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Note

On the non-uniqueness of q-cones of matroids

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Abstract

Let M be a rank-r simple GF(q)-representable matroid. A q-cone of M is a matroid M' that is constructed by embedding M in a hyperplane of PG(r,q), adding a point p of PG(r,q) not on H, and then adding all the points of PG(r,q) that are on lines joining p to an element of M. If $r(M)>2$ and M is uniquely representable over GF(q), then M' is unique up to isomorphism. This note settles a question made explicit by Kung by showing that if $r(M)=2$ or if M is not uniquely representable over GF(q), then M' need not be unique.

MSC: 05B35

1. Introduction

The matroid terminology used here will follow Oxley [5] with one exception that will be discussed in detail beginning in the third paragraph. The construction, described in the abstract, of a q-cone of a simple GF(q)-representable matroid M is a natural one. It was introduced by Whittle [6], who called the construction a *q-lift*. He showed that every q-cone of a tangential k-block over GF(q) is a tangential $(k+1)$-block. The operation also appears in [3, p. 36] where it is called *framing*. Implicit in Whittle's paper is the question of whether non-isomorphic matroids can arise as q-cones of the same matroid M. This problem was made explicit by Kung [4, p. 103]. The purpose of this note is to solve this problem.

If M is a rank-r simple GF(q)-representable matroid, then M' is a *q-cone* of M with *base* E and *apex* p if the following conditions hold:

(i) E is a set of points of PG(r,q) such that $M \cong \text{PG}(r,q)|E$;

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(ii) p is a point of $\mathrm{PG}(r,q)$ that is not contained in the subspace of $\mathrm{PG}(r,q)$ spanned by E; and
(iii) the elements of M' are all of the points of $\mathrm{PG}(r,q)$ that lie on lines through p and some element of E.

Kung [4, p. 102] noted that it is easy to see that, for fixed E, altering the choice of p subject to (ii) produces a matroid isomorphic to M'. One can also change E and still obtain a matroid isomorphic to M' but, to state this observation more precisely, we shall need to discuss equivalent representations of matroids. Our discussion is somewhat extended since we wish to clarify the relationship between several notions of equivalence in the literature. Let M be a rank-r matroid on the set $\{e_1,e_2,\ldots,e_n\}$ where $r\geqslant 1$. Let A_1 and A_2 be $r\times n$ matrices over $\mathrm{GF}(q)$ with the columns of each being labelled, in order, by $e_1,e_2,\ldots,e_n$. Assume that, for each i in $\{1,2\}$, the identity map on $\{e_1,e_2,\ldots,e_n\}$ is an isomorphism between M and $M[A_i]$, the vector matroid of A_i. We define A_1 and A_2 to be *algebraically equivalent* $\mathrm{GF}(q)$-representations of M if A_2 can be obtained from A_1 by a sequence of operations each consisting of an elementary row operation, a column scaling, or, for some arbitrary automorphism γ of $\mathrm{GF}(q)$, the replacement of every matrix entry by its image under γ. Moreover, we define A_1 and A_2 to be *geometrically equivalent* $\mathrm{GF}(q)$-representations of M if the map that takes each column of A_1 to the corresponding column of A_2 is induced by an automorphism of the matroid corresponding to $\mathrm{PG}(r-1,q)$. Such an automorphism of $\mathrm{PG}(r-1,q)$ is a permutation of the set of subspaces that preserves dimension and inclusion. Equivalently, it is a permutation of the set of points of $\mathrm{PG}(r-1,q)$ that maps lines to lines. The last definition accounts for the name *collineation* for such maps. It is a consequence of the Fundamental Theorem of Projective Geometry (see, for example, [1, p. 44] or [2, p. 655]) that, when $r\neq 2$, the representations A_1 and A_2 are algebraically equivalent if and only if they are geometrically equivalent. Thus, when $r\neq 2$, these two notions of equivalence coincide and it is conventional to refer to this common notion as simply *equivalence* (or sometimes *projective equivalence* [4]). However, when $r=2$, the situation is less clear. The collineation group of $\mathrm{PG}(1,q)$ is the symmetric group and therefore two representations may be geometrically equivalent without being algebraically equivalent. Although Oxley [5, pp. 185–189] used 'equivalent' to mean 'geometrically equivalent', we shall use *equivalent* here to mean 'algebraically equivalent'. To complete the picture, we note that there is yet another notion of equivalence: A_1 and A_2 are *strongly equivalent* if A_2 can be obtained from A_1 by a sequence of the matrix operations described above without applying a field automorphism. Thus A_1 and A_2 are strongly equivalent if and only if there is a linear transformation σ of $V(r,q)$ and a sequence $c_1,c_2,\ldots,c_n$ of non-zero elements of $\mathrm{GF}(q)$ such that $\underline{v}_j^{(2)}=c_j\sigma(\underline{v}_j^{(1)})$ for all j where $\underline{v}_j^{(i)}$ is the jth column of A_i. The last assertion remains true if we replace 'strongly equivalent' and 'linear transformation' by 'equivalent' and 'semilinear transformation' [5, p. 186].

Kung [4, p. 102] noted that the q-cones of two equivalent $\mathrm{GF}(q)$-representations of a matroid M are isomorphic. The question that he asked is whether two inequivalent

GF(q)-representations of M must always produce isomorphic q-cones. We answer this question negatively in the next section. Because, as described above, the rank-2 case is special, we give two examples, one when M has rank 2, and a second when M has rank 3.

2. The examples

Let M_α be a 4-point line represented over GF(9) by the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where α is in GF(9) $- \{0\}$. Let $p=(0,0,1)^{\mathrm{T}}$ and let M'_α be the 9-cone of M_α having base $E(M_\alpha)$ and apex p.

Theorem 2.1. *If* $\alpha \in$ GF(9) $-$ GF(3), *then* M'_α *and* M'_{-1} *are non-isomorphic* 9-*cones of a* 4-*point line.*

Proof. Assume the contrary. Since p is the unique point in each of M'_α and M'_{-1} lying on four 10-point lines, the isomorphism between M'_α and M'_{-1} must map p to p. Clearly M'_{-1} has a restriction that is isomorphic to PG(2,3) and uses p. Hence, M'_α has a restriction N that is isomorphic to PG(2,3) and uses p.

The 12 points of $E(N)-\{p\}$ lie, three to a line, on the four lines L_1, L_2, L_3, and L_4 through p and each of $(0,1,0)^{\mathrm{T}}, (1,0,0)^{\mathrm{T}}, (1,1,0)^{\mathrm{T}}$, and $(1,\alpha,0)^{\mathrm{T}}$, respectively. Let $(1,\alpha,a)^{\mathrm{T}}$ be a point of N from L_4, and let $(0,1,b_1)^{\mathrm{T}}, (0,1,b_2)^{\mathrm{T}}$, and $(0,1,b_3)^{\mathrm{T}}$ be the points of $N \setminus p$ on L_1. Then it is not difficult to check that, for each i in $\{1,2,3\}$, the line through $(0,1,b_i)^{\mathrm{T}}$ and $(1,\alpha,a)^{\mathrm{T}}$ meets L_2 and L_3 in $(1,0,a-\alpha b_i)^{\mathrm{T}}$ and $(1,1,a+b_i-\alpha b_i)^{\mathrm{T}}$, respectively (see Fig. 1). Then, since $N \cong$ PG(2,3), without loss of generality, $(0,1,b_1)^{\mathrm{T}}$, $(1,0,a-\alpha b_2)^{\mathrm{T}}$, and $(1,1,a+b_3-\alpha b_3)^{\mathrm{T}}$ are collinear. This implies that $(0,1,b_2)^{\mathrm{T}}$, $(1,0,a-\alpha b_1)^{\mathrm{T}}$, and $(1,1,a+b_3-\alpha b_3)^{\mathrm{T}}$ are collinear. The first of these two lines implies that $b_3(1-\alpha)=b_2-\alpha b_1$, while, by symmetry, the second implies that $b_3(1-\alpha)=b_1-\alpha b_2$. Combining these two equations gives $b_2-\alpha b_1=b_1-\alpha b_2$, so $(1+\alpha)b_2=(1+\alpha)b_1$. As $\alpha \in$ GF(9) $-$ GF(3), we deduce that $b_2=b_1$; a contradiction. □

Let N_1 and N_2 be the rank-3 matroids for which geometric representations are shown in Fig. 2. For all prime powers $q \geqslant 4$, both N_1 and N_2 are GF(q)-representable. For each i in $\{1,2\}$, let $M_i = N_i|\{a_1,a'_1,b_1,b'_1,c_1,c'_1\}$. Evidently, $M_1=M_2$. For each i, let N'_i be the q-cone of N_i with apex p and base $E(N_i)$. For each d in $\{a,b,c\}$, let the lines through p and d_1 and through p and d'_1 be $\{d_1,d_2,\ldots,d_q,p\}$ and $\{d'_1,d'_2,\ldots,d'_q,p\}$, respectively.

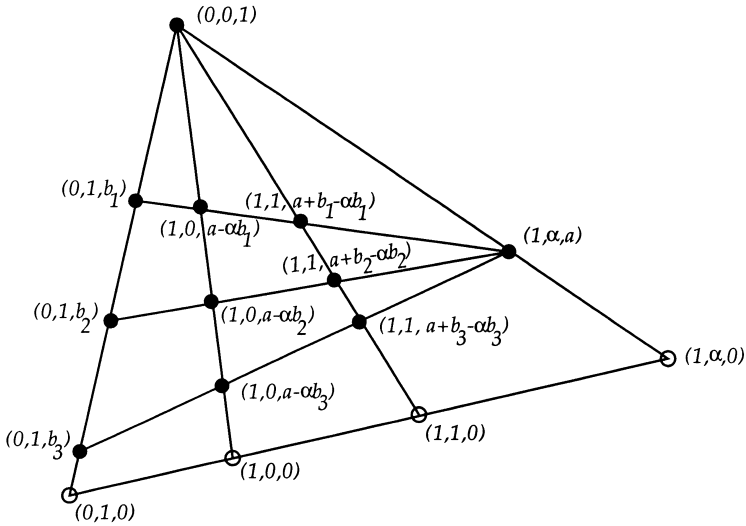

Fig. 1. The points of N and some of its lines.

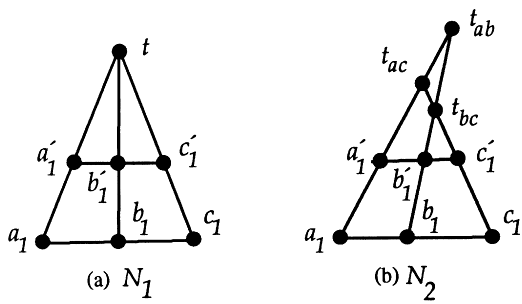

Fig. 2. The matroids N_1 and N_2.

Lemma 2.2. *In M_1', suppose that both $\{a_i, a_j', b_k, b_l'\}$ and $\{b_k, b_l', c_m, c_n'\}$ are circuits. Then so is $\{a_i, a_j', c_m, c_n'\}$.*

Proof. The plane P_{ab} of N_1' spanned by $\{a_i, a_j', b_k, b_l'\}$ meets the line spanned by $\{p, t\}$ in a single point, t'. Since t' also lies in the plane P_{pb} of N_1' spanned by $\{p, b_k, b_l'\}$, we deduce that $\{t', b_k, b_l'\} \subseteq P_{ab} \cap P_{pb}$, so t', b_k, and b_l' are collinear. Similarly, t', a_i, and a_j' are collinear, and t', c_m, and c_n' are collinear. We deduce that $\{a_i, a_j', c_m, c_n'\}$ is a circuit of M_1'. □

Theorem 2.3. *M_1' and M_2' are non-isomorphic q-cones of M_1.*

Proof. It suffices to show that M_2' does not satisfy the condition in the preceding lemma. In N_2', consider the lines through p and t_{ab}, through p and t_{ac}, and through p and t_{bc}. Let t_{ab}', t_{ac}', and t_{bc}' be points of these lines different from p. The plane P' spanned by $\{t_{ab}', t_{ac}', t_{bc}'\}$ meets the plane spanned by $\{p, a_1, a_1'\}$ in the line spanned by t_{ab}' and t_{ac}', and this line contains a unique a_i and a unique a_j'. Likewise, P' contains a unique b_k, a unique b_l', a unique c_m, and a unique c_n'.

Let t_{ab}'' be a point on the line spanned by p and t_{ab} that is different from both p and t_{ab}'. Then the plane P'' spanned by $\{t_{ab}'', t_{ac}', t_{bc}'\}$ contains $\{c_m, c_n'\}$ and $\{a_u, a_v'\}$ for some u and v distinct from i and j, respectively, where t_{ab}'', t_{ac}', a_u, and a_v' are collinear. Since t_{ab}', t_{bc}', b_k, and b_l' are collinear, the set $\{a_u, a_v', b_k, b_l'\}$ spans $\{t_{ab}'', t_{ab}', t_{ac}', t_{bc}'\}$. But the last set has rank 4, so $\{a_u, a_v', b_k, b_l'\}$ is not a circuit of M_2'. However, both $\{a_u, a_v', c_m, c_n'\}$ and $\{b_k, b_l', c_m, c_n'\}$ are circuits of M_2'. We conclude that M_2' fails to satisfy the condition of the last lemma, so $M_2' \ncong M_1'$. □

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