Print Your Name Here: ____________________

- **Show all work** in the space provided. We can give credit *only* for what you write! *Indicate clearly if you continue on the back side.*
- **No** books or notes (paper or electronic) or communication devices (smart/cell phones, internet-connected devices such as laptops, tablets, or I-watches) are allowed. A scientific calculator (*not capable* of graphing or symbolic calculations) is allowed—but it is not needed. If you use a calculator, you *must still write out all operations performed* on the calculator. ***Do not*** replace precise answers such as $\sqrt{2}$, $\frac{1}{3}$, or π with decimal approximations. *Keep your eyes on your own paper!*
- The maximum score possible is 100 points.

1. (10 points) *Evaluate* the limit *or show* that it *does not exist* by evaluating along two different paths through the origin: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 - 3y^2}{2x^2 + y^2}$.

2. (20 points) Use the Chain Rule to find the partial derivatives $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ if $z = xy$, $x = e^{st}$, and $y = \dfrac{s}{t}$.

3. (20 points) Let $f(x, y) = e^x \sin y$. Find the gradient, ∇f, and the directional derivative of f in the *direction* of the vector $\vec{v} = \langle 3, 4 \rangle$ at the point $\left(0, \frac{\pi}{3}\right)$.

4. (20 points) Let $F(x, y, z) = e^{2z} - x^2yz$ and let S be the surface defined by $F(x, y, z) = 0$..

a. If z is a differentiable function of x and y satisfying $F(x, y, z) = 0$, use implicit differentiation to find $\frac{\partial z}{\partial x}$ at $(e, 1, 1)$.

b. Find $\nabla F(e, 1, 1)$ and use this to write an equation for the tangent plane to the surface S defined by $F(x, y, z) = 0$ at $(e, 1, 1)$.

5. (30 points) Let $f(x, y, z) = x^2yz$ and $g(x, y, z) = x^2 + y^2 + z^2$.

a. (10) Find the gradient vectors $\nabla f(x, y, z)$ and $\nabla g(x, y, z)$.

b. (20) Use the method of Lagrange multipliers to find the *absolute maximum value* and the *absolute minimum value* of $f(x, y, z)$ on the sphere which is the level surface $g(x, y, z) = 8$. (Hint: $f(x, y, z) = x^2yz$ has both positive and negative values on the given sphere. So x, y and z must be nonzero at the extreme points.)

Solutions

1. This limit does not exist. The limit along the x-axis as $x \to 0$ is $\displaystyle\lim_{(x,0)\to(0,0)} \frac{x^3}{2x^2} = 0$, but the limit along the y-axis as $y \to 0$ is $\displaystyle\lim_{(0,y)\to(0,0)} \frac{-3y^2}{y^2} = -3$. This problem was *modified homework problem* 14.2/15. There were some bizarre choices of paths. Always try the simplest methods possible first! The work evaluating the two path-dependent limits needs to be correct.

2. $\displaystyle\frac{\partial z}{\partial s} = z_x x_s + z_y y_s = e^{st}\left(s + \frac{1}{t}\right)$ and $\displaystyle\frac{\partial z}{\partial t} = z_x x_t + z_y y_t = e^{st}\left(\frac{s^2}{t} - \frac{s}{t^2}\right)$. Compare with 14.5/11-15(odd). The answers should be expressed in terms of s and t. When using a letter "d" to indicate a *partial* derivative, one needs to use the round-backed German ∂ which is the official partial derivative symbol. Its name in LaTeX is " forward slash partial". Make obvious simplifications such as $t\frac{s}{t} = s$. Take note that the question requires using the Chain Rule. Knowing the Chain Rule in several variables is required in this course.

3. $\nabla f\left(0, \frac{\pi}{3}\right) = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$ and corresponding to $\vec{v}$ is the unit vector $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$. Hence $D_{\vec{u}} f\left(0, \dfrac{\pi}{3}\right) = \nabla f\left(0, \dfrac{\pi}{3}\right) \cdot \vec{u} = \dfrac{3\sqrt{3}+4}{10}$. Compare with assigned homework problems 14.6/13-19(odd). Please *read* the instructions: No decimal approximations! For the values of the trigonometric functions of $\frac{\pi}{3}$ draw an equilateral triangle of side 1 and drop an altitude to one side. Be sure to distinguish between a vector and a scalar!

4. Let $F(x, y, z) = e^{2z} - x^2yz$.

a. If $z = z(x, y)$ satisfies $F(x, y, z) = 0$ then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{2xyz}{2e^{2z} - x^2y} = \frac{2}{e}$$

at the indicated point $(e, 1, 1)$. An alternative but longer way of writing the solution is to differentiate with respect to x on both sides of $e^{2z} = x^2yz$, treating z as a function of x and y and using the chain rule and product formula correctly. Compare with assigned homework problem 14.5/33. Note: Please do not embarrass yourself by leaving $\dfrac{e}{e^2}$ or $2e^2 - e^2$ unsimplified. It is like broadcasting that one is insecure with middle school algebra. Also, if you introduce a function F write what $F(x, y, z)$ equals in terms of x, y and z. This is important because some students memorize formulas in terms of letters without knowing what the letters represent, and this leads to multiple errors.

b. $\nabla f(e, 1, 1) = \langle -2e, -e^2, e^2 \rangle$, which is of course a *vector* and *not a scalar*. An equation for the tangent plane at this point is

$$\langle -2e, -e^2, e^2 \rangle \cdot \langle x - e, y - 1, z - 1 \rangle = 0.$$

Note that if you omit the "=0" at the end, then you have no equation at all! That would be serious. Simplifying,

$$2x + ey - ez = 2e.$$

Compare with 14.6/51. Of course the equation for a plane *MUST* be a *LINEAR* equation! If you got a nonlinear equation you should recognize that this cannot possibly be correct. Also, please cancel a common factor of e when possible.

5.

a. $\nabla f(x,y,z) = \langle 2xyz, x^2z, x^2y\rangle$ and $\nabla g(x,y,z) = \langle 2x, 2y, 2z\rangle$. Once again, it is essential to understand that the gradient of a scalar-valued function is *always a vector, not a scalar.*

b. This is a slight modification of problem 14.8/5. Since f is continuous and the level surface $x^2+y^2+z^2 = 8$ is a sphere of radius $\sqrt{8}$, and is therefore closed and bounded, f must achieve an absolute maximum and an absolute minimum value. So we set $\nabla f(x,y,z) = \lambda\nabla g(x,y,z)$. Since f has both positive and negative values on the level surface, x, y, z and λ are ***all nonzero at the extrema***. We solve the system of equations

$$\begin{aligned} xyz &= \lambda x \\ x^2z &= 2\lambda y \\ x^2y &= 2\lambda z \\ x^2+y^2+z^2 &= 8, \end{aligned}$$

finding $\dfrac{y}{x} = \dfrac{x}{2y}$. and $\dfrac{z}{y} = \dfrac{y}{z}$.Thus $z^2 = y^2$ and $x^2 = 2y^2$, which we substitute into the last of the four equations. Thus $x = \pm 2,\ y = \pm\sqrt{2}$, and $z = \pm\sqrt{2}$. So the maximum value of $f(x,y,z) = x^2yz$ is 8 and the minimum is -8. Note that this problem has nothing to do with classifying critical points as *local* extrema or as saddle points in the *interior* of a domain. The second derivative test for functions of two variables is *not directly* applicable to this problem concerning the function f of three variables with the given constraint. Notice that I gave you all the observations you should actually make for yourself, justifying the use of division of equals by nonzero equals so as to solve the system easily.

Class Statistics

% Grade	Test#1	Test#2	Test#3	Final Exam	Final Grade
90-100 (A)	13				
80-89 (B)	6				
70-79 (C)	6				
60-69 (D)	5				
0-59 (F)	0				
This Test Avg	84.8%	%	%	%	%
HW Avg	87.5%	%	%	%	%
HW/TST Corr	0.61				

The Correlation Coefficient is the cosine of the angle between two data vectors in $\mathbb{R}^{31}$–one dimension for each student enrolled. Thus this coefficient is between 1 and -1, with coefficients above 0.6 being considered strongly positive. The correlation coefficient shown indicates that the test grades in the course have a strongly positive correlation with performance on the homework.