6.2 Fubini's Theorem

Theorem 6.2.1. *(Fubini's theorem - first form) Let $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ be complete σ-finite measure spaces. Let $\mathfrak{C} = \mathfrak{A} \bigotimes \mathfrak{B}$. Then for each $\mu \times \nu$-measurable set $C \in \mathfrak{C}$ the section ${}_xC$ is measurable for almost all x, the function $f_C(x) = \nu({}_xC)$ is $\mathfrak{A}$-measurable, and*

$$(\mu \times \nu)(C) = \int_X f_C(x)\, d\mu(x). \tag{6.2}$$

Proof. Note that $(X \times Y, \mathfrak{A} \bigotimes \mathfrak{B}, \mu \times \nu)$ must be σ-finite as well, so that

$$X \times Y = \bigcup_{n \in \mathbb{N}} X_n \times Y_n$$

where $\mu(X_n)\nu(Y_n) < \infty$ and $X_n \times Y_n$ is an ascending chain of sets that are rectangular and thus elementary and Borel as well. We remark that the first form of Fubini's theorem expresses the product measure $\mu \times \nu$ of a set $C \in \mathfrak{C}$ as the integral with respect to μ of the ν-measures of the x-sections of C. This includes the possibility of both sides of Equation 6.2 being infinite.

Observe that the theorem follows easily from the definition of the product measure on the field $\mathfrak{E}$ of elementary sets in the special case in which C is an elementary set in the product space. For the latter sets, each section ${}_xC$ is measurable as well, being a union of finitely many measurable subsets of Y.

Next we wish to prove the theorem in the case that C is a Borel set. By Theorem 2.1.2 we see that the theorem would be true for all Borel sets C provided that the family $\mathcal{F}$ of sets for which the theorem is true forms a *monotone class.*

i. We will prove first that $\mathcal{F}$ is closed under the operation of forming the union of an increasing chain of sets in $C_n \in \mathcal{F}$. This conclusion will follow from the Monotone Convergence theorem, together with the fact that the pointwise limit almost everywhere of measurable functions must be measurable in a complete measure space. Let $C = \bigcup_n C_n$. We let $f_n(x) = \nu({}_xC_n)$, which is an monotone increasing sequence of measurable functions defined almost everywhere. We observe that $f_n \to f$ where $f(x) = \nu({}_xC)$ because ${}_xC$ is the union of the increasing chain ${}_xC_n$ of measurable sets, and because ν is countably additive. Hence

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu = \lim_{n\to\infty} (\mu \times \nu)(C_n) = (\mu \times \nu)(C)$$

by countable additivity of the product measure $\mu \times \nu$.

ii. For a decreasing nest C_n, we limit ourselves first to a typical subspace $S_N = X_N \times Y_N$ of finite product-measure. Thus we assume at first that each $C_n \subseteq S_N$, which has finite measure. Define $f_n(x) = \nu({}_xC_n)$ as before, and observe that f_1 is an integrable function dominating the decreasing sequence f_n, and $f_n(x) \to f(x) = \nu({}_xC)$ for almost all x. Then the Lebesgue Dominated Convergence theorem implies that

$$\int_X f\,d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu = \lim_{n\to\infty} (\mu \times \nu)(C_n) = (\mu \times \nu)(C).$$

This shows that for each $N \in \mathbb{N}$ we have $\mathcal{F} \cap \mathfrak{P}(S_N)$ is a monotone class within the power set of S_N, and thus also a σ-field. Hence $\mathcal{F}$ contains all the Borel sets in S_N. However, the Borel sets of $S = X \times Y$ will be the unions of their own intersections with the Borel sets in each of the Borel sets S_N. Hence $\mathcal{F} \supseteq \mathfrak{A} \otimes \mathfrak{B}$ by part (i), in which we showed that $\mathcal{F}$ is closed under forming unions of ascending chains. Moreover, every section ${}_xC$ of a Borel set must be measurable, as shown in the proof of part (ii) of Theorem 6.1.3.

To complete the proof of this theorem for all measurable sets C, recall that each measurable set differs from a Borel set by a null-set. Thus it would suffice to prove that the theorem applies to all C that are null sets[2] with respect to the measure ν. If $\nu(C) = 0$ then there exists a Borel set $D \supseteq C$ such that $\nu(D) = 0$ also. By the previous part of this proof, it follows that

$$(\mu \times \nu)(D) = \int_X \nu({}_xD)\,d\mu = 0.$$

Hence $\nu({}_xD) = 0$ for almost all x, and thus ${}_xC \subseteq {}_xD$ is both a measurable set and a ν-null set for almost all x. It follows that $(\mu\times\nu)(C) = \int_X \nu({}_xC)\,d\mu$, and the proof is complete. □

Theorem 6.2.2. *(Fubini's theorem - main form) Let* $(X, \mathfrak{A}, \mu)$ *and* $(Y, \mathfrak{B}, \nu)$ *be two complete* σ*-finite measure spaces. Suppose* f *is an integrable function on* $X \times Y$*. Then*

[2] One should note here that it is not necessary for each cross section of a null set in the product measure to be measurable. For example, if M is non-measurable in Y and if N is a null set in X, the $N \times M$ is a null set in $X \times Y$. Recall that every set of positive measure contains a non-measurable set, by Theorem 3.4.4.

i. For almost all $x \in X$, the function f_x given by $f_x(y) = f(x,y)$ is integrable on Y.

ii. For almost all $y \in Y$, the function f_y given by $f_y(x) = f(x,y)$ is integrable on X.

iii. The function $\int_Y f(x,y)\, d\nu(y)$ is integrable on X.

iv. The function $\int_X f(x,y)\, d\mu(x)$ is integrable on Y.

v. $\int_X \left(\int_Y f\, d\nu\right) d\mu = \int_{X\times Y} f\, d(\mu\times\nu) = \int_Y \left(\int_X f\, d\mu\right) d\nu.$

Proof. It will suffice to prove (1), (3), and the first equality in (5). If the conclusions are true for two functions then they are true also for the difference of the two functions. Hence it suffices to prove the statements listed for non-negative functions, because we can write $f = f^+ - f^-$. It follows easily from Theorem 6.2.1 that the claims are true if f is the indicator function of a measurable set of finite measure. Thus the theorem is true if f is a special simple function. By Exercise 5.4.5 we know that if f is measurable and non-negative then f is the pointwise limit of a monotone increasing sequence of special simple functions:

$$f = \lim_n \phi_n \tag{6.3}$$

a monotone increasing sequence of special simple functions.

The function f_x is a measurable function of y for almost all x, being the pointwise limit of a sequence of functions $(\phi_n)_x$ that are measurable for almost all x. There is a different null-set S_n for each n of values of x for which $(\phi_n)_x$ is not measurable, but the union $\bigcup_{n\in\mathbb{N}} S_n$ of countably many null sets is still a null set. It follows that

$$\int_Y f(x,y)\, d\nu(y) = \lim_n \int_Y \phi_n(x,y)\, d\nu(y)$$

by the monotone convergence theorem for almost all x.

Thus the integral is a measurable function of x and it follows again from monotone convergence that

$$\int_X \left(\int_Y f\, d\nu\right) d\mu = \lim_n \int_X \left(\int_Y \phi_n\, d\nu\right) d\mu$$

$$= \lim_n \int_{X\times Y} \phi_n \, d(\mu \times \nu) = \int_{X\times Y} f \, d(\mu \times \nu).$$

□

Corollary 6.2.1. *Theorem 6.2.2 is true without the assumption that f is integrable on $X \times Y$ provided that f is both measurable and non-negative.*

Proof. These hypotheses are sufficient for the validity of Equation 6.3. The remainder of the proof is based on the Monotone Convergence theorem, which does not require integrability. □

Remark 6.2.1. Corollary 6.2.1 has an important practical consequence. In order to use Theorem 6.2.2 we need a way to confirm whether or not the measurable function f is integrable. Since $|f|$ is non-negative, we can calculate whether or not

$$\int_{X\times Y} |f| \, d(\lambda \times \mu) < \infty$$

by calculating the iterated integral in either order, according to convenience. Thus, if either

$$\int_X \left(\int_Y |f| \, d\nu \right) d\mu < \infty$$

or

$$\int_Y \left(\int_X |f| \, d\mu \right) d\nu < \infty$$

then f is an integrable function on the product space and the full strength of the Fubini theorem can be applied to f. If one of the two orders of iteration yields a finite result, this must be true of the other order and of the integral over the product space, because of Corollary 6.2.1.

Fubini's theorem is one of the most powerful tools in real analysis. The reason is that the interchange of order of iteration of a double integral is an interchange of order of two limit operations of the most delicate kind - namely, Lebesgue integration. Several important applications are contained in the following exercises.

Exercise 6.2.1. Suppose both f and g are L^1 functions on $\mathbb{R}^n$. In the following problems, you may use the translation-invariance of both Lebesgue measure (Exercises 3.2.5 and 3.5.2) and the Lebesgue integral on $\mathbb{R}^n$ (Exercise 5.2.3).

a. Show that

$$h(x, y) = f(x - y)g(y)$$

is an L^1 function on $\mathbb{R}^{2n}$.

b. Show that the *convolution* denoted and defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\, dl(y)$$

is defined almost everywhere in x.

c. Show that $f * g$ is an integrable function on $\mathbb{R}^n$.

d. Show that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

e. Show that $f * g = g * f$. (See Exercise 5.2.4.)

f. Show that $\widehat{(f * g)}(\alpha) = \hat{f}(\alpha)\hat{g}(\alpha)$. (See Exercise 5.6.2.)

Exercise 6.2.2. Let $g \in L^1(\mathbb{R}^n, \mathcal{L}, l)$ and define the mapping

$$T : L^1(\mathbb{R}^n, \mathcal{L}, l) \to L^1(\mathbb{R}^n, \mathcal{L}, l)$$

by $T(f) = f * g$. Prove that if $f_n \to f$ in the L^1-norm, then $T(f_n) \to T(f)$ in the L^1-norm. That is, prove that T is a continuous mapping.

Exercise 6.2.3. Let f and g be in $L^1(\mathbb{R})$, and suppose also that g is bounded: for some $M \in \mathbb{R}$ we have $|g(x)| \leq M$ for all $x \in \mathbb{R}$.

a. Prove that $f * g(x)$ is a continuous real-valued function defined for all $x \in \mathbb{R}$. That is, show that

$$|f * g(x) - f * g(x_0)| \to 0$$

as $x \to x_0$.

b. Let

$$f(x) = \frac{1}{\sqrt{|x|}} 1_{[-1,0)\cup(0,1]}(x)$$

for all x. Show that $f \in L^1(\mathbb{R})$ but $f * f$ is not continuous at $x = 0$.

Exercise 6.2.4. Suppose A and $B = -A$ be in $\mathfrak{L}$, the family of Lebesgue measurable subsets of the real line. Suppose $0 < l(A)l(B) < \infty$. Let $f = 1_A$ and let $g = 1_B$.

a. Prove that $g * f(0) > 0$. (Hint: See Exercise 5.5.5.)

b. Prove Steinhaus's theorem: There exists an interval

$$(-\delta, \delta) \subseteq A - A = \{x - z \mid x \in A,\, z \in A\}.$$

(Hint: Compare with Exercise 3.4.5.)

Exercise 6.2.5. Let $f : X \to \mathbb{R}$ be a measurable function on the complete σ-finite measure space $(X, \mathfrak{A}, \mu)$. Suppose $g(x, y) = f(x) - f(y)$ is integrable on $X \times X$. Show that f is integrable on X and calculate the numerical value of $\int_{X \times X} g\, d(\mu \times \mu)$.

Exercise 6.2.6. Suppose $(X, \mathfrak{A}, \mu)$ and $(Y, \mathfrak{B}, \nu)$ are both σ-finite complete measure spaces. Suppose $f \in L^1(X)$ and $g \in L^1(Y)$. Define

$$h(x, y) = f(x)g(y)$$

and prove that $h \in L^1(X \times Y, \mathfrak{A} \bigotimes \mathfrak{B}, \mu \times \nu)$.

Exercise 6.2.7. We investigate what is called the *essential uniqueness* of translation-invariant measures.

a. Let $(\mathbb{R}^n, \mathcal{L}, l)$ be the standard Euclidean measure space with Lebesgue measure l defined on the σ-field of Lebesgue measurable sets. By Exercise 3.5.2 we know that l is *translation-invariant*. Suppose that μ is any *other* σ-finite measure defined and translation-invariant on $\mathcal{L}$. Use Fubini's theorem to prove that $\mu = cl$ for some constant c. This is called the *essential uniqueness* of translation-invariant measure.

   Hint: To prove this with Fubini's theorem, let $E \in \mathcal{L}$ be any set of finite measure, let Q be the *unit cube*, and write

$$\mu(E) = \int_{\mathbb{R}^n} 1_Q(y)\, dl(y) \int_{\mathbb{R}^n} 1_E(x) d\mu(x).$$

   Then write this as a double integral over the product space $\mathbb{R}^{2n}$ and play with the translation-invariance of both measures. This proof is modeled on the proof of a more general case published by Shizuo Kakutani [6].

b. Let $\nu(E)$ be the number of elements (cardinality) of E, for each $E \in \mathcal{L}$. Is ν translation-invariant? Is ν a constant multiple of l? Do we have a counter-example to the essential uniqueness of translation-invariant measure on $\mathbb{R}^n$?

Exercise 6.2.8. Suppose $(X, \mathfrak{A}, \mu)$ is a complete σ-finite measure space and let $f \in L^1(X, \mathfrak{A}, \mu)$ be a real-valued integrable function. Let l denote Lebesgue measure on the real line. Apply Fubini's theorem to the space $X \times \mathbb{R}$ to prove that

$$\int_X f \, d\mu(x) = \int_{\mathbb{R}} \mu\left(f^{-1}(\alpha, \infty)\right) - \mu\left(f^{-1}(-\infty, -\alpha)\right) \, dl(\alpha).$$

The use of a powerful tool such as Fubini's theorem can produce serious errors if the tool is applied in cases that do not satisfy the hypotheses of the theorem. Here are some examples.

Exercise 6.2.9. Let $X = Y = \mathbb{N}$ the set of all natural numbers, and let $\mathfrak{A} = \mathfrak{P}(X)$, the power set of the set of natural numbers. Let $\mu = \nu$ be the ordinary *counting measure* on $\mathfrak{A}$, so that $\mu(A)$ equals the number of elements in A. Clearly, $(X, \mathfrak{A}, \mu) = (Y, \mathfrak{A}, \nu)$. Show that both spaces are σ-finite. Define $f(x, y) = \begin{cases} 2 - 2^{-x} & \text{if } x = y \\ -2 + 2^{-x} & \text{if } x = y + 1 \\ 0 & \text{if } x \notin \{y, y+1\} \end{cases}$. Show that

$$\int_X \int_Y f(x, y) \, d\nu(y) \, d\mu(x) \neq \int_Y \int_X f(x, y) \, d\mu(x) \, d\nu(y)$$

and explain why this does not violate Fubini's theorem.

Exercise 6.2.10. For $x \in \mathbf{R}^1$ and $t > 0$, let

$$f(x, t) = \frac{1}{\sqrt{2\pi t}} \, e^{-\frac{x^2}{2t}}.$$

It is well-known that for each $t > 0$, $\int_{-\infty}^{\infty} f(x, t) dx = 1$. It is also known that $2\dfrac{\partial f}{\partial t} = \dfrac{\partial^2 f}{\partial x^2}$.

If $g(x, t) = \dfrac{\partial f}{\partial t}$, prove or disprove:

$$\int_{-\infty}^{\infty} \int_s^{\infty} g(x, t) dt \, dx \neq \int_s^{\infty} \int_{-\infty}^{\infty} g(x, t) dx \, dt.$$

What is the relevance of this example to the Fubini theorem?

Exercise 6.2.11. Let $X = Y = [0,1]$. Let μ be Lebesgue measure on X and let λ be counting measure on Y. Let $f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$. Show that $\int_X \int_Y f(x,y)\, d\mu(x)\, d\lambda(y) \neq \int_Y \int_X f(x,y)\, d\lambda(y)\, d\mu(x)$ and explain why this does not violate Fubini's theorem.

6.3 Comparison of Lebesgue and Riemann Integrals

Riemann integration corresponds to the concept of Jordan measure in a manner that is similar (but not identical) to the correspondence between the Lebesgue integral and Lebesgue measure. Although it is possible for an unbounded function to be Lebesgue integrable, this cannot occur with Riemann integration. For simplicity we illustrate the comparison between the two types of integrals on the unit interval of the real line.

Let f be any bounded real-valued function on $[a,b]$. Since $f = f^+ - f^-$, a difference between two non-negative functions, it will suffice to deal with the Riemann integration of non-negative bounded functions f. Let Δ denote a set of finitely many partitioning points: $\Delta = \{a = x_0 < x_1 < \ldots < x_n = b\}$. Let $\Delta x_i = x_i - x_{i-1}$. On each of the n intervals $[x_{i-1}, x_i]$ we let $m_i = \inf f(x)$ and $M_i = \sup f(x)$. We form the so-called lower and upper sums

$$s(\Delta) = \sum_1^n m_i \Delta x_i$$

and

$$S(\Delta) = \sum_1^n M_i \Delta x_i$$

Then we define the so-called lower and upper Riemann integrals by

$$\underline{\int_a^b} f(x)\, dx = \sup_\Delta s(\Delta)$$

which is a supremum over all possible finite partitions Δ of $[a,b]$ and

$$\overline{\int_a^b} f(x)\, dx = \inf_\Delta S(\Delta).$$