Exercise 0.0.1. Prove the following theorem of Steinhaus. Suppose $A \subset \mathbb{R}$ is Lebesgue measurable and suppose its Lebesgue measure $l(A) > 0$. Denote $A - A = \{x - y \mid x \in A, y \in A\}$. Prove that $A - A$ contains an open interval around 0. (Hint: Let $f(x) = l((x + A) \cap A)$. Show that f is a continuous function of x, and find $f(0)$.)

With the help of Exercise 0.0.1 we can prove a generalization of Example **??**. We will show that every subset $P \subseteq \mathbb{R}$ of strictly positive measure must contain a non-measurable set.

Theorem 0.0.1. *If $P \in \mathcal{L}(\mathbb{R})$ and if $l(P) > 0$, then there exists a non-measurable subset of P.*

Proof. Let $\mathbb{Q}$ denote the set of all rational numbers, as usual. We will define again an equivalence relation on $\mathbb{R}$ by

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}.$$

By the Axiom of Choice, we can find a cross-section Γ [1] of the quotient space $\mathbb{R}/\sim$. Thus we can express the real line as the following disjoint union:

$$\mathbb{R} = \dot{\bigcup}_{q \in \mathbb{Q}} (\Gamma + q)$$

since if $\gamma + q = \gamma' + q'$ then $\gamma - \gamma' \in \mathbb{Q}$. This would make $\gamma \sim \gamma'$ and thus $\gamma = \gamma'$ since Γ is a cross-section of $\mathbb{R}/\sim$.

If $P \cap (\Gamma + q)$ were not Lebesgue measurable for some $q \in \mathbb{Q}$, then we would be finished because P would have a non-measurable subset. However, if $P \cap (\Gamma + q) = P_q \in \mathcal{L}(\mathbb{R})$ for each $q \in \mathbb{Q}$ then we observe that

$$P_q - P_q \subseteq (\Gamma + q) - (\Gamma + q) = \Gamma - \Gamma$$

where $\Gamma - \Gamma$ is disjoint from the dense set $\mathbb{Q} \setminus \{0\}$ for the reasons explained just above. Thus the difference $P_q - P_q$ of a supposedly Lebesgue measurable set with itself fails to include any open interval around 0. By Steinhaus's theorem (Exercise 0.0.1) P_q must have measure zero. Hence P itself is the union of countably many disjoint null sets, which contradicts the hypothesis that $l(P) > 0$. $\square$

[1] That is, Γ contains exactly one element of each equivalence class in $\mathbb{R}$.