Print Your Name Here: ______________________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded*. **No** books, notes, smart phones, cell phones, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

For the first two questions, and for each non-negative integer n, let X_n be the set of polynomials of degree equal to n, and let $P_n = \bigcup_{k=0}^{n} X_k$.

1. True or give a counterexample: The set X_n is a vector space.

2. True or give a counterexample: The set P_n a vector space.

3. Find $\|f\|_{\sup}$ if

a. $f(x) = x$ on $\left(-1, \frac{1}{2}\right)$.

b. $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ -x^2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

4. Let $f_n(x) = xe^{-nx^2}$ for all $x \in \mathbb{R}$. Find $\|f_n\|_{\sup}$. Does f_n converge uniformly on $\mathbb{R}$, and if so, to what limit function?

5. Let $f_n(x) = x^n$ for all $x \in \mathbb{R}$. Does f_n converge uniformly on $[0, 0.999]$?

6. Let $f_n(x) = x^n$ for all $x \in \mathbb{R}$. Does f_n converge uniformly on $[0, 1)$?

7. Let $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, \\ 0 & \text{if } x \in [0,1] \setminus \{\frac{1}{2}\}. \end{cases}$ Does there exist a Riemann sum $P(f, \{\bar{x}_k\})$ that is greater than or equal to all Riemann sums for this function? If so, find the numerical value of that Riemann sum.

8. True or False: If $f \in R[0,1]$ and if $f(x) \leq g(x)$ for all $x \in [0,1]$, then $\int_0^1 f(x)\,dx \leq \int_0^1 g(x)\,dx$.

9. True or False: If $f \in R[0,1]$, then $\inf\{f(x)|x \in [0,1]\} \leq \int_0^1 f(x)\,dx \leq \sup\{f(x)|x \in [0,1]\}$.

10. Express $\displaystyle\lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \cos\left(1 + \frac{2k}{n}\right)$ as an integral. (Be sure to include the lower and upper limits of integration.)

11. Let f be any real-valued function on a domain $D \subseteq \mathbb{R}$. Define $f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{if } f(x) < 0. \end{cases}$ and let $f^-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0, \\ 0 & \text{if } f(x) \geq 0 \end{cases}$ for all $x \in D$. Express both $f(x)$ and $|f(x)|$ in terms of $f^+(x)$ and $f^-(x)$

12. Let f be any bounded function on $[a,b]$, and suppose $U(f,P) - L(f,P) < .001$. True or False: If $P' \supset P$ is another partition of $[a,b]$, then $U(f,P') - L(f,P') < .001$.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Let $p(x) = a_{2n}x^{2n} + \cdots + a_1x + a_0$ be any polynomial of *even degree*. Prove: If $a_{2n} > 0$, then p has a *minimum* value on $\mathbb{R}$.

B. Let $f_n(x) = nxe^{-nx}$ for all $x \in [0, \infty)$.

(**i**) Find the pointwise limit of $f_n(x)$ for all $x \geq 0$.

(**ii**) Find $\|f_n\|_{\text{sup}}$ for all n.

(**iii**) Determine whether or not f_n converges uniformly on $[0, \infty)$, and prove your claim.

(Hint: You may use your prior knowledge of differential calculus for this problem.)

C. Prove the *Mean Value Theorem for Integrals*. That is, let $f \in \mathcal{C}[a, b]$. Prove: There exists $\bar{x} \in [a, b]$ such that $\displaystyle\int_a^b f(x)\,dx = f(\bar{x})(b - a)$. (Hints: You will need to use both the Extreme Value Theorem and the Intermediate Value Theorem for continuous functions.)

Solutions and Class Statistics

1. Counterexample: If $n > 0$ then X_n is not closed under multiplication by the scalar 0, or under vector addition. However, X_0 *is* a vector space.

2. True

3.

a. $\|f\|_{\text{sup}} = 1$

b. $\|f\|_{\text{sup}} = \infty$

4. $\|f_n\|_{\text{sup}} = \frac{1}{\sqrt{2ne}}$ and the convergence is uniform to 0.

5. Yes

6. No

7. Yes. Let $P = \{0, 1\}$ and let $\frac{1}{2}$ be the evaluation point. Then the Riemann sum will be 1, and that is the maximum over all possible Riemann sums for this function.

8. False: g might not be Riemann integrable.

9. True

10. $\lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \cos\left(1 + \frac{2k}{n}\right) = \int_1^3 \cos x \, dx = \sin 3 - \sin 1.$

11. $f(x) = f^+(x) - f^-(x)$ and $|f(x)| = f^+(x) + f^-(x)$ for all $x \in D$.

12. True

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: The idea is to show that for $|x|$ sufficiently large, meaning $|x| > B$ for some B that you will specify, $p(x) > p(0) = a_0$. Then show the minimum for f on $[-B, B]$ is also a minimum for f on $\mathbb{R}$.

B: Show that $\|f_n\|_{\text{sup}} = e^{-1}$. Show that the pointwise limit is zero but $\|f_n - 0\|_{\text{sup}} = e^{-1}$ does not converge to zero, so that the convergence is not uniform.

C: The key is that the Extreme Value Theorem tells us that there exist points in $[a, b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$. Then integrate each of the three functions from a to b in order to prove that, if $a < b$, $f(x_m) \leq \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\, dx \leq f(x_M)$. Then invoke the Intermediate Value theorem to to show the existence of the needed point $\bar{x}$. Attempts to be vague or informal result in botching the proof. Make your proofs quantitative and definitive.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	2	3	4		
80-89 (B)	6	4	2		
70-79 (C)	2	2	4		
60-69 (D)	1	2	2		
0-59 (F)	1	1	0		
Test Avg	81.3%	80.2%	79.4%	%	%
HW Avg	7.8	7.0	7.2		
HW/Tst Correl	0.71	0.70	0.74		

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{12}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.