Families of curves with nontrivial endomorphisms in their Jacobians

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Let X be a projective nonsingular algebraic curve of genus g (defined over a field of characteristic 0). Let $A = \mathrm{Jac}(X)$ be its Jacobian. This is a principally polarized abelian variety (ppav) of dimension g defined over the same field as X.

Moduli spaces

Let $\mathfrak{M}_g$ be the moduli space (coarse) of smooth projective curves of genus g. This has dimension $3g-3$ if $g \geq 2$.
Let $\mathfrak{A}_g$ be the moduli space (coarse) of ppav of dimension g. This has dimension $g(g+1)/2$.
The map $X \mapsto \mathrm{Jac}(X) : \mathfrak{M}_g \to \mathfrak{A}_g$ is an injection (Torelli).

When $g = 2, 3$, we have $3g - 3 = g(g+1)/2$, so that in these cases, $\mathfrak{M}_g$ and $\mathfrak{A}_g$ are birationally equivalent.

Recall: for any abelian variety A, $\mathrm{End}(A) \otimes \mathbb{Q}$ is a finite-dimensional semisimple algebra with involution (usually just $\mathbb{Q}$). The different possible types were classified by A. A. Albert.

Consider the set of isomorphism classes of data (A, ϕ, θ, r) where

1. A is an abelian variety of dimension g.
2. ϕ is a polarization of A, of a fixed type.
3. $\theta : R \to \mathrm{End}(A)$ is a homomorphism from an order in a semi simple algebra of finite dimension over $\mathbb{Q}$; θ is compatible with ϕ in a suitable sense.
4. r is a rigidification, typically a marking of a finite set of points of finite order on A.

This data is parametrized by a Shimura variety (of PEL type) $S(g, \phi, R, r)$.

Shimura Varieties

1. As a complex manifold, a Shimura variety is a quotient

$$\Gamma \backslash D$$

where D is a Hermitian symmetric domain and $\Gamma \subset \mathrm{Aut}(D) = G$ is an arithmetic group. G is the set of real points of a reductive algebraic group defined over $\mathbb{Q}$.

2. As an algebraic variety, they have canonical models over specific number fields.

3. While not all Shimura varieties have straightforward moduli interpretations, those of PEL type do. In particular, there is a universal family

$$\pi : A(g, \phi, R, r) \rightarrow S(g, \phi, R, r)$$

Problem 1. Describe $\pi : A(g, \phi, R, r) \to S(g, \phi, R, r)$ as algebraic varieties. As complex manifolds they were made explicit by Kuga and Shimura.

Problem 2. There are canonical subvarieties $H \subset S(g, \phi, R, r)$ of Hodge type. Describe these algebro-geometrically. Example: find the algebraic coordinates of CM points.

Problem 3. Sometimes an interesting family of varieties is known. Determine the endomorphism structure of the corresponding Picard varieties. Example: (generalized) hypergeometric families.

Problem 4. In Problem 3 replace Picard varieties by motives of any weight.

Classical modular curves

$S(g = 1, \phi, \mathrm{End} = \mathbb{Z}, r)$ are the classical modular curves. $D = \mathfrak{H}_1$ is the complex upper half plane. $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup. The algebraic variety structure is mediated by automorphic forms/functions, e.g., the j-function

$$S(1, \phi, \mathbb{Z}, r = \emptyset) = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}_1 \overset{j}{\cong} \mathbf{P}^1(\mathbb{C}).$$

Values $j(\tau)$ at CM points $\tau \in \mathfrak{H}_1$ are algebraic integers. Their arithmetic is very interesting, c.f., the Gross-Zagier formula.

Siegel modular varieties

$S(g, \phi, \mathrm{End} = \mathbb{Z}, r)$ are the Siegel modular varieties. $D = \mathfrak{H}_g$ is the Siegel half space. $\Gamma \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ is a congruence subgroup. The algebraic variety structure is mediated by automorphic forms/functions. There are embeddings

$$S(g, \phi, \mathbb{Z}, r) = \Gamma \backslash \mathfrak{H}_g \hookrightarrow \mathbf{P}^N(\mathbb{C})$$

given by theta constants, but these are cumbersome, N is big and there are many equations (determined by Mumford).

Shimura curves

$S(g=2,\phi,R,r)$. Where R is an order in an indefinite quaternion division algebra B over $\mathbb{Q}$. $D=\mathfrak{H}_1$ is the complex upper halfspace. $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ is a Fuchsian subgroup determined by the units of norm 1 in R.

These were first studied by Poincaré. They are called Shimura curves. The contrast to the case of classical modular curves, $\Gamma\backslash\mathfrak{H}_1$ is compact (no cusps).

Explicit equations for these have been written down in some cases, by various methods (Ihara, Kurihara, Jordan-Livné, Hashimoto-Murabayashii, Elkies, Yifan Yang, Fang-Ting Tu...)

Some universal families of genus 2 QM curves have also been found.

Quaternionic Shimura varieties

Let R be the ring of integers in an totally real numberfield K, with $[K:\mathbb{Q}] = d \geq 2$. Let B be a quaternion division algebra over K. Then

$$B \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{H}^g \times M_2(\mathbb{R})^{d-g}, \quad \mathbb{H} = \text{Hamilton's quaternions.}$$

If $g = 0$, the Shimura variety has a moduli interpretation as parametrizing a family of abelian varieties of dimension $2d$ with endomorphisms by an order in B.
If $1 \leq g \leq d-1$, there is a Shimura variety S, but it does not have a naive moduli interpretation. Nonetheless, Shimura constructed embeddings of S into moduli spaces of abelian varieties. In particular, there are families of abelian varieties parametrized by S.

Quaternionic Shimura varieties

These have been used to construct Galois representations attached to $B_{\mathbb{A}}$ (M. Ohta).

Examples arise from arithmetic triangle groups; they have been further investigated by P. Beazley-Cohen, Ling Long, Wolfart, and J. Voight.

Problem

Construct families of genus 2 curves

$$X : y^2 = f(x), \quad \deg\ f(x) = 5 \text{ or } 6.$$

such that $\mathrm{End}(\mathrm{Jac}(X)) \otimes \mathbb{Q}$ is nontrivial, i.e., larger than $\mathbb{Q}$.

Interesting cases

1. $\mathrm{End}(\mathrm{Jac}(X)) \otimes \mathbb{Q} =$ quartic CM field. These are isolated in moduli. Applications to cryptography (K.Lauter).
2. $\mathrm{End}(\mathrm{Jac}(X)) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{D})$ a real quadratic field. The Shimura variety is a Hilbert modular surface (a Humbert surface).
3. $\mathrm{End}(\mathrm{Jac}(X)) \otimes \mathbb{Q} = B$, an indefinite quaternion division algebra over $\mathbb{Q}$. This gives a Shimura curve.

Hilbert modular surface for $\mathbb{Q}(\sqrt{5})$

1. A point $\tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathfrak{H}_2$ with $\tau_1 = \tau_2 + \tau_3$ gives an abelian variety
$$A_\tau := \mathbb{C}^2/\mathbb{Z}^2 + \mathbb{Z}^2\tau$$
whose endomorphism ring contains $\mathbb{Q}(\sqrt{5})$ (Humbert).
2. The diagonal surface of Clebsch and Klein
$$\sum_{i=0}^{4} x_i = 0, \quad \sum_{i=0}^{4} x_i^3 = 0,$$
is isomorphic to the level 2 covering of the Hilbert modular surface for $\mathbb{Q}(\sqrt{5})$ (Hirzebruch).

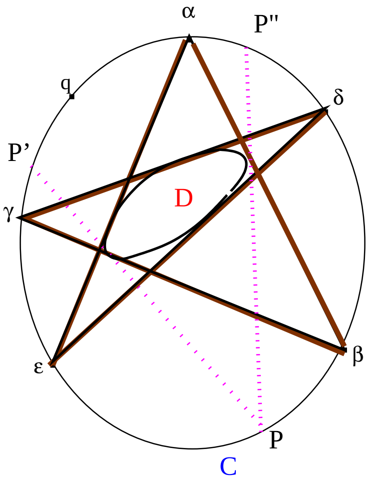
Hilbert modular surface for $\mathbb{Q}(\sqrt{5})$

1. Let

$$f(x;a,b,c) = x^6 - (4+2b+3c)x^5 + (2+2b+b^2-ac)x^4 - (6+4a+6b-2b^2+5c+2ac)x^3 + (1+b^2-ac)x^2 + (2-2b)x + (c+1).$$

   The $y^2 = f(x;a,b,c)$ is a universal family of genus 2 curves with RM by $\mathbb{Q}(\sqrt{5})$ (Brumer/Hashimoto).

2. These curves can be constructed from Poncelet 5-gons (Humbert/Mestre).



Humbert 5 = Poncelet 5

Pentagon αβγδε
inscribes conic C
circumscribes conic D

Genus 2 curve X is the
double cover of C branched
above α, β, γ, δ, ε and
a point q in C intersect D.

The correspompdence

P −> P'+P''

lifts to a correspondence
ϕ of X with $\phi^2 + \phi - 1 = 0$
in Jac(X).

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Shimura curve for B_6

1. The maximal order in B is $\mathcal{O} = \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\gamma$ where

$$\alpha^2 = -1, \beta^2 = 3, \alpha\beta = -\beta\alpha, \gamma = (1 + \alpha + \beta + \alpha\beta)/2.$$

2. $S(\mathbb{C}) = \mathcal{O}_1^* \backslash \mathfrak{H} \cong \mathbb{P}^1(\mathbb{C})$.
3. The canonical model is the projective conic $x^2 + y^2 + 3z^2 = 0$.
4. The graded ring of modular forms for $\Gamma = \mathcal{O}_1^*$ is generated by forms h_4, h_6, h_{12} subject to the relation:

$$h_{12}^2 + 3h_6^4 + h_4^6 = 0.$$

Shimura curve for B_6

Consider the family of genus 2 curves

$$y^2 = x(x^4 - Px^3 + Qx^2 - Rx + 1),$$

where

$$P = -2(s+t), \quad R = -2(s-t)$$
$$Q = \frac{(1+2t^2)(11-28t^2+8t^4)}{3(1-t^2)(1-4t^2)}$$

where $4s^2t^2 - s^2 + t^2 + 2 = 0$. This is a universal family of genus 2 curves whose Jacobians have QM by the maximal order in B_6 (Hashimoto and Murabayashii).

Method I: Automorphic Forms

1. Algebraic moduli of genus 2 curves $y^2 = f_6(x)$ are given by the invariant theory of binary sextic forms. These were determined by Clebsch.
2. One can reconstruct a genus 2 curve from its Clebsch/Igusa invariants: Mestre's algorithm.
3. Analytic moduli of genus 2 curves are given by a point in Siegel's spaces of degree 2: $\tau \in \mathfrak{H}_2$.
4. The bridge between analytic moduli and algebraic moduli is given by automorphic forms, specifically theta constants.

Method I: Automorphic Forms

1. The explicit expressions of the Igusa/Clebsch invariants as Siegel modular forms were given by Thomae, Bolza and Igusa.
2. Idea: one can convert the relatively simple formulas for Shimura subvarieties of $\mathfrak{H}_2$ into algebraic equations in the Igusa/Clebsch invariants. This has been implemented by Runge and Gruenewald.

Method I: Humbert surface for $D = 5$

1. The Satake compactification of $\mathfrak{A}_2[2]$ has a model in $\mathbf{P}^5$ given by

$$s_1 = 0, \quad s_2^2 - 4s_4 = 0, \quad s_k = \sum_{i=1}^{6} x_i^k,$$

where x_i is a linear combination of theta constants. Each s_i is a Siegel modular form of weight $2i$.

2. In $\mathfrak{A}_2[2]$ Humbert surfaces of discriminant 5 have equations

$$2p_{2,j} + p_{1,j}^2 = 0, \quad j = 1, ..., 6,$$

where $p_{k,j}$ is kth elementary symmetric function on the 5 coordinates excluding x_j.

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Method I: Shimura curves; A. Besser

1. In $\mathfrak{A}_2[2]$, Shimura curves of discriminant 6 have equations
$$3x_i^2 = s_2,\ x_i = -x_j, \quad 1 \leq i < j \leq 6.$$
2. In $\mathfrak{A}_2[2]$, Shimura curves of discriminant 10 have equations
$$x_i + 5x_j = 0,\ 3x_i^2 = s_2, \quad 1 \leq i \neq j \leq 6.$$
3. In $\mathfrak{A}_2[2]$, Shimura curves of discriminant 15 have equations
$$15(x_i + x_j)^2 = 4(s_2 + 3x_i x_j),\ 6x_i + 5x_j + 5x_k = 0,$$
$$1 \leq i \neq j \neq k \neq i \leq 6.$$

Method II: Kummer Surfaces. Elkies and Kumar

1. If X is a genus 2 curve then the Kummer surface $\text{Km}(X)$ is the nonsingular model of $\text{Jac}(X)/\pm id$. This is a K3 surface of high rank : $\text{rank}(\text{NS}(\text{Km}(X)) \geq 17$.
2. If $\text{Jac}(X)$ has additional endomorphisms, then the rank of $\text{Km}(X)$ should go up.

Dolgachev and A. Kumar proved:

Theorem

There is an isomorphism $\psi : \mathfrak{M}_2 \to \mathcal{E}_{E_8,E_7}$, where $\mathcal{E}_{E_8,E_7}$ is the moduli space of elliptic K3 surfaces with an E_8-fibre at ∞ and and E_7-fibre at 0.

Let A be the elliptic K3 surface with equation

$$y^2 = x^3 - t^3\left(\frac{I_4}{12} + 1\right)x + t^5\left(\frac{I_{10}}{4}t^2 + \frac{I_2 I_4 - 3I_6}{108}t + \frac{I_2}{24}\right),$$

which has fibres of type E_8 and E_7 respectively at $t = \infty$ and $t = 0$. Let C be the genus 2 curve with Igusa-Clebsch invariants $(I_2 : I_4 : I_6 : I_{10})$. Then A and $\mathrm{Km}(C)$ are Shioda-Inose twins.

Theorem

Consider the lattice of rank 18: $L_D := E_8(-1)^2 \oplus \mathcal{O}_D$. Let $\mathcal{F}_{L_D}$ be the moduli space of K3 surface that are lattice polarized by L_D. Then there is a surjective birational morphism $\mathcal{F}_{L_D} \to \mathcal{H}_D$.

Therefore, to construct the Humbert surface $\mathcal{H}_D$ for $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{D})$ one attempts to realize L_D as the Néron-Severi lattice of an elliptic K3 surface. One might have to modify this to a new elliptic K3 surface so as to have fibers of type E_7 and E_8 (2 and 3 neighbors).

Method II: Kummer Surfaces. $D = 5$

The elliptic surface is

$$y^2 = x^3 + \frac{1}{4}t^3(-3g^2t + 4)x - \frac{1}{4}t^5(4h^2t^2 + (4h + g^3)t + (4g + 1))$$

The Hilbert modular surface (double cover of the Humbert surface $\mathcal{H}_5$) is

$$z^2 = 2(6250h^2 - 4500g^2h - 1350gh - 108h - 972g^5 - 324g^4 - 27g^3)$$

The Igusa-Clebsch invariants are

$$(I_2 : I_4 : I_6 : I_{10}) = (6(4g + 1), 9g^2, 9(4h + 9g^3 + 2g^2), 4h^2).$$

Method II: Shimura curve with $D = 6$

The elliptic surface is

$$y^2 = x^3 + tx^2 + 2bt^3(t-1)x + b^2t^5(t-1)^2$$

The Shimura curve is $X(6)/\langle w_2, w_3\rangle \cong \mathbf{P}^1$ with coordinate b. This is the arithmetic triangle group (2,4,6). $X(6)$ has the model $s^2 + 27r^2 + 16 = 0$, where $b = r^2$.
The Igusa-Clebsch invariants are

$$(I_2 : I_4 : I_6 : I_{10}) = (24(b+1), 36b, 72b(5b+4), 4b^3).$$

There are CM points of discriminants $-3, -4, -24, -19$ respectively at $b = \infty, 0, -16/27, 81/64$.

Genus 3 curves

$\mathfrak{M}_3$ and $\mathfrak{A}_3$ are birationally equivalent, but now there is a distinction between hyperelliptic and nonhyperelliptic curves. A hyperelliptic curve has an equation

$$y^2 = f_8(x), \quad \deg f_8 = 8.$$

There are many models of nonhyperelliptic genus 3 curves, the simplest being the the canonical model, which is a smooth projective plane quartic

$$F_4(x, y, z) = 0.$$

Genus 3 curves: moduli

Algebraic moduli of genus 3 hyperelliptic curves is given by the invariant theory of binary octic forms. These were determined by Shioda.

As in the case of genus 2, these invariants can be expressed in terms of Siegel modular forms of degree 3 (theta constants: Thomae's formulas).

Algebraic moduli of genus 3 nonhyperelliptic curves is given by the invariant theory of ternary quartic forms.

Studied by many people, e.g., E. Noether, the complete determination of these is quite recent - Dixmier-Ohno invariants.

Genus 3 curves: moduli

In principle, these invariants can be expressed in terms of Siegel modular forms of degree 3.

The necessary formulas are implicit in 19th century works, especially Frobenius and Schottky, but to my knowledge, they are not in the modern literature (but see Dolgachev-Ortland and Looijenga).

Genus 3 curves: endomorphisms of Jacobians

Some interesting cases:

1. A degree 6 CM number field.
2. An imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ (Picard modular case).
3. A totally real cubic number field (Hilbert modular case).

Picard's family

Picard studied the family of genus 3 curves:

$$C_{a,b} : y^3 = x(x-1)(x-a)(x-b)$$

$\mathrm{End}(\mathrm{Jac}(C_{a,b}))$ contains $R = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$.
The parameter space is isomorphic to $\Gamma\backslash\mathbb{B}_2$ where $\Gamma \subset \mathrm{SU}(2,1;R)$ is a congruence subgroup, $\mathbb{B}_2 \subset \mathbb{C}^2$, the unit ball.
This is a generalized hypergeometric family.

A Hilbert modular family

Joint with:

Dun Liang

Zhibin Liang

Ryotaro Okazaki

Yukiko Sakai

Haohao Wang

We have constructed a universal (3-dimensional) family of nonhyperelliptic curves C with the property that $\mathrm{End}(\mathrm{Jac}(\mathrm{C}))$ contains $\mathbb{Z}[\zeta_7 + \bar{\zeta}_7]$, the integers in a cubic number field.

A Hilbert modular family

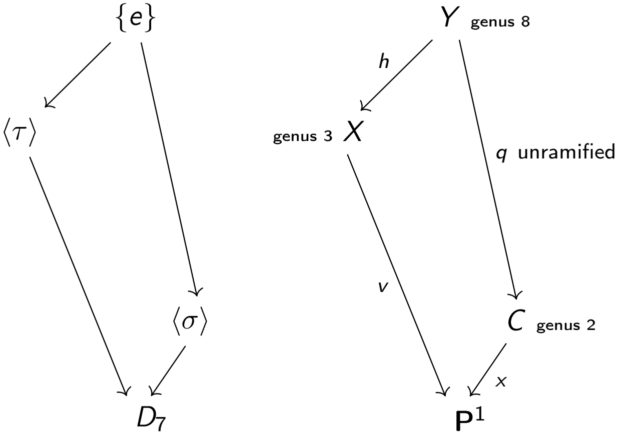
The construction is based on a method of Shimada and Ellenberg.
Basic idea: Let G be a finite group acting on a curve Y. If $H \subset G$ is a subgroup we let $X = Y/H$. We get an action of the "Hecke algebra" $\mathbb{Q}[H\backslash G/H]$ on Jac(X).
$\mathbb{Q}[H\backslash G/H] \subset \mathbb{Q}[G]$ is the subalgebra generated by $\tau_H g \tau_H$ where

$$\tau_H = \frac{1}{\#H} \sum_{h \in H} h.$$

Our case:

$$G = D_7 =< \sigma, \tau \mid \sigma^7 = \tau^2 = 1, \tau\sigma\tau = \sigma^6 >$$

and $H =< \tau >$. $\mathbb{Q}[H\backslash G/H] = \mathbb{Q}[\zeta_7^+]$.


{e}
⟨τ⟩
⟨σ⟩
D7
Y genus 8
h
genus 3 X
q unramified
v
C genus 2
x
P1

A Diophantine equation

Problem. Find solutions to the following equation:

$$a(x)^2 - s(x)b(x)^2 = c(x)^7$$

where a, b, c, s are polynomials in one variable of respective degrees 7, 4, 2, 6.

Why? Let $C : y^2 = s(x)$, a genus 2 curve. Let $\varphi = a(x) + b(x)y$, an element of its function field $k(C) = k(x, y)$. Then $k(x, y, \sqrt[7]{\varphi})$ is an unramified cyclic Galois extension of $k(x, y)$ of degree 7.

If X is a (smooth, projective) curve of genus g, say defined over $\mathbb{Q}$, there are l-adic representations

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbf{GSp}(H^1(X \otimes \overline{\mathbb{Q}}, \mathbb{Q}_l)) = \mathbf{GSp}_{2g}(\mathbb{Q}_l).$$

In general, one expects that the image is all of $\mathbf{GSp}_{2g}(\mathbb{Q}_l)$.
If $\mathrm{End}(\mathrm{Jac}(X)) \otimes \mathbb{Q}$ is larger than $\mathbb{Q}$, the Galois image will be smaller.
For instance, in our case (genus 3 with endomorphisms by a totally real cubic number field K) we get Galois representations of $\mathbf{GL}_2$-type.

A curve X with multiplication by $\mathbb{Z}[\zeta_7 + \bar{\zeta}_7]$

$$x^4 + \frac{345x^3y}{4} - \frac{16038x^3z}{7} + \frac{14499x^2y^2}{14} - \frac{553623}{4}x^2yz + \frac{4273137x^2z^2}{2}$$
$$+ \frac{2153679xy^3}{28} + \frac{28315359}{7}xy^2z + \frac{659015811}{7}xyz^2 - \frac{6866481456xz^3}{7}$$
$$- \frac{28405935y^4}{7} - 20973087y^3z - \frac{10692058320y^2z^2}{7} - \frac{205496736912yz^3}{7}$$
$$+ \frac{1321162646760z^4}{7} = 0$$

Zeta function and Galois representation for X

We compute the zeta function of the scheme $X/\mathbb{Z}$:

$$Z(X/\mathbb{F}_p, x) = \exp\left(\sum_{\nu \geq 1} N_\nu x^\nu/\nu\right)$$
$$= \frac{1 + a_p x + b_p x^2 + c_p x^3 + p b_p x^4 + p^2 a_p x^5 + p^3 x^6}{(1-x)(1-px)}$$

for the primes $p \neq 2, 3, 7, 73, 109, 829, 967$ where $N_\nu = \#X(\mathbb{F}_{p^\nu})$. The numerator in the above expression equals

$$h_p(x) := \det\left(1 - x\rho(\mathrm{Frob}_p) \mid H^1_{\mathrm{et}}(X \otimes \overline{\mathbb{Q}}, \mathbb{Q}_l)\right), \quad l \neq p$$

where $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSp}(H^1_{\mathrm{et}}(X \otimes \overline{\mathbb{Q}}, \mathbb{Q}_l))$ is the canonical Galois representation in étale cohomology, and Frob_p=Frobenius.

Zeta function and Galois representation for X

Since the Jacobian of X has endomorphisms in the field $K = \mathbb{Q}(\zeta_7 + \overline{\zeta}_7)$, this Galois representation is of $\mathbf{GL}_2$-type. This implies that the characteristic polynomials $h_p(x)$ factor as $g_p(x) g_p^{\sigma}(x) g_p^{\sigma^2}(x)$ for a quadratic polynomial $g_p(x) \in \mathbb{Z}_K[x]$, where $\mathbb{Z}_K = \mathbb{Z}[t]/(t^3 + t^2 - 2t - 1)$ is the ring of integers of K and σ generates the Galois group of K over $\mathbb{Q}$.

p	$g_p(x)$	Trace
5	$1 - tx + 5x^2$	-1
11	$1 - tx + 11x^2$	-1
13	$1 + (3 - t)x + 13x^2$	-10
17	$1 + (-1 - 4t)x + 17x^2$	-1
19	$1 + (6 - 3t - 2t^2)x + 19x^2$	-11
23	$1 + (8 - t - 3t^2)x + 23x^2$	-10
29	$1 + (8 - 5t - 6t^2)x + 29x^2$	1
31	$1 + (7 - t - 2t^2)x + 31x^2$	-12
37	$1 + (6 - 4t - 5t^2)x + 37x^2$	3
41	$1 + 8x + 41x^2$	-24
43	$1 + (4 - t - 2t^2)x + 43x^2$	-3

Table : Factorization of $h_p(x) = g_p(x)g_p^{\sigma}(x)g_p^{\sigma^2}(x)$, trace of Frob_p at good primes. $\mathbb{Z}_K = \mathbb{Z}[t]/(t^3 + t^2 - 2t - 1)$.

p	$g_p(x)$	Trace
47	$1+(10-t-4t^2)x+47x^2$	-11
53	$1+(6+2t-5t^2)x+53x^2$	9
59	$1+(10-6t-9t^2)x+59x^2$	9
61	$1+(-2+3t)x+61x^2$	9
67	$1+(4-t-2t^2)x+67x^2$	-3
71	$1+(10-4t-5t^2)x+71x^2$	-9
79	$1+(7-8t-9t^2)x+79x^2$	16
83	$1+(1-3t-6t^2)x+83x^2$	24
89	$1+(19-t-11t^2)x+89x^2$	-3

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