Problem Set 1b

Due: September 6, 2005

1. (Humphreys, Exercise 1.5.2) Let W be a reflection group with simple system Δ. Prove that no proper subset of the set of simple reflections can generate W.

2. We have seen that an essential reflection group on an n-dimensional real vector space can be generated by just n reflections (namely, the simple reflections with respect to some simple system). This is not always true over other fields.

   Let $V = \mathbb{C}^2$, and consider the matrices

$$s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad t = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad u = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

   Check that these are in fact reflections, and then answer the following questions.

   (a) Show that they generate a finite reflection group.

   (b) What might a "root system" for this group look like? (There is not necessarily a unique correct answer to this question. You should try to find a finite set of vectors in V that are permuted by the group, and then try to modify the definition of "root system" so that it makes sense in a complex vector space and is satisfied by the set of vectors you found earlier.)

   (c) For real reflection groups, a simple system has the following properties : (i) it is a linearly independent set of roots, and (ii) the corresponding reflections generate the group. (However, not every set with these two properties is a simple system.) Prove that no set of vectors in your "root system" from the previous part has both these properties. Thus, there is no "simple system" for this complex reflection group.