Print Your Name Here: ____________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart phones, cell phones, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. True or False: The polynomial $p(x) = \dfrac{x^3}{10000} + 2000x^2 + 100000$ has a real root.

2. True or False: Suppose $f \in \mathcal{C}[0,1]$ and suppose $0 \leq f(x) \leq 1$ for all $x \in [0,1]$. Then there exists $c \in [0,1]$ such that $f(c) = 1 - c^2$.

3. V be the set of polynomials of degree equal to 3. True or False: V is a vector space.

4. Find $\|f\|_{\text{sup}}$ if $f(x) = x$ on $\left(-1, \frac{1}{2}\right)$.

5. True or False: There exists a function $f \in C[-1,1]$ and a sequence $x_n \in [-1,1]$ such that $f(x_n) \geq \ln n$.

6. Let $f_n(x) = \dfrac{x^2}{x^2 + n}$ for each $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$.

a. Find the $f(x) = \lim_{n \to \infty} f_n(x)$, the pointwise limit.

b. Find $\|f_n - f\|_{\text{sup}}$. Is the convergence uniform?

7. Let $p, q \in [0,1]$ and define the *indicator function* $1_{\{p,q\}}(x) = \begin{cases} 1 & \text{if } x \in \{p,q\} \\ 0 & \text{if } x \notin \{p,q\}. \end{cases}$ If $\epsilon > 0$, how small must $\|P\|$ be to ensure that the Riemann sum $P(1_{\{p,q\}}, \{\bar{x}_i\}) < \epsilon$?

8. Express $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(k\frac{1}{n}\right)^2$ as a definite integral.

9. True or Give a Counterexample: The function $|f| \in \mathcal{R}[a,b] \Leftrightarrow f \in \mathcal{R}[a,b]$.

10. Give an example of a function f for which $\overline{\int_0^1} f(x)\,dx = 1$ but $\underline{\int_0^1} f(x)\,dx = 0$.

11. Define the *indicator function* $1_{\{0.5\}}(x) = \begin{cases} 1 & \text{if } x = 0.5, \\ 0 & \text{if } x \neq 0.5. \end{cases}$ Find a partition P of $[0,1]$ for which $U(f,P) - L(f,P) < \frac{1}{8}$.

12. Suppose f and g are two functions defined on $[a,b]$. Suppose $f+g$ and $f-g$ are both Riemann integrable on $[a,b]$. True or False: Then f and g are both Riemann integrable.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Let $f(x) = \frac{1}{x}$, for all $x \neq 0$. *Prove your answers* to the parts below, making use of the *definition* of uniform continuity.

- (**i**) Is f uniformly continuous on $(1, \infty)$?
- (**ii**) Is f uniformly continuous on $(0, \infty)$?

B. Let $f_n(x) = 1 - x^n$ for each $n \in \mathbb{N}$. Decide whether or not f_n *converges uniformly* on each interval below, and *prove* your conclusion.

- (**i**) (6) $[0, 1]$
- (**ii**) (10) $[0, b]$, where $0 \leq b < 1$
- (**iii**) (10) $[0, 1)$

C. If $f \in \mathcal{R}[a, b]$, *prove:* $\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$. (Hint: Write the left side by expressing $f = f^+ - f^-$ and use the fact that f^+ and f^- are both nonnegative functions.)

Solutions and Class Statistics

1. True, because of the odd degree. See exercise 2.35.

2. True: Consider $f(x) - (1 - x^2)$. See Exercise 2.37.

3. False: V is not closed under scalar multiplication and is not closed under vector addition.

4. $\|f\|_{\sup} = 1$.

5. False: f must be bounded.

6.

a. $f(x) = \lim_{n\to\infty} f_n(x) \equiv 0$

b. $\|f_n - f\|_{\sup} = 1$ for all $n \in \mathbb{N}$. The convergence is not uniform.

7. We need $\|P\| < \dfrac{\epsilon}{4}$.

8. $\displaystyle \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(k\frac{1}{n}\right)^2 = \int_0^1 x^2\, dx \left(= \frac{1}{3}\right)$.

9. Counterexample: Let $f(x) = 1_{\mathbb{Q}\cap[0,1]} - 1_{[0,1]\setminus\mathbb{Q}}$.

10. For example, let f be the indicator function of the set of rational numbers in $[0, 1]$.

11. For example, let $P = \{0 < 0.45 < 0.55 < 1\}$.

12. True, since $R[a, b]$ is a vector space.

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: (i) Let $\epsilon > 0$. *Show* that if $\delta = \epsilon$ and if $x > 1$ and $y > 1$ then $|x - y| < \delta$ implies that $\left|\dfrac{1}{x} - \dfrac{1}{y}\right| < \epsilon$.

(ii) Let $\epsilon > 0$. Show that for all $\delta > 0$, there exist $x, y > 0$ with $|x - y| < \delta$ yet $\left|\dfrac{1}{x} - \dfrac{1}{y}\right| \geq \epsilon$.

B: Remember that for each of the three parts you must give a reason for the convergence to be uniform or not. For (i) you could cite the relevant theorem, and for the other two parts you'll need to determine whether or not $\|f_n - f\|_{\sup} \to 0$ and $n \to \infty$. This means you'll need to *find* the norm, for each $n \in \mathbb{N}$. Most of the errors amounted omitting reasons for conclusions, and/or tangled language that obscured the reasoning.

C: Most solutions were fine. Just take care that all inequalities are true. For example, it is **false** that $|a - b| \le |a| - |b|$. Just try a few numbers!

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	7	12	13		
80-89 (B)	7	6	8		
70-79 (C)	8	3	2		
60-69 (D)	2	3	3		
0-59 (F)	6	5	1		
Test Avg	75.6%	79.8%	86.3%	%	%
HW Avg	7.2	7.62	7.4		
HW/Test Correl	0.79	0.66	0.57		

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{27}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a fairly strong positive correlation with performance on the homework.