Math 2090 | Spring 2005
Sections 4 and 7 | P. Achar

Hour Exam 3 Solutions

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Total points: 100 Time limit: 50 minutes

No calculators permitted. You must show all your work to receive full credit.

IMPORTANT: You **must** answer Problems 1–5 and Problem 11. For Problems 6–10, choose **three** out of the **five** to answer. **Note:** in order to do Problem 11, you must answer at least one of Problems 9 or 10.

1. (4 points) What is the definition of *dimension*?

   Solution: the number of vectors in a basis for the vector space

2. (6 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be vectors in a vector space V. In each of the following situations, what can you conclude about the dimension of V? Your answers may be inequalities like "$\dim V \geq 2$," precise answers like "$\dim V = 5$,", or the words "no information."

   (a) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.

   Solution: no information

   (b) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span V.

   Solution: $\dim V \leq 3$

   (c) $\mathbf{v}_1$, $\mathbf{v}_3$, and $\mathbf{v}_4$ span V and are linearly independent.

   Solution: $\dim V = 3$

3. (12 points) Consider the linear transformation $T : P_2 \to \mathbb{R}$, $T(p(x)) = \displaystyle\int_0^1 p(x)\,dx$.

   (a) What is $\dim \operatorname{Rng}(T)$? Justify your answer. (*Hint:* What are the possible subspaces of $\mathbb{R}$? What are their dimensions?)

   Solution: $\dim \operatorname{Rng}(T) = 1$. The range of T is a subspace of $\mathbb{R}$, whose only subspaces are the trivial subspace (dimension 0) and $\mathbb{R}$ itself (dimension 1). $\operatorname{Rng}(T)$ can't be the trivial subspace—that would mean $\int_0^1 p(x)\,dx = 0$ for all polynomials $p(x) \in P_2$—so it has to be all of $\mathbb{R}$. Therefore, its dimension is 1.

   (b) What is $\dim \operatorname{Ker}(T)$? Justify your answer. It is not necessary to find the kernel to answer this question.

   Solution: $\dim \operatorname{Ker}(T) = 1$. This comes from the fact that $\dim \operatorname{Ker}(T) + \dim \operatorname{Rng}(T) = \dim P_2$, and $\dim P_2 = 2$.

4. (12 points) Find the eigenvalues of the matrix $\begin{bmatrix} 5 & 3 \\ -2 & 0 \end{bmatrix}$. (You need not find the eigenvectors.)

   Solution:

   Characteristic polynomial:

$$\begin{aligned} p(\lambda) = \det \begin{bmatrix} 5-\lambda & 3 \\ -2 & -\lambda \end{bmatrix} &= (5-\lambda)(-\lambda) - (-2)3 \\ &= -5\lambda + \lambda^2 + 6 = \lambda^2 - 5\lambda + 6 \\ &= (\lambda - 2)(\lambda - 3) \end{aligned}$$

   Eigenvalues: $\lambda = 2, 3$.

5. Consider the following two mappings:

$$S : P_4 \to P_7, \quad S(p(x)) = p(x^2) \qquad \text{and} \qquad T : P_4 \to P_7, \quad T(p(x)) = p(x)^2$$

(a) (6 points) Find $S(3x^3 - 5x)$ and $T(3x^3 - 5x)$.

Solution:

$$\begin{aligned} S(3x^3 - 5x) &= 3(x^2)^3 - 5(x^2) \qquad \text{(plug in } x^2 \text{ for } x\text{)} \\ &= 3x^6 - 5x^2 \\ T(3x^3 - 5x) &= (3x^3 - 5x)^2 \\ &= 9x^6 - 30x^4 + 25x^2 \end{aligned}$$

(b) (8 points) Is S a linear transformation? Justify your answer.

Solution: Yes. First linearity condition:

$$\begin{aligned} S(p(x) + q(x)) &\stackrel{?}{=} S(p(x)) + S(q(x)) \\ p(x^2) + q(x^2) &\stackrel{?}{=} p(x^2) + q(x^2) \end{aligned}$$

This equation is true. Second linearity condition:

$$\begin{aligned} S(cp(x)) &\stackrel{?}{=} c\,S(p(x)) \\ cp(x^2) &\stackrel{?}{=} c\,p(x^2) \end{aligned}$$

This one is also true.

(c) (8 points) Is T a linear transformation? Justify your answer.

Solution: No. For example:

$$\begin{aligned} T(3x^3 - 5x) &\stackrel{?}{=} T(3x^3) + T(-5x) \\ (3x^3 - 5x)^2 &\stackrel{?}{=} (3x^3)^2 + (-5x)^2 \\ 9x^6 - 30x^4 + 25x^2 &\neq 9x^6 + 25x^4 \end{aligned}$$

so the first linearity condition fails.

Problems 6 and 7 deal with the following functions:

$$f(x) = 3x^2 + 2, \qquad g(x) = x - 3, \qquad h(x) = x^2 + 5$$

6. Show that these functions span P_3.

Solution: We need to set a linear combination of these functions equal to an arbitrary element of P_3, and then show that we can solve for the coefficients:

$$\begin{aligned} c_1(3x^2 + 2) + c_2(x - 3) + c_3(x^2 + 5) &= ax^2 + bx + c \\ 3c_1x^2 + 2c_1 + c_2x - 3c_2 + c_3x^2 + 5c_3 &= ax^2 + bx + c \\ (3c_1 + c_3)x^2 + c_2x + (2c_1 - 3c_2 + 5c_3) &= ax^2 + bx + c \end{aligned}$$

Equating coefficients, we have

$$\begin{aligned} 3c_1 \qquad\quad + \;\; c_3 &= a \\ c_2 \qquad\quad &= b \\ 2c_1 - 3c_2 + 5c_3 &= c \end{aligned}$$

Augmented matrix:

$$\begin{bmatrix} 3 & 0 & 1 & a \\ 0 & 1 & 0 & b \\ 2 & -3 & 5 & c \end{bmatrix} \xrightarrow{A_{1,3}(-1)} \begin{bmatrix} 1 & 3 & -4 & a-c \\ 0 & 1 & 0 & b \\ 2 & -3 & 5 & c \end{bmatrix} \xrightarrow{A_{3,1}(-2)} \begin{bmatrix} 1 & 3 & -4 & a-c \\ 0 & 1 & 0 & b \\ 0 & -9 & 13 & -2a+3c \end{bmatrix}$$

$$\xrightarrow{A_{3,2}(9)} \begin{bmatrix} 1 & 3 & -4 & a-c \\ 0 & 1 & 0 & b \\ 0 & 0 & 13 & -2a+9b+3c \end{bmatrix} \xrightarrow{M_3(\frac{1}{13})} \begin{bmatrix} 1 & 3 & -4 & a-c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & \frac{-2a+9b+3c}{13} \end{bmatrix}$$

Since the ranks of the augmented matrix and the coefficient matrix are both 3, which is equal to the number of variables, the system has a unique solution. Since there is a solution for c_1, c_2, c_3 (it doesn't matter for the span whether it's unique or not), the given functions do span P_3.

7. Let $f(x)$, $g(x)$, and $h(x)$ be the functions defined on the previous page.

   (a) (8 points) Use the Wronskian to determine whether they are linearly independent or not.

   Solution:

$$W[f,g,h](x) = \det \begin{bmatrix} 3x^2+2 & x-3 & x^2+5 \\ 6x & 1 & 2x \\ 6 & 0 & 2 \end{bmatrix} \qquad \text{(expand by minors on 2nd column)}$$

$$= (x-3)(-1)\det\begin{bmatrix} 6x & 2x \\ 6 & 2 \end{bmatrix} + 1(+1)\det\begin{bmatrix} 3x^2+2 & x^2+5 \\ 6 & 2 \end{bmatrix} + 0(-1)\det[\cdots]$$

$$= -(x-3)[6x\cdot 2 - 6\cdot 2x] + 1[(3x^2+2)2 - 6(x^2+5)]$$

$$= -(x-3)0 + (6x^2+4-6x^2-30) = 4-30 = -26.$$

   Since $W[f,g,h](x) = -26 \neq 0$, the functions are linearly independent.

   (b) (4 points) Do these functions form a basis for P_3?

   Solution: Yes. They span P_3 by Problem 6 and are linearly independent by part (a) of this problem.

8. (12 points) Find the kernel of the following linear transformation:

$$T : C^2(\mathbb{R}) \to C^0(\mathbb{R}), \qquad T(f(x)) = f''(x) - 3f'(x) + 2f(x)$$

Solution: To find the kernel, we have to find all functions $f(x)$ such that $T(f(x)) = 0$: in other words, we have to find solutions to the differential equation

$$y'' - 3y' + 2y = 0.$$

The auxilliary polynomial is $r^2 - 3r + 2 = (r-1)(r-2)$, so the general solution to the differential equation is

$$y = c_1e^x + c_2e^{2x}.$$

That is, the solutions are linear combinations of e^x and e^{2x}. In other words, Ker(T) is the span of $y = e^x$ and $y = e^{2x}$.

The remaining problems deal with the following linear transformation:

$$T : \mathbb{R}^3 \to \mathbb{R}^2, \qquad T(\mathbf{x}) = \begin{bmatrix} 4 & 2 & -6 \\ -2 & -1 & 3 \end{bmatrix}\mathbf{x}$$

9. (12 points) Find its kernel.

Solution: The kernel consists of solutions to the equation $T(\mathbf{x}) = \mathbf{0}$, or

$$\begin{bmatrix} 4 & 2 & -6 \\ -2 & -1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Augmented matrix:

$$\begin{bmatrix} 4 & 2 & -6 & 0 \\ -2 & -1 & 3 & 0 \end{bmatrix} \xrightarrow{A_{1,2}(2)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & -1 & 3 & 0 \end{bmatrix} \xrightarrow{P_{1,2}} \begin{bmatrix} -2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rewriting this as an equation, we get

$$-2x - y + 3z = 0.$$

The kernel is the plane in $\mathbb{R}^3$ described by this equation.

OR: to describe the kernel by giving a basis for it, we first observe that in the equation $-2x-y+3z = 0$, we can set y and z equal to anything and then solve for x. If we take $y = s$ and $z = t$, then $-2x - s + 3t = 0$, so $x = (3t - s)/2$. Therefore, points in the kernel have the form

$$\begin{bmatrix} \frac{3}{2}t - \frac{1}{2}s \\ s \\ t \end{bmatrix} = t \begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}.$$

In other words, the kernel is spanned by the vectors $(3/2, 0, 1)$ and $(-1/2, 1, 0)$. These vectors are also linearly independent, so in fact they are a basis for the kernel.

10. (12 points) Find its range.

Solution: For a matrix transformation, the range is just the span of the columns. Here, all the columns are multiples of one another, so any one of them (for instance, $\left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right]$ spans the range. That one vector by itself is also linearly independent, so in fact $\left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right]$ is a basis for the range.

For another description of the range, we observe that all the columns of the matrix lie along the line $x = -2y$, so in fact that is the equation of the line that they span.

11. (8 points) Find the dimensions of the kernel and range. (Remember that you **must** answer this question.)

Solution: Since the kernel is a plane (which has a basis consisting of two vectors), we have $\dim \operatorname{Ker}(T) = 2$.

Since the range is a line (and has a basis consisting of a single vector), $\dim \operatorname{Rng}(T) = 1$.

If you only worked out one of the two spaces, then you can use the fact that

$$\dim \operatorname{Ker}(T) + \dim \operatorname{Rng}(T) = \dim \mathbb{R}^3 = 3$$

to find the other one.