Weil representations over abelian varieties

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- They 'encode' the transformation laws of theta functions.

Example: one-variable theta functions of rank 1 lattices

Let $q = e^{2\pi i\tau}, \tau \in \mathfrak{h}, m \in 2\mathbb{Z}_{>0}$.

$$\theta_{m,0}(q) = \sum_{n\in\mathbb{Z}} q^{\frac{m}{2}n^2}$$

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$$\theta_{\mathrm{null},m}(q) = \left(\sum_{\substack{n\equiv\nu \mod m \\ n\in\mathbb{Z}}} q^{n^2/2m} \right)_{\nu\in\mathbb{Z}/m\mathbb{Z}}$$

- Let $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi \right) \in \mathrm{Mp}_2(\mathbb{Z}), \quad \phi^2 = c\tau + d.$

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  where

  $$\rho_m : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[\mathbb{Z}/m\mathbb{Z}])$$

  is the Weil representation attached to the quadratic form $x \mapsto mx^2/2$.

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$$\rho_m(T)(\delta_\nu) = e^{-\pi i \nu^2/m}\, \delta_\nu$$
$$\rho_m(S)(\delta_\nu) = \frac{\sqrt{i}}{\sqrt{m}} \sum_{\mu \in \mathbb{Z}/m\mathbb{Z}} e^{2\pi i \nu\mu/m}\, \delta_\mu$$

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Let $q = e^{2\pi i\tau}, \tau \in \mathfrak{h}$, (L, Q) a positive-definite rank r (even) lattice.

$$\theta_{L,0}(q) = \sum_{\lambda \in L} q^{Q(\lambda)}$$

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$$\rho_m(T)(\delta_\nu) = e^{-2\pi i Q(\nu)}\, \delta_\nu$$

$$\rho_m(S)(\delta_\nu) = \frac{\sqrt{i}^r}{\sqrt{|L'/L|}} \sum_{\mu \in L'/L} e^{2\pi i B(\nu,\mu)}\, \delta_\mu$$

Further examples

- Let $(\mathbb{C}^g/\Lambda, H)$ be a complex torus with a symmetric principal polarization.

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where $T \in \mathfrak{h}_g$.
- For $k \in 2\mathbb{Z}_{>0}$, let
$$\theta_{\mathrm{null},H^k} = \left\{ \sum_{\lambda \in \mathbb{Z}^g} e^{2\pi i \langle \lambda + c_1, T(\lambda + c_1) \rangle} \right\}_{c_1 \in \frac{1}{k}\mathbb{Z}^g/\mathbb{Z}^g}$$

Geometric interpretations

- André Weil, *sur certains groupes d'opérateurs unitaires* (1964):

> *A force d'habitude, le fait que les séries thêta définissent des fonctions modulaires a presque cessé de nous étonner. Mais l'apparition du groupe symplectique comme un* deus ex machina *dans les célèbres travaux de Siegel sur les formes quadratiques n'a rien perdu encore de son caractère mystérieux.*

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Question

Can we construct Weil representations *geometrically*?

Heisenberg groups

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- Let S be a noetherian scheme and let $H \to S$ be a commutative finite flat group scheme.
- $$\mathscr{G}_H := \mathbb{G}_m \times H \times \widehat{H},$$
  with group law given by
  $$(\lambda_1, x_1, y_1) \cdot (\lambda_2, x_2, y_2) = (\lambda_1 \lambda_2 \langle x_2, y_1 \rangle, x_1 + x_2, y_1 + y_2).$$

Lift H to a subgroup of $\mathscr{G}_H$:

$$\begin{aligned} H &\longrightarrow \mathscr{G}_H \\ x &\longmapsto (1, x, 0) \end{aligned}$$

Definition

The *Schrödinger representation* of $\mathscr{G}_H$ is the $\mathcal{O}_S$-module $\mathcal{S}_H$ of functions $f : \mathscr{G}_H \to \mathcal{O}_S$ such that, for all $g \in \mathscr{G}_H$:

(i) $f(\lambda g) = \lambda f(g)$, for all $\lambda \in \mathbb{G}_m$,

(ii) $f(hg) = f(g)$, for all $h \in H \subseteq \mathscr{G}_H$,

together with $\mathscr{G}_H$-action $\rho : \mathscr{G}_H \longrightarrow \underline{\mathrm{GL}}(\mathcal{S}_H)$ given by

$$\rho(g')f(g) := f(gg').$$

Functoriality of Schrödinger representations

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathscr{G}_H & \xrightarrow{p} & H \times \widehat{H} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \bar{\sigma}} & & \\ 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathscr{G}_{H'} & \xrightarrow{p'} & H' \times \widehat{H'} & \longrightarrow & 0 \end{array}$$

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Theorem (Stone-von Neumann)

There is an invertible $\mathcal{O}_S$-module $\mathcal{I}$ with trivial $\mathscr{G}_H$-action and a $\mathscr{G}_H$-module isomorphism

$$\mathcal{S}_H \otimes \mathcal{I} \simeq \mathcal{S}_{H'}$$

intertwining ρ and $\rho' \circ \sigma$.

Definition

Let $\mathscr{G}_H$ be a Heisenberg group. The *Schrödinger algebra* of $\mathscr{G}_H$ is the $\mathscr{G}_H \times \mathscr{G}_H$-module given by

$$\mathcal{A}_H := \mathrm{End}_{\mathcal{O}_S}(\mathcal{S}_H).$$

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Theorem

Let $\sigma : \mathscr{G}_H \to \mathscr{G}_{H'}$ be a morphism of Heisenberg groups. Then σ induces a canonical $\mathcal{O}_S$-algebra isomorphism

$$\sigma_{\mathcal{A}} : \mathcal{A}_H \xrightarrow{\simeq} \mathcal{A}_{H'},$$

intertwining the $\mathscr{G}_H \times \mathscr{G}_H$-actions.

Canonical involutions

Any Heisenberg group is equipped with a canonical order 2 automorphism:

$$\begin{aligned} \iota : \mathscr{G}_H &\longrightarrow \mathscr{G}_H \\ (\lambda, x, y) &\longmapsto (\lambda^{-1}, -x, y). \end{aligned}$$

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There is a canonical $\mathscr{G}_H$-module isomorphism

$$\mathcal{S}_H^{\iota} \simeq \mathcal{S}_H^{\vee}$$

intertwining $\rho \circ \iota$ and $\rho^{\vee}$.

Refining stone-von Neumann

Suppose

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commutes with the involutions (σ is *symmetric*):

$$\begin{array}{ccc} \mathscr{G}_H & \xrightarrow{\sigma} & \mathscr{G}_{H'} \\ \downarrow \iota & & \downarrow \iota \\ \mathscr{G}_H & \xrightarrow{\sigma} & \mathscr{G}_{H'} \end{array}$$

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Sketch.

$$\mathcal{S}_H^{\iota} \otimes \mathcal{I} \simeq \mathcal{S}_{H'}^{\iota} \simeq \mathcal{S}_{H'}^{\vee} \simeq \mathcal{S}_H^{\vee} \otimes \mathcal{I}^{-1} \simeq \mathcal{S}_H^{\iota} \otimes \mathcal{I}^{-1}$$

and take H-invariants. □

Azumaya algebras point of view

- To an Heisenberg group $\mathscr{G}_H$, we have functorially attached a (trivial) Azumaya algebra
$$\mathcal{A}_H : S \longrightarrow B\mathrm{PGL}$$
- If morphisms $\mathscr{G}_H \to \mathscr{G}_{H'}$ are involution-preserving, then we have functorially attached a 'order 2 Azumaya algebra':
$$\mathcal{A}_H : S \longrightarrow B\mathrm{GL}/\{\pm 1\}.$$

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- Locally (for the étale topology)

$$\mathscr{G}(\mathcal{L}) \simeq \mathscr{G}_H$$

where

$$K(\mathcal{L}) \simeq H \times \widehat{H}$$

Glueing Schrödinger algebras

Definition

The *theta algebra* $\mathcal{A}_{\mathcal{L}}$ is the $\mathcal{O}_S$-algebra with $\mathscr{G}(\mathcal{L})$-action obtained by glueing the Schrödinger algebras

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Theorem

Let $\mathcal{L}$ be totally symmetric. Then $\mathcal{A}_{\mathcal{L}}^{\otimes 2}$ is the endomorphism algebra of a vector bundle over S.

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Let $\mathcal{L}$ be totally symmetric. Then $\mathcal{A}_\mathcal{L}^{\otimes 2}$ is the endomorphism algebra of a vector bundle over S.

Proof.

$$\mathcal{A}_\mathcal{L}^{\otimes 2} \simeq \mathrm{End}_{\mathcal{O}_S}(\mathcal{S}_H^{\otimes 2})$$

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- Azumaya algebra of 'order n': $\mathcal{A}_1^{\otimes n} \simeq \mathrm{End}_{\mathcal{O}_S}(\mathcal{V})$.

Theta algebras of order 2

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- If $\mathcal{L}$ is totally symmetric,

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i.e. $\mathcal{A}_{\mathcal{L}}$ is of order 2.

Torsor-lifting

Question

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$$\begin{array}{ccc} S_{\mathcal{L}} := S \times_{\mathcal{A}_{\mathcal{L}}} B\mathrm{GL} & \xrightarrow{\mathcal{W}_{\mathcal{L}}} & B\mathrm{GL} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mathcal{A}_{\mathcal{L}}} & B\mathrm{GL}/\{\pm 1\} \end{array}$$

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Definition

The vector bundle $\mathcal{W}_{\mathcal{L}}$ over the $\{\pm 1\}$-gerbe $S_{\mathcal{L}}$ is the *Weil bundle* attached to $\mathcal{L}$.

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Examples

- E.g. $g = 1$, $\mathcal{E} \to \mathscr{M}_1$, $m \in 2\mathbb{Z}_{>0}$, $\mathcal{L} = \mathcal{O}_{\mathcal{E}}(m0_{\mathcal{E}})$ (+ normalization),

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Examples

- E.g. $g = 1$, $\mathcal{E} \to \mathscr{M}_1$, $m \in 2\mathbb{Z}_{>0}$, $\mathcal{L} = \mathcal{O}_{\mathcal{E}}(m0_{\mathcal{E}})$ (+ normalization),

$$\rho_{\mathcal{L}} = \rho_m : \mathrm{Mp}_2(\mathbb{Z}) \longrightarrow \mathrm{GL}(\mathbb{C}[\mathbb{Z}/m\mathbb{Z}])$$

- E.g. $g = r$, (L, Q) any (even) lattice of rank r, $\mathcal{A} = \mathcal{E}^r \to \mathscr{M}_1$, $\mathcal{L} = \mathcal{L}_Q$

$$\rho_{\mathcal{L}} = \rho_L : \mathrm{Mp}_2(\mathbb{Z}) \longrightarrow \mathrm{GL}(\mathbb{C}[L'/L])$$

- E.g. $\mathcal{L} = H^k$, even powers of a symmetric principal polarization.

Mumford's algebraic theta functions

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>
> *There are several interesting topics which I have not gone into in this paper, but which can be investigated in the same spirit: for example, [...] a discussion of the transformation theory of theta-functions.*

The Ideal Theorem

Let $\mathcal{L}$ be a normalized, totally symmetric, relatively ample line bundle over an abelian scheme (stack) $\pi : \mathcal{A} \to S$.

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Theorem (Ideal Theorem)

There is a canonical isomorphism

$$\mathcal{W}_{\mathcal{L}}^{\vee} \otimes \underline{\omega}_{\mathcal{L}}^{-1/2} \simeq \pi_* \mathcal{L}$$

of locally free modules of rank d over S, where $\underline{\omega}_{\mathcal{L}}^{-1/2}$ *is a square root of the inverse of the Hodge bundle*

$$\underline{\omega} := \det(\pi_* \Omega^1_{A/S})$$

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- (Dual of the) Ideal Theorem:

Transformation Laws of Theta Functions

$$\mathcal{W}_{\mathcal{L}} \otimes \underline{\omega}_{\mathcal{L}}^{1/2} \simeq (\pi_*\mathcal{L})^\vee$$

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Theorem (Ideal Theorem, extended)

There is a canonical isomorphism

$$\mathcal{W}_{\mathcal{L}}^{\vee} \otimes \underline{\omega}_{\mathcal{L}}^{-1/2} \simeq R^{i(\mathcal{L})}\pi_* \mathcal{L}$$

of locally free modules of rank d over S, where $i(\mathcal{L})$ is the index of the line bundle.

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- By SVN:

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$$\begin{aligned} \mathcal{W}_{\mathcal{L}} \otimes \mathcal{I} &\simeq R^{g-i(\mathcal{L})}\pi_*\mathcal{L}^{-1} \\ &\simeq (R^{i(\mathcal{L})}\pi_*\mathcal{L})^{\vee} \otimes \underline{\omega}^{-1} \\ &\simeq \mathcal{W}_{\mathcal{L}} \otimes \mathcal{I}^{-1} \otimes \underline{\omega}^{-1} \end{aligned}$$

Take H-invariants: $\mathcal{I}^{\otimes 2} \simeq \underline{\omega}^{-1}$.