18.014–ESG Notes 1

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1 Limits and Continuity

The notation $\lim_{x\to p} f(x) = L$ means that for any $\epsilon > 0$, you can choose a $\delta > 0$ such that whenever $0 < |x - p| < \delta$, we are guaranteed that $|f(x) - L| < \epsilon$. N.B.: We do not care how f behaves *at* $x = p$, just near it. (f need not even be defined at p.)

To say that f is *continuous* at p is to say that $\lim_{x\to p} f(x) = f(p)$. Another way of saying this is that for any $\epsilon > 0$, you can choose a $\delta > 0$ such that whenever $|x - p| < \delta$, we are guaranteed that $|f(x) - f(p)| < \epsilon$. The most important difference between the statements "f has a limit as x approaches p" and "f is continuous at p" is that in the latter statement, we *do* care about how f behaves at p.

A function can be either continuous or discontinuous at each point of its domain. At points not in its domain, the function is neither continuous nor discontinuous—just undefined.

Whenever $f : A \to B$ has the property that it is one-to-one (i.e. if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$) and onto (i.e. for every $y_0 \in B$ there is an $x_0 \in A$ such that $f(x_0) = y_0$), we can define the inverse function $f^{-1} : B \to A$ of f[1].

In particular, if $f : A \to \mathbb{R}$ is strictly monotonic, then it is one-to-one. If B is the image of f, we can restrict its codomain to B; now $f : A \to B$ is one-to-one and onto, so it has an inverse $f^{-1} : B \to A$. For example, if A is a closed interval $[a, b]$, and B is another closed interval $[c, d]$, then we have the following theorem.

Theorem 1.1. *If $f : [a, b] \to [c, d]$ is strictly monotonic and continuous, then $f^{-1} : [c, d] \to [a, b]$ exists and is also strictly monotonic and continuous.*

Theorem 1.2 (Sign-Preserving Property). *If f is continuous and nonzero at c, there is a $\delta > 0$ such that if $|x - c| < \delta$, then $f(x)$ has the same sign as $f(c)$.*

For each of the following theorems, assume that $f : [a, b] \to \mathbb{R}$ is continuous.

Theorem 1.3. *f is bounded.*

Theorem 1.4. *f is integrable.*

Remember, integrable functions need not be continuous—notably, step functions are usually not continuous!

Theorem 1.5 (Bolzano's Theorem). *If $f(a)$ and $f(b)$ have opposite signs, then there is some point $c \in (a, b)$ such that $f(c) = 0$.*

Theorem 1.6 (Intermediate-Value Theorem). *If w is any number between $f(a)$ and $f(b)$, then there is some point $c \in (a, b)$ such that $f(c) = w$.*

Theorem 1.7 (Extreme-Value Theorem). *There exist points c and d in $[a, b]$ such that f has a maximum at c and a minimum at d.*

[1] The grown-up words for *one-to-one* and *onto* are *injective* and *surjective*, respectively. A function that is both injective and surjective is called *bijective*. Only bijective functions have inverses, although sometimes you can fix up a non-bijective function to have an "inverse" by shrinking its domain or codomain. For example, we define arcsin after shrinking the domain of sin.

Theorem 1.8 (Small-Span Theorem). *There is a partition*

$$P = \{x_0 = a, x_1, \dots, x_{n-1}, x_n = b\}$$

of $[a, b]$ *such that* $\max f - \min f < \epsilon$ *on each subinterval* $[x_{i-1}, x_i]$.

2 Differentiation

If $f : A \to \mathbb{R}$, we define a new function Df or f', called the *derivative* of f whose value at x_0 is given by either of the formulæ

$$\lim_{h\to 0} \frac{f(x_0 = h) - f(x_0)}{h} \qquad \text{or} \qquad \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

The domain of the derivative is the set of points of A where the above limits exist. We have the following rules for computing derivatives:

$$D(f+g) = Df + Dg \qquad D(f-g) = Df - Dg$$
$$D(fg) = f\,Dg + g\,Df \qquad D(f \circ g) = (Df \circ g) \cdot Dg$$
$$D(f/g) = \frac{g\,Df - f\,Dg}{g^2} \qquad D(f^{-1}) = \frac{1}{Df \circ f^{-1}}$$

For the following theorems, assume that $f : [a, b] \to \mathbb{R}$ is continuous, and that its derivative is defined on (a, b).

Theorem 2.1 (Rolle's Theorem). *If* $f(a) = f(b)$, *then there is a point* $c \in (a, b)$ *such that* $f'(c) = 0$.

Theorem 2.2 (Mean-Value Theorem). *There is a point* $c \in (a, b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Be careful about the assumptions of the following theorems. Remember that many discontinuous functions are integrable (e.g., step functions).

Theorem 2.3 (First Fundamental Theorem of Calculus). *Let* $F : [a, b] \to \mathbb{R}$ *be the indefinite integral of* f*:*

$$F(x) = \int_a^x f.$$

Then F *is differentiable on* (a, b), *and* $DF = f$.

Theorem 2.4 (Second Fundamental Theorem of Calculus). *Assume that* $f : [a, b] \to \mathbb{R}$ *is continuous and differentiable on* (a, b)*; moreover, suppose that its derivative* Df *is continuous too. Then*

$$\int_a^x Df = f(x) - f(a).$$

The important thing to remember about both these theorems is that in the equations

$$D\int_a^x f = f \qquad \text{and} \qquad \int_a^x Df = f(x) - f(a),$$

the integrands are assumed to be continuous.

Theorem 2.5. *There exists a unique pair of functions* u, v *satisfying the following equations:*

$$Du = v \qquad Dv = -u$$
$$u(0) = 0 \qquad v(0) = 1$$

We name these functions $u(x) = \sin x$ *and* $v(x) = \cos x$. *They have the following properties:*

$$\sin^2 x + \cos^2 x = 1 \qquad \sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin x,\ \cos x \in [-1.1] \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$