Print Your Name Here: ______________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your **name** at the **top** of the *scratch* sheet *if you will hand it in to be graded.* **No** books, notes, smart phones, cell phones, communication devices, internet devices, or electronic devices are allowed except for a scientific calculator—which is not needed. The maximum total score is 100.

Part I: Short Questions. Answer **8** of the 12 short questions: 6 points each. **Circle** the **numbers** of the 8 questions that you want counted—*no more than 8*! Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 48 points.

1. True or Give a Counterexample: Every subset of $\mathbb{R}$ is the union of a family of closed sets.

2. True or False: There is only one *closed, dense* subset of $\mathbb{R}$.

3. True or False: The set of irrational numbers of the form $\frac{p}{q}\sqrt{2}$ is uncountable.

4. True or False: The set of all irrational numbers is an *open* subset of $\mathbb{R}$.

5. Find the set of all cluster points of the set $\mathbb{Q}$ of all rational numbers.

6. Let $\epsilon > 0$. Find a $\delta > 0$ such that for all pairs $x, x' \in \mathbb{R}$ if $|x - x'| < \delta$ then $\big||x| - |x'|\big| < \epsilon$.

7. Let $f(x) = \begin{cases} 1+x & \text{if } x \in \mathbb{Q}, \\ 1-x^2 & \text{if } x \notin \mathbb{Q}. \end{cases}$ Find all points p at which f is continuous.

8. True or False: The polynomial equation $x^{19} + x^{11} + x^3 + 4 = 0$ has a root in the interval $(-1, \infty)$.

9. True or False: $f(x) = \frac{1}{x}$ is uniformly continuous on $[1, \infty)$.

10. True or False: The function $f(x) = \sin\left(\frac{1}{x}\right)$ is *uniformly continuous* on $[0.001, 1000]$.

11. Suppose f is *uniformly continuous* on D and suppose $E \subset D$. True or False: f is *uniformly continuous* on E.

12. True or Give a Counterexample: If f is *uniformly continuous* on $[-1, 0)$ and also on $[0, 1]$ then f is *uniformly continuous* on $[-1, 1]$.

Part II: Proofs. *Prove carefully* **2** of the following 3 theorems for 26 points each. **Circle** the *letters* of the 2 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 52 points.

A. Let $E = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$. Find an open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ of E that has no finite subcover, and prove that $\mathcal{O}$ is an open cover and that $\mathcal{O}$ has no finite subcover.

B. Suppose that f *monotone increasing* on $\mathbb{R}$, meaning that $x_1 < x_2 \implies f(x_1) \leq f(x_2)$. Prove: For all $a \in \mathbb{R}$, $\lim_{x \to a+} f(x)$ exists and is a real number. (Hint: Let $S = \{f(x) \mid x > a\}$. Show that $\inf(S)$ is a real number L, and then prove that $\lim_{x \to a+} f(x) = L$.)

C. Prove the following *fixed point theorem*: Suppose $f \in \mathcal{C}[0,1]$ and suppose $0 \leq f(x) \leq 1$ for all $x \in [0,1]$. Then there exists $c \in [0,1]$ such that $f(c) = c$. (Hint: Consider $g(x) = f(x) - x$.)

Solutions and Class Statistics

1. True, since each $S = \bigcup_{x \in S} \{x\}$ and a singleton set is closed.

2. True, since a closed dense subset of $\mathbb{R}$ must be all of $\mathbb{R}$.

3. False, since the set of rational numbers is countable.

4. False, since the set $\mathbb{Q}$ is dense in $\mathbb{R}$.

5. $\mathbb{R}$

6. Any $\delta \leq \epsilon$ works.

7. $p \in \{0, -1\}$

8. False

9. True, despite the fact that the interval is not finite.

10. True, since f is continuous on a closed finite interval.

11. True

12. Counterexample: Let $f(x) = 1$ if $x \geq 0$ and let $f(x) = 0$ if $x < 0$.

Remarks about the proofs

Proofs are graded for logical coherence. If you have questions about the grading of the proofs on this test, or if you are having difficulty writing satisfactory proofs, *please bring me your test and also the graded homework from which the questions in Part II came.* This will help us to see how you use the corrections to your homework in order to learn to write better proofs. Also *please bring your notebook showing how we presented the same proof in class* after the homework was graded. It is important to learn from both sources.

A: Take care not to confuse an open cover with the union of its members. That union is just one set, whereas the open cover is often a set of infinitely many open sets. Take care to use inequalities correctly. I think it would help you to use the language I suggested: Let F be any finite subset of $\mathbb{N}$, the index set for the elements of your open cover. Show why $\bigcup_{n \in F} O_n$ cannot cover E.

B: The language needs to be very clear for this proof. $\inf\{x \mid x > a\} = a$ but $f(a)$ need not be L, the infimum of S. Note that f is merely monotone increasing: it need not be continuous. f can have jump discontinuities, and L need not be a value of f. The key is to make good use of monotonicity and the meaning of L being the *greatest* lower bound of S.

C: Be very careful with inequalities! Here are the key steps to justify: (1) g is continuous on [0,1]. (2) $g(0) \geq 0$ and $g(1) \leq 0$. (3) special case: suppose $g(0) = 0$. Then $f(0) = 0$ (4) special case: suppose $g(1) = 0$. Then $f(1) = 1$. (5) The only remaining case is $g(0) > 0 > g(1)$. Then invoke the intermediate value theorem and explain how that yields some $c \in (0, 1)$ such that $g(c) = 0$ and then $f(c) = c$.

Class Statistics

Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	2	3			
80-89 (B)	6	4			
70-79 (C)	2	2			
60-69 (D)	1	2			
0-59 (F)	1	1			
Test Avg	81.3%	79.7%	%	%	%
HW Avg	7.8	7.0			
HW/Tst Correl	0.71	0.70			

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{12}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.