Bonus Problem B10

We will define a monotone increasing function $f : [0,1] \to \mathbb{R}$ such that $f(0) = 0$, $f(1) = 1$, and f has a jump discontinuity at each rational point in $(0,1]$.

Write the countable set $\mathbb{Q} \cap (0,1] = \{q_k \mid k \in \mathbb{N}\}$. Let $f(0) = 0$, and for each $x \in (0,1]$ define

$$f(x) = \sum_{\{k \,|\, q_k \leq x\}} \frac{1}{2^k}$$

Show that f is monotone increasing from the value 0 to the value 1 on $[0,1]$ with a jump discontinuity of magnitude $\frac{1}{2^k}$ at each rational point q_k in $(0,1]$. Show that f is continuous at each irrational point in $[0,1]$.