Print Your Name Here: ____________________

Show all work in the space provided and *keep your eyes on your own paper. Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart/cell phones, I-watches, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. For each of the following sets, label it *countable* or ***un**countable*:

a. The set of all irrational numbers

b. The set of all ordered pairs (m,n) where $m, n \in \mathbb{Z}$

c. The set of all infinite sequences of the digits $0, 1, \ldots, 9$.

2. Find the set of all *cluster* points of the set $\mathbb{Z}$ of all integers.

3. Find a sequence $x_n \to 0+$ such that $\sin \dfrac{1}{x_n} = (-1)^n$ for all $n \in \mathbb{N}$.

4. Find $\displaystyle\lim_{x\to\infty} \frac{a_n x^n + \cdots a_1 x + a_0}{b_n x^n + \cdots b_1 x + b_0}$ provided that $b_n \neq 0$.

5. Let $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Find *all* points p at which f is *continuous*.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be any *continuous* function such that $f(x+y) \equiv f(x) + f(y)$. If $f(2) = 1$, find $f(3)$.

7. If $\epsilon > 0$, find a $\delta > 0$ such that $|x - a| < \delta \implies \big||x| - |a|\big| < \epsilon$.

8. Give an example of a uniformly continuuous function ϕ defined on $\mathbb{R}$ which lacks a derivative at exactly one point in $\mathbb{R}$.

9. Let $f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$ True or False:

a. f has the Intermediate Value Property on $[0, 1]$.

b. f is *uniformly continuous* on $[0, 1]$.

10. Let $f(x) = x^2$, for all $x \in \mathbb{R}$. True or False: f is uniformly continuous on

a. $\mathbb{R}$.

b. $\left[-10^6, 10^6\right]$

11. Let $f(x) = \frac{1}{x}$, for all $x \in (1, \infty)$. True or False: f is *uniformly* continuous on $(1, \infty)$?

12. Let $p(x) = x^4 + x^3 - 2x^2 + x + 1$. True or False: the polynomial equation $p(x) = 0$ has a root in $(-1, 0)$.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. A subset $E \subset \mathbb{R}$ is called *closed* if and only if its complement $\mathbb{R} \setminus E$ is *open*. *Prove* that a closed set E that is also *dense* in $\mathbb{R}$ must be all of $\mathbb{R}$. (Hint: Suppose the claim were false, so that $\mathbb{R} \setminus E$ is a nonempty open set. Deduce a contradiction.)

B. Let $Q(x) = \sqrt{x}$, which is defined for all $x \geq 0$. *Prove:* $Q \in \mathcal{C}[0, \infty)$. (Hint: If $a \geq 0$, and $\epsilon > 0$, we seek $\delta > 0$ such that $x \geq 0$ and $|x - a| < \delta$ implies $|Q(x) - Q(a)| < \epsilon$. *Begin by proving* that $|\sqrt{x} - \sqrt{a}|^2 \leq |x - a|$.)

C. Suppose $f \in \mathcal{C}[0, 1]$ and suppose $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$. *Prove* that there exists $c \in [0, 1]$ such that $f(c) = c$. The point c is then called a *fixed point* for the function f. (Hint: Consider $g(x) = f(x) - x$.)

Solutions and Class Statistics

1.

a. uncountable

b. countable

c. uncountable

2. $\emptyset$

3. $\dfrac{2}{(2n+1)\pi}$

4. $\dfrac{a_n}{b_n}$

5. $p \in \{\pm 1\}$.

6. $\dfrac{3}{2}$

7. Any $0 < \delta \leq \epsilon$ will suffice.

8. $\phi(x) = |x|$ is uniformly continuous on $\mathbb{R}$ but $\phi'(0)$ does not exist.

9.

a. True

b. False. See Exercise 2.47.

10.

a. False: See exercise 2.43.

b. True: See Theorem 2.32.

11. True: see the homework question 2.42.

12. True. Use the Intermediate Value Theorem.

Remarks about the proofs

Proofs are graded for logical coherence. The sequence of steps should be logical, and *each claim should be justified explicitly* to show that you understand your own proof. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part*

II came. This will help us to see how you use the corrections to your homework in order to learn to write better proofs.

In proof (A), if you follow the outline in the hint, the hypothesis that $\mathbb{R} \setminus E$ is nonempty implies that there exists a point $p \in E$. Then invoke openness to find an open interval centered on p that lies in $\mathbb{R} \setminus E$. Next use the density of E to obtain a contradiction to complete the indirect proof. Several students decided on an alternate route to go directly from $\mathbb{R} \setminus E$ to an open interval contained therein. Remember that even the empty set $\emptyset$ is open and it can be expressed as an open interval. To justify this alternate route, be sure to cite the theorem that every open subset of $\mathbb{R}$ is the union of a possibly infinite family of open intervals, and then use the hypothesis that $\mathbb{R} \setminus E$ is nonempty to justify that at least one of those intervals is nonempty. Then invoke the density of E to obtain a contradiction.

In proof (B), it is important to *prove* the stated inequality. Use the very simple 1-line technique explained in class to do this easily. You should show that for all $x, a \geq 0$ and for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - a| < \delta$ implies tht $|Q(x) - Q(a)| < \epsilon$. If you follow the simple technique explained in class, you should find that δ depends only on ϵ which establishes uniform continuity, which is stronger than ordinary continuity.

In proof (C), the key is to explain why g is continuous and why $g(0) \geq 0$ and why $g(1) \leq 0$. This leaves one with 3 *mutually exclusive* and *exhaustive* cases: $g(0) = 0$, $g(1) = 0$, and $g(0) > 0 > g(1)$. Be explicit and clear about why these three cases are mutually exclusive and exhaustive. Then invoke the Intermediate Value Theorem with regard to g and deduce the natural conclusion regarding f.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	6	6			
80-89 (B)	2	2			
70-79 (C)	5	5			
60-69 (D)	1	3			
0-59 (F)	7	2			
Test Avg	73.5%	78.4%	%	%	%
HW Avg	8.26	7.8			
HW/Test Correl	0.90	0.83			

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{20}$– one dimension for each student currently enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a very strongly positive correlation with performance on the homework.