GEOMETRIC CONSTRUCTIONS WITH ELLIPSES

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ABSTRACT. Geometric construction with a straight edge, compass, and other curves and devices was a major force in the development of mathematics. In this paper the authors examine constructions with ellipses. Trisections using hyperbolas were known to Papus and a trisection construction using a parabola was found by Descartes. The authors give a trisection using ellipses.

1. Introduction

Mathematicians and philosophers of Ancient Greece studied the problem of trisecting a general angle, doubling the cube, and squaring the circle. They tried to accomplish these constructions using only a straight edge and compass. While these methods were unsuccessful, they also examined allowing other constructions, devices, and curves. For example, Archimedes is usually credited with a mechanical device to trisect angles and with a construction using the spiral of Archimedes. Pappus wrote of trisection constructions by Apollonius using conics, and he gave two constructions with hyperbolas. Menaechmus, the discoverer of conic sections, is supposed to have made his discovery while working on the problem of doubling the cube. A collection of these constructions are available on the University of St. Andrew's website [1] There were ancient solutions to the problems of trisecting a general angle and doubling the cube, using a hyperbola and parabola. In this paper we examine constructions using ellipses. Among the results we show is that one can trisect a general angle and double the cube using ellipses.

Another trisection construction is due to René Descartes in his 1637 *La Geometrie*. This construction uses a parabola and a circle relying on the the triple angle formula [2]. It is similar in spirit to the trisection via ellipses given in this paper. A version of Descartes's trisection is shown in Figure 1. The construction uses the parabola defined by $y = 2x^2$ and the circle through the origin with center $(1/2\cos(\theta), 1)$. Solving

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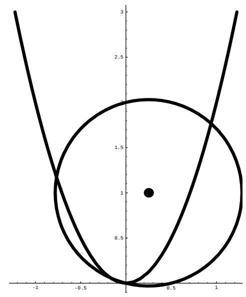

FIGURE 1. Descartes's trisection of $\theta = \frac{\pi}{3}$.

for the x-coordinates of the points of intersection gives the equation $x(4x^3 - 3x - \cos(\theta)) = 0$ or $x(x - \cos(\theta))(x - \cos(\theta + 2\pi/3))(x - \cos(\theta + 4\pi/3))$, since a triple angle formula is $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$. The trisection for $\pi/3$ is shown in Figure 1. There are four distinct points of intersection although two are very close together.

Recent work by Carlos R. Videla explored the concept of constructability when all conics are allowed. Extending straight edge and compass constructions with all the conics, gives Videla's notion of conically constructible. Videla allows the construction of a conic when the focus, directrix, and eccentricity are constructible. The conically-constructible numbers may be obtained by parabolas and hyperbolas alone. These conic constructions include ancient constructions of Papus and Menaechmus. Videla made the case that allowing a noncircular conic to be constructed if its directrix, focus, and eccentricity are constructible is consistent with the constructions of the Ancient Greeks. Ellipses are treated as extraneous as all the constructions can be accomplished with parabolas and hyperbolas. Among the results of this paper, is that all the conic constructions can be accomplished with ellipses alone.

We propose a mechanical device that will allow us to draw ellipses. The device consists of pins and string and allows the construction of an ellipse given the foci and a point on the ellipse. This approach is one way to construct ellipses. Allowing the construction of an ellipse given its directrix, focus, and eccentricity is another. These two approaches will be shown to allow the same constructions.

2. Classical Constructions

The proof that it is impossible to trisect a general angle, double the cube, or square the circle with a straight edge and compass alone is a wonderful demonstration of the power of abstraction and beauty of modern mathematics. We think of the plane as the Cartesian plane with coordinates. Start with an initial set of points P in the plane. The initial set of points should include $(0,0)$ and $(1,0)$. The set of points one may derive using only a straight edge and compass will be called *classically constructible points derived from* P. If the set of initial points P is clear or if $P = \{(0,0),(1,0)\}$, then we suppress the reference to P. The set of all numbers that arise as the ordinate or abscissa of classically constructible points is the set of *classically constructible numbers*. Consider, for example, the question of whether one can trisect an angle of measure θ. If $A = (\cos(\theta), \sin(\theta))$ is in the original set of points P, then the angle determined by $\angle ABC$ where $B = (0,0)$ and $C = (1,0)$ has measure θ. The angle can be trisected if $(\cos(\theta/3)$ is a classically constructible number from P because then one may construct the point $A' = (\cos(\theta/3), \sin(\theta/3))$ and $\angle A'BC$.

We recall some of the facts about classically constructible numbers taking some of the background notions from Hungerford [4]. It is one of the founding observations that starting with an initial set of points P the classically constructible numbers form a field, *i.e.*, using straight edge and compass constructions one may start with two numbers and construct the sum, difference, product and quotient. Furthermore, if (x,y) is a classically constructible point, it is a simple exercise to show that (y,x) is a classically constructible point. It is useful to introduce the notion of the plane of a field. If F is a subfield of the real numbers $\mathbb{R}$, then the *plane of* F is the subset of $\mathbb{R}^2$ consisting of all (x,y) with $x, y \in F$. If P and Q are distinct points in the plane of F then the line determined by P and Q is a *line in* F. Similarly, if P and Q are distinct points in F then the circle with center P and containing Q is a *circle in* F. It is a straightforward calculation that the intersection points of two lines in F are points in the plane of F. Furthermore if a circle in F is intersected with either a line in F or a circle in F, then the intersection points are in the plane of $F(r)$, where r is the square root of an element of F [4].

Suppose one starts with an initial set of points P. The coordinates of these points generate a subfield K of $\mathbb{R}$. Since P contains $(0,0)$ and $(1,0)$, K contains the rational numbers $\mathbb{Q}$. The field of classically constructible numbers derived from P can be completely characterized. It is the smallest subfield of $\mathbb{R}$ containing K in which every positive number has a square root. We will also call this field the *field of classically constructible numbers derived from* K.

3. Elliptic Constructions

Consider the following description of a mechanical device that will allow the construction of an ellipse when one has the foci and a point on the ellipse accessible. The device consists of a length of string with a clip, which may be clipped to form a loop, and some pins.

(1) Given three points, one may insert pins in the three points, tighten the string around the pins, and remove one pin. Keep the string taut and use a pen to draw the curve around the two pins. The result is a curve whose distance from any point on the curve to one of the pins plus the distance from the same point on the curve to the other pin is constant—an ellipse with the pins located at the foci, which contains the third point.

Remark 3.1. If an ellipse is in standard position then its equation can be given as

$$Ax^2 + Cy^2 + F = 0,$$

where $C > A > 0$. Let $a = \sqrt{\frac{F}{A}}$, $b = \sqrt{\frac{F}{C}}$ and $c = \sqrt{a^2 - b^2}$. Then the foci of the ellipse are at $(\pm c, 0)$ and $(a, 0)$ is a point on the ellipse. The tightly stretched string wraps around the foci and reaches to a point on the ellipse. It has length $2c + 2a$. The directrix is given by $x = \frac{a^2}{c}$ and the eccentricty $e = \frac{c}{a}$.

(2) Given two points, one may insert pins in the two points, tighten the string around the pins, remove one pin, and use a pen to draw the circle with center one point and the other on the circle.

In addition we may use the straight edge.

(3) Given two points, one may draw the line through the two points.

The operations 2 and 3 have the same result as the use of a compass and straight edge. Which points can be reached by constructions with a straight edge and pins and string? We may obtain the constructible points as follows. Start with a set of points $P = P_0$. Perform every geometric construction on the set P_0 allowed by 1, 2, and 3. Adjoin to P_0 all intersections of the ellipses, circles, and lines to obtain a set of points P_1. We can repeat the process an arbitrary number of times. Once P_i is constructed, perform every geometric construction on the set P_i allowed by 1, 2, and 3. We now adjoin to P_i all intersections of the ellipses, circles, and lines to obtain a set of points P_{i+1}. The *elliptically constructible points derived from* P is the union of the P_i's. We will use the term *constructible* to mean elliptically constructible and use the term classically constructible for straight edge and compass constructions.

We call the set of numbers that are the ordinate or abscissa of points obtainable using a straightedge and pins and string, elliptically constructible. Every straightedge and compass construction is included in construction with a straightedge and pins and string. Hence given a set of points that includes $(0, 0)$ and $(1, 0)$, the elliptically constructible numbers will be a field.

Analogous to the previous definitions for classical constructions is the following. Suppose F is a subfield of the real numbers $\mathbb{R}$. If O, P and Q are distinct points in the plane of F, then the ellipse containing O with foci P and Q is an ellipse in F.

Lemma 3.2. *Suppose that F is a subfield of $\mathbb{R}$ in which every positive number has a square root. If $\cos(\theta) \in F$, then rotation of the plane by θ induces a bijection on the plane of F. If (r, s) is in the plane of F, then translation of the Cartesian plane by (r, s) induces a bijection on the plane of F.*

Proof. Note that if $\cos(\theta)$ is in F, then $\sin(\theta)$ is in F since $\sin^2\theta = 1 - \cos^2\theta$. Let f be the rotation of the plane by an angle θ, then $f(x, y) = (x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta)$. If (x, y) is in the plane of F, then the image is in the plane of F. The inverse transformation of f is rotation by $-\theta$, therefore f is a bijection.
If g is translation by (r, s), then $g(x, y) = (x + r, y + s)$ is a bijection of the plane of F with inverse translation by $(-r, -s)$. □

Lemma 3.3. *Suppose that F is a subfield of $\mathbb{R}$ in which every positive number has a square root. Suppose an ellipse is described by the equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ with $a, b, c, d, e, f \in F$. If $\cos(\theta) \in F$, then the ellipse rotated about the origin by θ can be expressed by an equation with coefficients in F.*

Proof. Let θ be the angle of rotation. The transformed ellipse is described by the equation $a'x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0$, where the coefficients are:

$$\begin{aligned}
a' &= a\cos^2(\theta) + b\sin(\theta)\cos(\theta) + c\sin^2(\theta),\\
b' &= 2(c-a)\sin(\theta)\cos(\theta) + b(\cos^2(\theta) - \sin^2(\theta),\\
c' &= a\sin^2(\theta) - b\sin(\theta)\cos(\theta) + c\cos^2(\theta),\\
d' &= d\cos(\theta) + e\sin(\theta),\\
e' &= e\cos(\theta) - d\sin(\theta),\\
f' &= f,
\end{aligned}$$

which are all field operations of elements of F. Therefore the coefficients are in F. □

Proposition 3.4. *Suppose that F is a subfield of $\mathbb{R}$ in which every positive number has a square root. Consider an ellipse E described by the equation $ax^2 + bxy + y^2 + dx + ey + f = 0$. The ellipse E is in F if and only if $a, b, d, e,$ and f are in F. If E is in F, then it may also be rotated and translated to an ellipse in F in standard position.*

Proof. Assume the ellipse is in F. Let the points (f_x, f_y) and (g_x, g_y) be the focci and (x_0, y_0) be a point on the ellipse. Let $d = \sqrt{(f_x - x_0)^2 + (f_y - y_0)^2} + \sqrt{(g_x - x_0)^2 + (g_y - y_0)^2}$ which is an element of F. The coordinates of a point on the ellipse will satisfy the equation $\sqrt{(f_x - x)^2 + (f_y - y)^2} + \sqrt{(g_x - x)^2 + (g_y - y)^2} = d$

This equation is equivalent to $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where the coefficients are:

$$
\begin{aligned}
A &= 8 + 4d^2 - 4f_x^2 - 8f_x g_x + 4g_x^2 \\
B &= 8 + 4d^2 - 8f_x^2 + 4f_y^2 - 8f_y g_y + 4g_y^2 \\
C &= 8f_x f_y - 8f_y g_x - 8f_x g_y + 8g_x g_y \\
D &= 8f_x + 4d^2 f_x - 4f_x^3 + 4f_x f_y^2 + 8g_x + 4d^2 g_x - 12f_x^2 g_x - 4f_y^2 g_x - 4f_x g_x^2 + \\
&\quad 4g_x^3 - 4f_x g_y^2 + 4g_x g_y^2 \\
E &= 8f_y + 4d^2 f_y - 4f_x^2 f_y + 4f_y^3 - 4f_y g_x^2 + 8g_y + 4d^2 g_y - 12f_x^2 g_y - 4f_y^2 g_y + \\
&\quad 4g_x^2 g_y - 4f_y g_y^2 + 4g_y^3 \\
F &= 4 + 4d^2 + d^4 - 4f_x^2 - 2d^2 f_x^2 + f_x^4 + 4f_y^2 + 2d^2 f_y^2 - 2f_x^2 f_y^2 + f_y^4 + 4g_x^2 + \\
&\quad 2d^2 g_x^2 - 6f_x^2 g_x^2 - 2f_y^2 g_x^2 + g_x^4 + 4g_y^2 + 2d^2 g_y^2 - 6f_x^2 g_y^2 - 2f_y^2 g_y^2 + 2g_x^2 g_y^2 + g_y^4
\end{aligned}
$$

Since the coefficients are obtained by field operations of elements in F, the coefficients are in F.

For the converse, let $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, be the equation of the ellipse E with coefficients in F.

We rotate and translate the ellipse to standard form. The rotation will be by an angle $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. Since the coefficients A, B, and C are in the field F the $\cot(2\theta) = \frac{A-C}{B}$ is in F. Note that if $B = 0$ then θ is 0 or $\frac{\pi}{2}$. So $\sin(2\theta) = \frac{1}{\sqrt{1+\cot^2(2\theta)}}$ is also in F. Therefore $\sin(\theta)$ is in F, and thus $\cos(\theta)$ is as well. Also $\cos(-\theta)$ and $\sin(-\theta)$ are in F, so by using Lemma 3.3 the ellipse rotated by $-\theta$ also has coefficients in F. The equation of this rotated ellipse can be written as follows:

$$\frac{(x + x_0)^2}{a^2} + \frac{(y + y_0)^2}{b^2} = 1,$$

where x_0, y_0, a and b are all gotten by field operations on the coefficients of the equation of the rotated ellipse. Translation by (x_0, y_0) gives the ellipse in standard form so by Remark 3.1 E is in F. □

Theorem 3.5. *Suppose that F is a subfield of $\mathbb{R}$ in which every positive number has a square root. If E_1 and E_2 are ellipses in F then the coordinates of the points of intersection of E_1 and E_2 are in a field $F(R)$, where R is the set of real roots of a quartic polynomial with coefficients in F. If E is an ellipse in F and L is a line in F then the coordinates of the points of intersection of E and L are in the field F.*

Proof. First consider the intersection of two ellipses. Any two constructible ellipses E_1 and E_2 have equations of the form:

$$a_1x^2 + b_1xy + y^2 + d_1x + e_1y + f_1 = 0 \tag{3.1}$$

$$a_2x^2 + b_2xy + y^2 + d_2x + e_2y + f_2 = 0 \tag{3.2}$$

By Proposition 3.4 the coefficients are in F. Subtracting these two equations leaves us with

$$(a_1 - a_2)x^2 + (b_1 - b_2)xy + (d_1 - d_2)x + (e_1 - e_2)y + f_1 - f_2 = 0$$

Solving for y,

$$y = -\frac{f_1 - f_2 + (d_1 - d_2)x + (a_1 - a_2)x^2}{(b_1 - b_2)x + (e_1 - e_2)} \tag{3.3}$$

Putting this expression back into Equation 3.1 we get

$$Ax^4 + Bx^3 + Cx^2 + Dx + E = 0 \tag{3.4}$$

Where the coefficients are:

$$\begin{aligned}
A &= -e_1f_2 + f_2^2 + e_2f_1 - 2f_2f_1 + f_1^2 \\
B &= d_1e_2 - d_2e_1 - b_1f_2 + 2d_2f_2 - 2d_1f_2 + b_2f_1 - 2d_2f_1 + 2d_1f_1 \\
C &= -b_1d_2 + d_2^2 + b_2d_1 - 2d_2d_1 + d_1^2 + a_1e_2 - a_2e_1 + 2a_2f_2 - 2a_1f_2 \\
&\quad -2a_2f_1 + 2a_1f_1 \\
D &= a_1b_2 - a_2b_1 + 2a_2d_2 - 2a_1d_2 - 2a_2d_1 + 2a_1d_1 \\
F &= a_2^2 - 2a_2a_1 + a_1^2
\end{aligned}$$

So the x-coordinates of points of intersection are real roots of a quartic polynomial with coefficients in F. If we let R be the set of real roots of Equation 3.4 then the x-coordinates of the intersection of the two ellipses is in $F(R)$. Inserting those roots into Equation 3.3 gives us the y-coordinates, but as field operations of the x-coordinates and the coefficients which are in F, so each y-coordinate is already in $F(R)$.

For the case of the intersection of a line and an ellipse, the points of intersection are roots of a quadratic polynomial in the coefficients, so they remain in F.

□

To algebraically analyze the constructible numbers, we consider a slightly different decomposition of the constructible points. Start with an arbitrary set of points P. Allow arbitrary use of the constructions using 2 and 3 to obtain the set of classically constructible points derived from P. Call the set Q_0. Allow constructions of ellipses and lines in Q_0 and adjoin the points of intersection to obtain a set of points Q'_0. Then allow arbitrary constructions using 2 and 3 to obtain the set of classically constructible points derived from Q'_0. Call this set of points Q_1. Now repeat the process. Start with the points Q_i. Allow constructions of an ellipses and lines in Q_i and adjoin the points of intersection to obtain a set of points Q'_i. Then allow arbitrary constructions using 2 and 3 to obtain the set of classically constructible points derived from Q'_i. Call this set of points Q_{i+1}. The union $\bigcup Q_i$ are the constructible points.

We analyze the constructed numbers as follows. We construct a nested sequence of fields F_i. Suppose one starts with an initial set of points P. The field of classically constructible numbers derived from P is a field F_0. Let R be the set of real roots of quartic polynomials with coefficients in F_i obtained from the intersections of ellipses in F_i. Let F_{i+1} be the classically constructible numbers derived from $R \cup F_i$. The elliptically constructible numbers derived from P is the union $\bigcup F_i$. In the chain of fields that union to the constructible numbers,

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_i \subset F_{i+1} \subset \cdots \tag{3.5}$$

classical constructions in the plane of F_i can only produce numbers in F_i while elliptic constructions in the plane of F_i may produce numbers in F_{i+1}. Each of the fields F_i in the chain has the property that every positive number in the field F_i has its square root in the field F_i.

Lemma 3.6. *Suppose that a field F has the property that every positive number in F has its square root in F. If an ellipse is in F, then the string length and major axis are in F.*

Proof. If the ellipse is in F then its focci are constructible. The line L passing through the focci intersects the x-axis or runs parallel to the x axis. If L is parallel to the x-axis we say that θ equals 0, otherwise $\cos(\theta)$, where θ is the angle between L and the x-axis, is constructible. By Lemma 3.4, if we rotate and translate the ellipse, we get another ellipse in F, which is in standard position and whose equation is

$$Ax^2 + Cy^2 + F = 0$$

with coefficients in F. By Remark 3.1, the major axis and string length are in F. □

Theorem 3.7. *Suppose that a field F has the property that every positive number in F has its square root in F. An ellipse is in F if and only if the directrix and focus are in the plane of F and the eccentricity is in F.*

Proof. We may rotate and translate an ellipse in F to get another ellipse in F, which is in standard position (as we did in the proof of Lemma 3.6). By Remark 3.1 the directrix and eccentricity are in F.

For the converse, let (f_x, f_y) be the coordinate of the focus. Consider the line perpendicular to the directrix that passes through the focus. Let (d_x, d_y) be the coordinate of the point on the directrix that the perpendicular intersects. Two points on the ellipse satsify the following two equations:

$$(f_x - x)^2 - (f_y - y)^2 = e^2((d_x - x)^2 + (d_y - y)^2)$$

$$y - d_y = \frac{d_y - f_y}{d_x - f_x}(x - d_x)$$

From these equations we get two points on the ellipse whose coordinates are the roots of quadratic equations with coefficients in F, so they are in the plane of F. On the line that passes through the focci, the distance between one point and the known focus is the same as the distance between the other point and the other focus, so the other focus is in the plane of F. Therefore, the ellipse is in F. □

Theorem 3.7 shows the equivalence of allowing the construction of an ellipse given its foci and a point on the ellipse and allowing the construction given the directrix, focus, and eccentricity. This observation is important since it may be considered the link between the ancient rules and the pins and string construction, showing that pins and string is a valid ancient Greek construction.

4. Trisection of the general angle

Start with an angle of measure θ. The points $B = (0, 0)$ and $C = (1, 0)$ are constructible points, and we may construct the angle $\angle ABC$, with $A = (\cos(\theta), \sin(\theta))$. Hence , $\cos(\theta)$ is constructible. To obtain the angle of measure $\theta/3$, we will produce the number $\cos(\theta/3)$. The number the $\sin(\theta/3)$ can then be produced as $\sqrt{1 - \cos^2(\theta/3)}$.

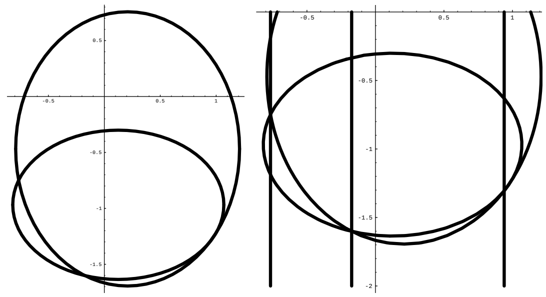

FIGURE 2. Trisection of $\theta = \frac{\pi}{3}$

The angle $\angle A'BC$ with $A' = (\cos(\theta/3), \sin(\theta/3))$ trisects the $\angle ABC$. Therefore given $q = \cos(\theta)$, we must produce $\cos(\theta/3)$. By a triple angle formula, $\cos(\theta/3)$ is a real solution to the equation

$$4x^3 - 3x - q = 0.$$

The other roots are $\cos(\theta/3 + 2\pi/3)$ and $\cos(\theta/3 + 4\pi/3)$.

Theorem 4.1. *The general angle can be trisected.*

Proof. Suppose $\cos(\theta) \in F_i$, where F_i is one of the fields in the chain 3.5 that union to the constructible numbers. We show that $\cos(\theta/3) \in F_{i+1}$. Let $q = \cos(\theta)$ and consider the following ellipses with coefficients in F_i:

$$2x^2 + 4y^2 - qx + 2py + 2 = 0$$

$$6x^2 + 4y^2 + (2p-4)y - (2+q)x - p - 1 = 0. \tag{4.1}$$

We can use Equation 3.4 to find a polynomial whose real roots are the x-coordinates of the points of intersection. It is

$$4x^4 - 4x^3 - 3x^2 + (3-q)x + q = 0$$

which factors as $\frac{1}{4}(4x^3 - 3x - q)(x - 1) = 0$. One of the solutions is $\cos(\theta/3)$. □

The example of the trisection of $\theta = \frac{\pi}{3}$ is shown in Figure 2. In Equations 4.1, $q = \frac{1}{2}$. There are four distinct points of intersection, although two are very close together.

5. Cube roots and doubling the cube

The ability to construct a cube whose volume is double a given cube is the same as the ability to multiple a side length by the $\sqrt[3]{2}$. Of course, $2 \in \mathbb{Q} \subset F_0$. where F_0 is the first field in 3.5.

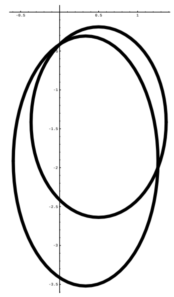

FIGURE 3. Constructing the $\sqrt[3]{2}$

Theorem 5.1. *If a is a constructible number, then so is $\sqrt[3]{a}$.*

Proof. Suppose $a \in F_i$. Consider the following equations of ellipses with coefficients in F_i:

$$2x^2 + y^2 - ax + 2\sqrt{2}y + 1 = 0 \tag{5.1}$$

$$3x^2 + y^2 - ax + (1 + 2\sqrt{2})y + \sqrt{2} = 0$$

We can use Equation 3.4 to find the polynomial whose real roots are the x-coordinates of the points of intersection:

$$x^4 - ax = 0.$$

The real roots are 0 and $\sqrt[3]{a}$. □

To doubling the cube, we need to let $a = 2$ in Equations 5.1. This gives the two ellipses shown in Figure 3.

6. Regular Polygons and the Constructible Numbers

Gauss determined that a regular n-gon can be constructed with a straight edge and compass if $\phi(n)$ is a power of 2, where ϕ is the Euler ϕ function. The first complete proof that a regular n-gon is classically constructible if and only if $\phi(n)$ is a power of 2 was by Pierre Wantzel in 1837. The regular polygons that can be constructed with a straight edge and pins and string can also be determined.

To outline which points in the plane are constructible requires a foray into the complex numbers. This argument is the same as in [3]. We interpret the points in the plane as the Gaussian plane of complex numbers, *i.e.*, we interpret (x, y) as $x + iy$. The essential needed points are that we can trisect the general angle and construct cube roots of a constructible (real) number. We can then construct the cube roots of a constructible (complex) number, *e.g., one* $\sqrt[3]{re^{\theta i}}$ *is* $\sqrt[3]{r}\, e^{\frac{\theta}{3}i}$*. We can* *construct the square root by ruler and compass. Since fourth degree polynomials can be solved by use of cube and square roots, the field of constructible complex numbers is the smallest subfield of* $\mathbb{C}$ *closed under conjugation, square roots and cube roots.*

Theorem 6.1. *A regular polygon with* n *sides is constructible if and only if* $\phi(n) = 2^s 3^t$.

The proof follows the familar proof quadratic extensions and is given in [3]*. The regular heptagon is elliptically constructible but not classically constructible*

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