and let $\eta \in \widehat{\mathbb{F}_p^\times}$ of order N. Let $A = \eta^i, B = \eta^j, C = \eta^k$ be characters such that none of $A, B, C, \overline{A}C, \overline{B}C$ are trivial. If $J(\eta^j, \eta^{k-j})/J(\eta^i, \eta^{k-i})$ is a character for each prime p with $p \equiv 1 \mod N$, then $B(\frac{j}{N}, \frac{k-j}{N})/B(\frac{i}{N}, \frac{k-i}{N})$ is an algebraic number.

Example

Let p be a prime such that $10 \mid p-1$ and $\eta \in \widehat{\mathbb{F}_p^\times}$ of order 10. Then

$$J(\eta, \eta^6)/J(\eta^2, \eta^5) = \eta(-1)J(\eta, \eta^5)/J(\eta^2, \eta^4) = \eta^8(2).$$

In comparison,

$$B\left(\frac{1}{10}, \frac{6}{10}\right)/B\left(\frac{2}{10}, \frac{5}{10}\right) = 2^{\frac{4}{5}}.$$

In conclusion, if $\mathrm{End}(J_\lambda^{new})$ contains a quaternion algebra, then

$$J(\eta_N^i, \eta_N^j)/J(\eta_N^{-k}, \eta_N^{(i+j+k)})$$

has to be a character. Hence,

$$B\left(\frac{i}{N}, \frac{j}{N}\right)/B\left(\frac{N-k}{N}, \frac{(i+j+k)}{N}\right) \in \overline{\mathbb{Q}},$$

equivalently, $B\left(\frac{N-i}{N}, \frac{N-j}{N}\right)/B\left(\frac{N-k}{N}, \frac{2N-i-j-k}{N}\right)$ has to be algebraic.

$X_\lambda^{[N;1,N-1,1]}$

- A period of $X_\lambda^{[N;1,N-1,1]}$ is

$$B\left(\frac{1}{N}, 1-\frac{1}{N}\right) {}_2F_1\left[\begin{matrix}\frac{N-1}{N} & \frac{1}{N} \\ & 1\end{matrix}; \lambda\right].$$

- Using the relation

$${}_2F_1\begin{pmatrix}A & \overline{A} \\ & \varepsilon\end{pmatrix}; \lambda\Big)_q = {}_2F_1\begin{pmatrix}\overline{A} & A \\ & \varepsilon\end{pmatrix}; \lambda\Big)_q,$$

one can deduce that the $G_{\mathbb{Q}(\lambda,\zeta_N)}$ representation $\sigma_n(\lambda)$ is isomorphic to $\sigma_{N-n}(\lambda)$.

- If $\sigma_n(\lambda)$ is absolutely irreducible, it can be descended to a 2-dimensional representation for $G_{\mathbb{Q}(\lambda,\zeta_N+\zeta_N^{-1})}$.

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$X_\lambda^{[3;1,2,1]}$

Example

Let ρ be the 4-dimensional Galois representation of $G_\mathbb{Q}$ arising from the genus-2 curve $y^3 = x(x-1)^2(1-\lambda x)$. Let ρ' be the Galois representation of $G_\mathbb{Q}$ arising from the elliptic curve $y^2 + xy + \frac{\lambda}{27} = x^3$. For any $\lambda \in \mathbb{Q}$ such that the elliptic curve does not have complex multiplication, ρ is isomorphic to $\rho' \oplus (\rho' \otimes \chi_{-3})$ where χ_{-3} is the quadratic character of $G_\mathbb{Q}$ with kernel $G_{\mathbb{Q}(\sqrt{-3})}$.

$y^5 = x(1-x)^4(1-2x)$ and Hilbert modular forms

For the curve $y^5 = x(1-x)^4(1-2x)$, one can predict that its L-function is related to two Hilbert modular forms, which differ by embeddings of $\mathbb{Q}(\sqrt{5})$ to $\mathbb{C}$. From numeric data, we identified two Hilbert modular forms, which are labeled by Hilbert Cusp Form 2.2.5.1-500.1-a in the LMFDB online database.

p	$L_p(C(\lambda), T)$ over $\mathbb{Q}(\sqrt{5})$	Hecke eigenvalues
7	$(49T^4+10T^2+1)(49T^4-10T^2+1)$	-10
11	$(11T^2-2T+1)^4$	$2, 2$
13	$(169T^4+1)^2$	0
17	$(289T^4-20T^2+1)(289T^4+20T^2+1)$	20
19	$\left(19T^2-5\left(\frac{1+\sqrt{5}}{2}\right)T+1\right)\left(19T^2-5\left(\frac{1-\sqrt{5}}{2}\right)T+1\right)$ $\left(19T^2+5\left(\frac{1+\sqrt{5}}{2}\right)T+1\right)\left(19T^2+5\left(\frac{1-\sqrt{5}}{2}\right)T+1\right)$	$5\left(\frac{1\pm\sqrt{5}}{2}\right)$
31	$\left(\left(31T^2+\left(\frac{1+5\sqrt{5}}{2}\right)T+1\right)\left(31T^2+\left(\frac{1-5\sqrt{5}}{2}\right)T+1\right)\right)^2$	$\frac{-1\pm5\sqrt{5}}{2}$
41	$\left(\left(41T^2+\left(\frac{1+5\sqrt{5}}{2}\right)T+1\right)\left(41T^2+\left(\frac{1-5\sqrt{5}}{2}\right)T+1\right)\right)^2$	$\frac{-1\pm5\sqrt{5}}{2}$

$X_\lambda^{[12;9,5,1]}$

- The arithmetic group $\Gamma = (2,6,6)$ can be realized as the monodromy group of a period on $J_\lambda^{[12;9,5,1]}$.
- $H_\Gamma = B_6$

The corresponding periods of J_λ^{new} are

$$\tau_1 = \int_0^1 \omega_1 = B(1/4, 7/12)\, {}_2F_1\left[\begin{matrix} \frac{1}{12} & \frac{1}{4} \\ & \frac{5}{6} \end{matrix}; \lambda\right], \quad \int_{1/\lambda}^{\infty} \omega_1$$

$$\tau_2 = \int_0^1 \omega_{11} = B(5/12, 3/4)\, {}_2F_1\left[\begin{matrix} \frac{3}{4} & \frac{11}{12} \\ & \frac{7}{6} \end{matrix}; \lambda\right], \quad \int_{1/\lambda}^{\infty} \omega_{11}$$

$$\tau_3 = \int_0^1 \omega_5 = B(1/4, 4/12)\, {}_2F_1\left[\begin{matrix} \frac{1}{4} & \frac{5}{12} \\ & \frac{7}{6} \end{matrix}; \lambda\right], \quad \int_{1/\lambda}^{\infty} \omega_5$$

$$\tau_4 = \int_0^1 \omega_7 = B(3/4, 1/12)\, {}_2F_1\left[\begin{matrix} \frac{7}{12} & \frac{3}{4} \\ & \frac{5}{6} \end{matrix}; \lambda\right], \quad \int_{1/\lambda}^{\infty} \omega_7$$

For the Gaussian hypergeometric functions, we have the identities:

$$
\begin{aligned}
{}_2F_1\begin{pmatrix} \eta & \eta^3 \\ & \eta^{-2} \end{pmatrix}; \lambda\Big)_p &= \eta^2(\lambda)\, {}_2F_1\begin{pmatrix} \eta^5 & \eta^3 \\ & \eta^2 \end{pmatrix}; \lambda\Big)_p \\
&= \eta\left(-27(1-\lambda)^6\right) {}_2F_1\begin{pmatrix} \eta^{-5} & \eta^{-3} \\ & \eta^{-2} \end{pmatrix}; \lambda\Big)_p \\
&= \eta\left(-27\lambda^2(1-\lambda)^6\right) {}_2F_1\begin{pmatrix} \eta^{-1} & \eta^{-3} \\ & \eta^{2} \end{pmatrix}; \lambda\Big)_p,
\end{aligned}
$$

where η is a multiplicative character of $\mathbb{F}_p^\times$ of order 12.

In this case,

$$\int_0^1 \omega_1 \Big/ \int_{\frac{1}{\lambda}}^\infty \omega_{11} = B(1/4, 7/12)/B(1/12, 3/4) = \sqrt{\frac{2\sqrt{3}}{3} - 1}.$$

For the subvariety J_λ^{new}, the lattice $\Lambda(\lambda)$ is generated by

$$\begin{array}{cccc|ccc}
\tau_1, & \zeta\tau_1, & \zeta^2\tau_1, & i\tau_1, & i\lambda^{\frac{1}{6}}\alpha\tau_2, & \zeta i\lambda^{\frac{1}{6}}\alpha\tau_2, & \zeta^2 i\lambda^{\frac{1}{6}}\alpha\tau_2, \\
\tau_2, & \tau_2/\zeta, & \tau_2/\zeta^2, & -i\tau_2, & i\frac{2+\sqrt{3}}{\alpha\lambda^{\frac{1}{6}}}\tau_1, & i\frac{2+\sqrt{3}}{\alpha\zeta\lambda^{\frac{1}{6}}}\tau_1, & i\frac{2+\sqrt{3}}{\alpha\zeta^2\lambda^{\frac{1}{6}}}\tau_1, \\
\alpha\tau_2, & \zeta^5\alpha\tau_2, & \alpha\tau_2/\zeta^2, & i\alpha\tau_2, & i\tau_1/\lambda^{\frac{1}{6}}, & \zeta^5 i\tau_1/\lambda^{\frac{1}{6}}, & i\tau_1/\zeta^2\lambda^{\frac{1}{6}}, \\
\frac{2+\sqrt{3}}{\alpha}\tau_1, & \frac{2+\sqrt{3}}{\alpha\zeta^5}\tau_1, & \frac{2+\sqrt{3}}{\alpha\zeta^{-2}}\tau_1, & \frac{2+\sqrt{3}}{i\alpha}\tau_1, & i\lambda^{\frac{1}{6}}\tau_2, & i\lambda^{\frac{1}{6}}\tau_2/\zeta^5, & \zeta^2 i\lambda^{\frac{1}{6}}\tau_2,
\end{array}$$

where

$$\tau_1 = B(1/4, 7/12)\,{}_2F_1\left[\begin{matrix}\frac{1}{12} & \frac{1}{4} \\ & \frac{5}{6}\end{matrix};\lambda\right], \tau_3 = B(5/12, 3/4)\,{}_2F_1\left[\begin{matrix}\frac{3}{4} & \frac{11}{12} \\ & \frac{7}{6}\end{matrix};\lambda\right],$$

$$\alpha = (1-\lambda)^{1/2}\sqrt{9+6\sqrt{3}}/3.$$

$\mathrm{End}_0(J_\lambda^{new})$ is generated by the endomorphisms

$$A = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & 1/\zeta & 0 & 0 \\ 0 & 0 & \zeta^5 & 0 \\ 0 & 0 & 0 & 1/\zeta^5 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & i/\lambda^{\frac{1}{6}} & 0 \\ 0 & 0 & 0 & i\lambda^{\frac{1}{6}} \\ i\lambda^{\frac{1}{6}} & 0 & 0 & 0 \\ 0 & i/\lambda^{\frac{1}{6}} & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & i\frac{2+\sqrt{3}}{\alpha\lambda^{\frac{1}{6}}} & 0 & 0 \\ i\lambda^{\frac{1}{6}}\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i\lambda^{\frac{1}{6}}}{\alpha} \\ 0 & 0 & i\frac{\alpha\lambda^{-\frac{1}{6}}}{2+\sqrt{3}} & 0 \end{pmatrix}.$$

$End_0(J_\lambda^{new})$ contains the quaternion algebra $\left(\frac{-1,3}{\mathbb{Q}}\right) \simeq H_\Gamma$, which is generated by B, and $A + A^{-1}$.

Theorem (Wüstholz)

Let A be an abelian variety isogenous over $\overline{\mathbb{Q}}$ to the direct product $A_1^{n_1} \times \cdots \times A_k^{n_k}$ of simple, pairwise non-isogenous abelian varieties A_μ defined over $\overline{\mathbb{Q}}$, $\mu = 1, \ldots, k$. Let $\Lambda_{\overline{\mathbb{Q}}}(A)$ denote the space of all periods of differentials, defined over $\overline{\mathbb{Q}}$, of the first kind and the second on A. Then the vector space $\widehat{V}_A$ over $\overline{\mathbb{Q}}$ generated by 1, $2\pi i$, and $\Lambda_{\overline{\mathbb{Q}}}(A)$, has dimension

$$\dim_{\overline{\mathbb{Q}}} \widehat{V}_A = 2 + 4\sum_{\nu=1}^{k} \frac{\dim A_\nu^2}{\dim_{\mathbb{Q}}(End_0 A_\nu)}.$$

$X_\lambda^{[10;2,7,7]}$

- The arithmetic triangle group Γ is $(5, 10, 10)$.
- H_Γ is quaternion algebra defined over $\mathbb{Q}(\sqrt{5})$ with discriminant $\mathfrak{p}_2$.

The corresponding periods of J_λ^{new} are

$$\tau_1 = \int_0^1 \omega_1 = B(3/10, 4/5)\, {}_2F_1\left[\begin{matrix}\frac{7}{10} & \frac{4}{5} \\ & \frac{11}{10}\end{matrix}; \lambda\right],$$

$$\tau_2 = \int_0^1 \omega_9 = B(7/10, 1/5)\, {}_2F_1\left[\begin{matrix}\frac{3}{10} & \frac{1}{5} \\ & \frac{9}{10}\end{matrix}; \lambda\right],$$

$$\tau_3 = \int_0^1 \omega_3 = B(9/10, 2/5)\, {}_2F_1\left[\begin{matrix}\frac{1}{10} & \frac{2}{5} \\ & \frac{13}{10}\end{matrix}; \lambda\right],$$

$$\tau_4 = \int_0^1 \omega_7 = B(1/10, 3/5)\, {}_2F_1\left[\begin{matrix}\frac{9}{10} & \frac{3}{5} \\ & \frac{7}{10}\end{matrix}; \lambda\right],$$

$$\tau_1' = \int_1^\infty \omega_1 = \frac{\sqrt{5}-1}{2\alpha_1(\lambda)\beta_1}\tau_2,\ \tau_2' = \int_1^\infty \omega_9 = \alpha_1(\lambda)\beta_1\tau_1$$
$$\tau_3' = \int_1^\infty \omega_3 = \frac{-\sqrt{5}-1}{2\alpha_1(\lambda)\beta_2}\tau_4,\ \tau_4' = \int_1^\infty \omega_7 = \alpha_2(\lambda)\beta_2\tau_3,$$

where

$$\alpha_1(\lambda) = (-1)^{7/5}\lambda^{1/10}(1-\lambda)^{2/5},\ \beta_1 = B\left(7/10, 2/5\right)/B\left(3/10, 4/5\right),$$
$$\alpha_2(\lambda) = (-1)^{1/5}\lambda^{3/10}(1-\lambda)^{-4/5},\ \beta_2 = B\left(1/10, 1/5\right)/B\left(9/10, 2/5\right).$$

- By using Gaussian hypergeometric functions, one knows that the subrepresentations σ_m and σ_{N-m} differ by a character. Thus β_1, β_2 are both algebraic.
- σ_1 and σ_3 do not differ by a character.
- Combining with Wüstholz's result we know that for a generic $\lambda \in \overline{\mathbb{Q}}$, the 4-dimensional abelian variety J_λ^{new} is simple, and $\Lambda_{\overline{\mathbb{Q}}}(J_\lambda^{new})$ is 10-dimensional.

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The algebra $End_0(J_\lambda^{new})$ contains the endomorphisms

$$A = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & \zeta^{-1} & 0 & 0 \\ 0 & 0 & \zeta^3 & 0 \\ 0 & 0 & 0 & \zeta^{-3} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \alpha_1(\lambda)\beta_1 & 0 & 0 \\ \frac{\sqrt{5}-1}{2\alpha_1(\lambda)\beta_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2(\lambda)\beta_2 \\ 0 & 0 & \frac{-\sqrt{5}-1}{2\alpha_2(\lambda)\beta_2} & 0 \end{pmatrix}$$

The algebra $End_0(J_\lambda^{new})$ contains the quaternion algebra $\left(\frac{\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}}{\mathbb{Q}(\sqrt{5})}\right) \simeq H_{(5,10,10)}$.