FLEXIPATHS IN MATROIDS

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Abstract. Although the unavoidable minors of large 3-connected matroids were found nearly thirty years ago, there has been little progress on solving the corresponding problem for large 4-connected matroids. This paper aims to take a step towards solving that problem. The objects of study here are 4-paths, that is, sequences $(L, P_1, P_2, \dots, P_n, R)$ of sets that partition the ground set of a matroid so that the union of any proper initial segment of parts is 4-separating. Viewing the ends L and R as fixed, we call such a partition a 4-flexipath if $(L, Q_1, Q_2, \dots, Q_n, R)$ is a 4-path for all permutations $(Q_1, Q_2, \dots, Q_n)$ of $(P_1, P_2, \dots, P_n)$. A straightforward simplification enables us to focus on $(4, c)$-flexipaths for some c in $\{1, 2, 3\}$, that is, those 4-flexipaths for which $\lambda(Q_i) = c$ and $\lambda(Q_i \cup Q_j) > c$ for all distinct i and j. Our main result is that the only non-trivial case that arises here is when $c = 2$. In that case, there are essentially only two possible dual pairs of $(4, c)$-flexipaths when $n \geq 5$. A key technique in the proof of this result is of independent interest. We construct the clonal core of a partitioned matroid. From the relatively simple structure of this clonal core, we can deduce many properties of the original partitioned matroid from the local connectivities between unions of parts of the partition.

1. Introduction

The problem that motivated this paper arose from a project of ours whose goal is to extend the results of [2] to find the unavoidable minors of 4-connected matroids. In essence, the strategy for finding such minors is to use extremal techniques to gradually refine the structure of the matroid until we are finally left with the unavoidable minors themselves. At an intermediate stage, one arrives at a matroid with an ordered partition $(L, P_1, P_2, \dots, P_n, R)$ of its ground set into many parts where this partition induces a nested sequence of 4-separations. In general, permuting the members of $(P_1, P_2, \dots, P_n)$ destroys this property, but understanding the structures for which the property is preserved under such permutations plays an important role in our search for unavoidable minors.

Before we can describe our results, we need some definitions. Our notation and terminology will follow [5]. For a positive integer n, we write $[n]$ for $\{1, 2, \dots, n\}$. Let M be a matroid on a set E. The *connectivity function*

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λ_M of M is the function that is defined on all subsets X of M by $\lambda_M(X) = r(X) + r(E - X) - r(M)$. If X and Y are disjoint subsets of E, then $\kappa(X, Y) = \min\{\lambda_M(Z) : X \subseteq Z \subseteq E - Y\}$.

In the definitions that follow, we focus on the specific cases relevant to this paper. A *path of* 4*-separations in* M is an ordered partition $(L, P_1, P_2, \dots, P_n, R)$ of $E(M)$ such that

(i) $\kappa(L, R) = 3$, and
(ii) $\lambda_M(L \cup P_1 \cup P_2 \cup \dots \cup P_i) = 3$ for all i in $\{0, 1, \dots, n\}$.

For such a path $\mathbf{P}$ of 4-separations, the members of $\mathbf{P}$ are *steps*, and L and R are *end steps* while $P_1, P_2, \dots, P_n$ are *internal steps*.

The path $\mathbf{P}$ is a 4*-flexipath* if $(L, Q_1, Q_2, \dots, Q_n, R)$ is also a path of 4-separations whenever $(Q_1, Q_2, \dots, Q_n)$ is a permutation of $(P_1, P_2, \dots, P_n)$. For a positive integer c, the 4-flexipath $\mathbf{P}$ is a $(4, c)$*-flexipath* if $\lambda_M(P_i) = c$ for all i in $[n]$, and $\lambda_M(P_i \cup P_j) > c$ for all distinct i, j in $[n]$. Imposing these two additional constraints on 4-flexipaths simplifies the analysis. Moreover, descriptions of all 4-flexipaths follow straightfowardly from those for $(4, c)$-flexipaths by noting that if $(L, Q_1, Q_2, \dots, Q_n, R)$ is a 4-flexipath $\mathbf{Q}$, then so is $(L, Q_1, Q_2, \dots, Q_{i-1}, Q_{i+1}, Q_{i+2}, \dots Q_n, Q_i \cup R)$. In this transformation, we say that Q_i has been *absorbed into the right end of* $\mathbf{Q}$.

We show in Lemma 4.4 that, when $c \geq 3$, a $(4, c)$-flexipath has at most two internal steps. The case when $c = 1$ is also straightforward. If we add the additional constraint that M is 3-connected, then all internal steps are singletons and these singletons are either in the closure or coclosure of both L and R. The full description of $(4, 1)$-flexipaths follows routinely from these observations and is given in Corollary 4.9.

This brings us to $(4, 2)$-flexipaths, the most interesting case. The *local connectivity* between disjoint sets X and Y in a matroid M is given by $\sqcap_M(X, Y) = \sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y)$. We write $\sqcap^*(X, Y)$ for $\sqcap_{M^*}(X, Y)$. Let $\mathbf{Q}$ be the $(4, 2)$-flexipath $(L, Q_1, Q_2, \dots, Q_n, R)$.

The flexipath $\mathbf{Q}$ is *spike-reminiscent* if all of the following hold:

(i) $\sqcap(L, R) = 1$ and $\sqcap^*(L, R) = 2$;
(ii) $\sqcap(Q_i, Q_j) = 1$ and $\sqcap^*(Q_i, Q_j) = 0$ for all distinct i and j in $[n]$; and
(iii) $\sqcap(Q_i, L) = \sqcap(Q_i, R) = 1 = \sqcap^*(Q_i, L) = \sqcap^*(Q_i, R)$ for all i in $[n]$.

The flexipath $\mathbf{Q}$ is *paddle-reminiscent* if all of the following hold:

(i) $\sqcap(L, R) = 2$ and $\sqcap^*(L, R) = 1$;
(ii) $\sqcap(Q_i, Q_j) = 0$ and $\sqcap^*(Q_i, Q_j) = 1$ for all distinct i and j in $[n]$; and
(iii) $\sqcap(Q_i, L) = \sqcap(Q_i, R) = 1 = \sqcap^*(Q_i, L) = \sqcap^*(Q_i, R)$ for all i in $[n]$.

Illustrations of spike-reminiscent and paddle-reminiscent flexipaths are shown in Figure 1(i) and (ii), respectively.

The flexipath $\mathbf{Q}$ is *squashed* if all of the following hold:

(i) $\sqcap(L, R) = 3$ and $\sqcap^*(L, R) = 0$;
(ii) $\sqcap(Q_i, Q_j) = 1$ and $\sqcap^*(Q_i, Q_j) = 0$ for all distinct i and j in $[n]$; and

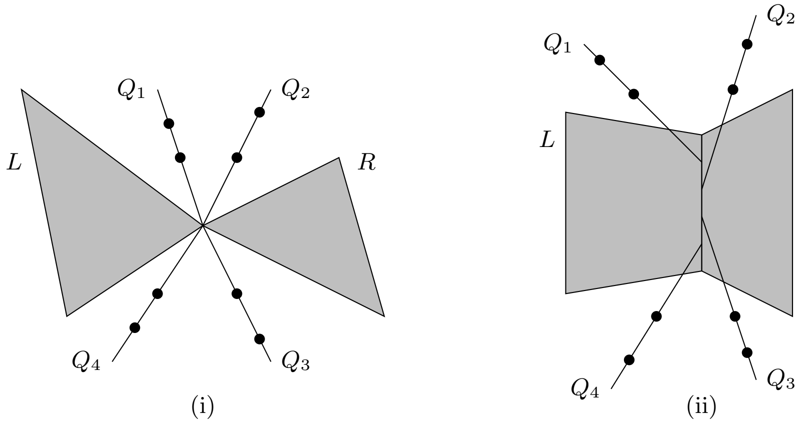

FIGURE 1. (i) A rank-7 matroid with a spike-reminiscent flexipath $(L, Q_1, Q_2, Q_3, Q_4, R)$. (ii) A rank-7 matroid with a paddle-reminiscent flexipath $(L, Q_1, Q_2, Q_3, Q_4, R)$.

(iii) $\sqcap(Q_i, L) = \sqcap(Q_i, R) = 2$, and $\sqcap^*(Q_i, L) = \sqcap^*(Q_i, R) = 0$ for all i in $[n]$.

The flexipath $\mathbf{Q}$ is *stretched* if all of the following hold:

(i) $\sqcap(L, R) = 0$ and $\sqcap^*(L, R) = 3$;
(ii) $\sqcap(Q_i, Q_j) = 0$ and $\sqcap^*(Q_i, Q_j) = 1$ for all distinct i and j in $[n]$; and
(iii) $\sqcap(Q_i, L) = \sqcap(Q_i, R) = 0$, and $\sqcap^*(Q_i, L) = \sqcap^*(Q_i, R) = 2$ for all i in $[n]$.

In $\mathbf{Q}$, the step Q_i is *specially placed* if either $\sqcap(L, R) = 2$ and $\sqcap(L, Q_i) = 2 = \sqcap(R, Q_i)$, or $\sqcap^*(L, R) = 2$ and $\sqcap^*(L, Q_i) = 2 = \sqcap^*(R, Q_i)$. Figure 2 illustrates a rank-7 matroid in which $\{a, b\}$ is a specially placed step of the first type. In Lemma 5.1, we show that any $(4, 2)$-flexipath has at most one specially placed step.

The next theorem follows from Theorem 5.15, the main result of the paper.

Theorem 1.1. *Let $\mathbf{Q}$ be a $(4, 2)$-flexipath with at least five internal steps. When $\mathbf{Q}$ has no specially placed steps, let $\mathbf{Q}'$ be $\mathbf{Q}$; otherwise let $\mathbf{Q}'$ be obtained from $\mathbf{Q}$ by absorbing its specially placed step into its right end. Then $\mathbf{Q}'$ is spike-reminiscent, paddle-reminiscent, squashed or stretched.*

In fact, $\mathbf{Q}$ is spike-reminiscent in M if and only if $\mathbf{Q}$ is paddle reminiscent in M^*, and $\mathbf{Q}$ is stretched in M if and only if $\mathbf{Q}$ is squashed in M^*. It follows that, after any specially placed step is absorbed, there are at least four remaining internal steps and, up to duality, there are only two outcomes for $(4, 2)$-flexipaths. A variety of other outcomes appear for $(4, 2)$-flexipaths with two or three internal steps. While these outcomes are less interesting, it turns out to be useful to understand them for our work on unavoidable minors, so we give a full description of them in Theorem 5.15.

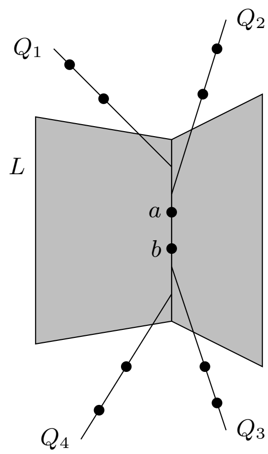

FIGURE 2. A rank-7 matroid in which $\{a, b\}$ is a specially placed step in the flexipath $(L, Q_1, Q_2, \{a, b\}, Q_3, Q_4, R)$.

Finally, we make an observation on the techniques of this paper. Say we have a matroid M with a partition $\{Z_1, Z_2, \dots, Z_n\}$ of the ground set and assume that, to make an argument, we only need to use sets that respect this partition. Then, by making the argument for matroids where the sets have the lowest rank possible given their connectivities, we can deduce the result in general. An advantage of this approach is that it enables us to capture and exploit the intuition that arises when dealing with low-rank sets.

This observation does not appear to have been previously formalised in the literature. Specifically, we move from the partition $\{Z_1, Z_2, \dots, Z_n\}$ of $E(M)$ to an associated matroid M' where each set Z_i is replaced by a clonal class of size $\lambda_M(Z_i)$. We describe sufficient conditions that enable us to make arguments in the matroid M' from which we can draw conclusions about properties of M. The construction of M', the clonal core of $(M, \{Z_1, Z_2, \dots, Z_n\})$, is described in Section 3. This technique is sufficiently general to be of independent interest.

2. Preliminaries

For a matroid M, it is well known that $\lambda_M(X) = r(X) + r^*(X) - |X|$ for all subsets X of $E(M)$. Hence $\lambda_{M^*} = \lambda_M$. When the underlying matroid is clear, we may abbreviate λ_M as λ. The following basic facts about the connectivity and local connectivity functions of a matroid will be used frequently throughout the paper. The first two appear as Lemmas 2.6 and 2.4 of [6] and are easily verified by rewriting everything in terms of ranks of sets in M. The third follows straightforwardly from the first.

Lemma 2.1. *For subsets X and Y of the ground set of a matroid M,*

$$\lambda(X \cup Y) = \lambda(X) + \lambda(Y) - \sqcap(X, Y) - \sqcap^*(X, Y).$$

In the next lemma, (ii) follows from (i) by taking D to be empty.

Lemma 2.2. *In a matroid M, let A, B, C, and D be disjoint subsets of $E(M)$. Then*

(i) $\sqcap(A \cup B, C \cup D) + \sqcap(A, B) + \sqcap(C, D) = \sqcap(A \cup C, B \cup D) + \sqcap(A, C) + \sqcap(B, D)$.
(ii) $\sqcap(A \cup B, C) + \sqcap(A, B) = \sqcap(A \cup C, B) + \sqcap(A, C)$.

Lemma 2.3. *Let $(L, Q_1, Q_2, \dots, Q_n, R)$ be a 4-flexipath. For distinct i and j in $[n]$,*

$$\lambda(Q_i \cup Q_j) \geq \lambda(Q_i).$$

Proof. Since we have a 4-flexipath, $\lambda(L \cup Q_i \cup Q_j) = 3 = \lambda(L \cup Q_i)$. Thus, by Lemma 2.1, $\lambda(L \cup Q_i \cup Q_j) = \lambda(L) + \lambda(Q_i \cup Q_j) - \sqcap(L, Q_i \cup Q_j) - \sqcap^*(L, Q_i \cup Q_j)$, and $\lambda(L \cup Q_i) = \lambda(L) + \lambda(Q_i) - \sqcap(L, Q_i) - \sqcap^*(L, Q_i)$. The lemma follows because the functions $\sqcap$ and $\sqcap^*$ are monotonic. □

3. The Clonal Core of a Matroid

The purpose of this section is to develop a versatile tool for dealing with connectivities and local connectivities of sets in a matroid. In particular, we shall define the clonal core of a matroid M whose ground set has a partition $(Z_1, Z_2, \dots, Z_n)$. This clonal core $(\widehat{M}, (\widehat{Z}_1, \widehat{Z}_2, \dots, \widehat{Z}_n))$ will replace each Z_i by an independent, coindependent set $\widehat{Z}_i$ of clones of size $\lambda(Z_i)$. We shall show that $\lambda_{\widehat{M}}(\widehat{Z}_i) = \lambda_M(Z_i)$ for all i in $[n]$ and that, more generally, for all disjoint subsets I and J of $[n]$, we have $\sqcap_{\widehat{M}}(\cup_{i \in I} \widehat{Z}_i, \cup_{j \in J} \widehat{Z}_j) = \sqcap_M(\cup_{i \in I} Z_i, \cup_{j \in J} Z_j)$.

We begin with a well-known concept. For a matroid M, let X and Y be subsets of $E(M)$. We call $\{X, Y\}$ a *modular pair* if

$$r_M(X) + r_M(Y) = r_M(X \cap Y) + r_M(X \cup Y).$$

A collection $\mathcal{F}$ of subsets of $E(M)$ is a *modular cut* of M if it satisfies the following conditions.

(i) If $X \subseteq Y \subseteq E(M)$ and $X \in \mathcal{F}$, then $Y \in \mathcal{F}$.
(ii) If $X, Y \in \mathcal{F}$ and (X, Y) is a modular pair, then $X \cap Y \in \mathcal{F}$.
(iii) If $Y \in \mathcal{F}$ and $X \subseteq Y$ with $r(X) = r(Y)$, then $X \in \mathcal{F}$.

In [5], a modular cut in a matroid M is defined to be a set $\mathcal{F}$ of flats of M obeying (i) and (ii). The definition just given extends that definition to arbitrary collections of subsets of $E(M)$.

Lemma 3.1. *Let M be a matroid and (R, S) be a partition of $E(M)$. Let $\mathcal{F}$ be the set of subsets X of $E(M)$ for which $\lambda_{M/X}(R - X) = 0$. Then $\mathcal{F}$ is a modular cut of M.*

This lemma is the basis of the proof of the following result.

Theorem 3.2. *Let M be a matroid and (Z, A) be a partition of $E(M)$. Suppose $\lambda_M(Z) > 0$. Let M' be the single-element extension of M by the element e corresponding to the modular cut $\{F \subseteq E(M) : \lambda_{M/F}(Z-F) = 0\}$. Then e is a non-loop element of $\mathrm{cl}_{M'}(Z) \cap \mathrm{cl}_{M'}(A)$ and $r_{M'}(X \cup \{e\}) = r_M(X)$ if and only if X is in the modular cut.*

We say that the matroid M' constructed from M in the last theorem has been obtained by *freely adding e into the guts of* (Z, A). We have symmetry between Z and A in the definition, but for our purposes it helps to focus on one side. Thus we will also say that M' has been obtained from M by *freely adding e into the guts of Z*. We may repeat the operation. Let $\{e_1, e_2, \dots, e_s\}$ be a set disjoint from $E(M)$. Let $M_0 = M$. For $i \geq 1$, inductively define M_i to be the matroid that is obtained from M_{i-1} by freely adding e_i into the guts of Z. Let $\mathcal{F}_{i-1}$ be the modular cut that generates M_i from M_{i-1}. It follows from Lemma 3.4 below that M_s is well defined in that the matroid M_s does not depend on the order in which the elements of $\{e_1, e_2, \dots, e_s\}$ are added. We say that M_s is the matroid obtained by *freely adding $\{e_1, e_2, \dots, e_s\}$ into the guts of Z*.

In the next sequence of lemmas, we shall develop some properties of the matroids obtained by extending freely into the guts of a partition. Throughout, we shall assume that $\lambda_M(Z) = t$.

Lemma 3.3. *If $F \in \mathcal{F}_i$, then $F \in \mathcal{F}_j$ for all $j \geq i \geq 0$.*

Proof. We argue by induction on $j - i$ noting that the result is immediate if $j - i = 0$. Assume the result holds for $j - i < n$ and let $j - i = n$. Then $F \in \mathcal{F}_{j-1}$. Thus $\lambda_{M_{j-1}/F}(Z - F) = 0$ and $r_{M_j}(F \cup \{e_j\}) = r_{M_{j-1}}(F)$. Hence e_j is a loop of M_j/F. Thus $\lambda_{M_j/F}(Z - F) = 0$, so $F \in \mathcal{F}_j$. We conclude, by induction, that the lemma holds. □

Lemma 3.4. *The elements $e_1, e_2, \dots, e_s$ are clones in M_s.*

Proof. We argue by induction on s showing first that e_1 and e_2 are clones in M_2. Assume that this fails. Then there is a subset S of $E(M)$ such that

(i) $e_1 \in \mathrm{cl}_{M_2}(S)$ but $e_2 \notin \mathrm{cl}_{M_2}(S)$; or
(ii) $e_2 \in \mathrm{cl}_{M_2}(S)$ but $e_1 \notin \mathrm{cl}_{M_2}(S)$.

In the first case, as $S \subseteq E(M_0)$ and $e_1 \in \mathrm{cl}_{M_2}(S)$, we deduce that $e_1 \in \mathrm{cl}_{M_1}(S)$. Thus $S \in \mathcal{F}_0$. By Lemma 3.3, $S \in \mathcal{F}_1$. But this implies that $e_2 \in \mathrm{cl}_{M_2}(S)$, a contradiction. In case (ii), $S \in \mathcal{F}_1$, so $\lambda_{M_1/S}(Z - S) = 0$. Thus $Z - S$ is a union of components of M_1/S that avoids e_1, so it is a union of components of $(M_1/S)\backslash e_1$, that is, of M_0/S. Thus $\lambda_{M_0/S}(Z - S) = 0$, so $e_1 \in \mathrm{cl}_{M_1}(S)$. Hence $e_1 \in \mathrm{cl}_{M_2}(S)$, a contradiction. We conclude that e_1 and e_2 are clones in M_2.

Now assume that $e_1, e_2, \dots, e_{s-1}$ are clones in M_{s-1}. By what we have just shown, e_s and e_{s-1} are clones in M_s. Say e_s and e_u are not clones in M_s for some $u \leq s - 2$. Then there is a subset V of $E(M_s) - \{e_u, e_s\}$ such that

(i) $e_u \in \mathrm{cl}_{M_s}(V)$ but $e_s \notin \mathrm{cl}_{M_s}(V)$; or
(ii) $e_s \in \mathrm{cl}_{M_s}(V)$ but $e_u \notin \mathrm{cl}_{M_s}(V)$.

In the first case, $e_u \in \mathrm{cl}_{M_s}(V)$ but $e_s \notin V$, so $e_u \in \mathrm{cl}_{M_{s-1}}(V)$. As e_u and e_{s-1} are clones in M_{s-1}, we deduce that $e_{s-1} \in \mathrm{cl}_{M_{s-1}}(V)$. Hence $e_{s-1} \in \mathrm{cl}_{M_s}(V)$. As e_{s-1} and e_s are clones in M_s, it follows that $e_s \in \mathrm{cl}_{M_s}(V)$, a contradiction. In the second case, $e_{s-1} \in \mathrm{cl}_{M_s}(V)$. As $e_s \notin V$, it follows that $e_{s-1} \in \mathrm{cl}_{M_{s-1}}(V)$. Hence $e_u \in \mathrm{cl}_{M_{s-1}}(V)$ and $e_u \in \mathrm{cl}_{M_s}(V)$, a contradiction. We conclude that $e_1, e_2, \ldots, e_s$ are clones in M_s and the lemma follows by induction. □

Lemma 3.5. $\lambda_{M_s}(Z) = t$.

Proof. Since $A \in \mathcal{F}_0$, we see that $e_1 \in \mathrm{cl}_{M_1}(A)$, so $e_1 \in \mathrm{cl}_{M_s}(A)$. As $e_1, e_2, \ldots, e_s$ are clones in M_s, we see that $\{e_1, e_2, \ldots, e_s\} \in \mathrm{cl}_{M_s}(A)$. Thus $\lambda_{M_s}(Z) = \lambda_M(Z) = t$. □

Lemma 3.6. $r_{M_s}(\{e_1, e_2, \ldots, e_s\}) \leq t$.

Proof. As $Z \in \mathcal{F}_0$, we see that $e_1 \in \mathrm{cl}_{M_1}(Z)$, so $\{e_1, e_2, \ldots, e_s\} \subseteq \mathrm{cl}_{M_s}(Z)$. By submodularity,

$$\begin{aligned} r_{M_s}(\{e_1, e_2, \ldots, e_s\}) &\leq r_{M_s}(A \cup \{e_1, e_2, \ldots, e_s\}) \\ &\quad + r_{M_s}(Z \cup \{e_1, e_2, \ldots, e_s\}) - r(M_s) \\ &= r_M(A) + r_M(Z) - r(M) = t. \end{aligned}$$

□

Lemma 3.7. *For all $u \leq t$, the set $\{e_1, e_2, \ldots, e_u\}$ is independent in M_s.*

Proof. Let $X_i = \{e_1, e_2, \ldots, e_i\}$. It suffices to prove that $e_{i+1} \notin \mathrm{cl}_{M_{i+1}}(X_i)$ when $i + 1 \leq s \leq t$. Assume the contrary. Then $X_i \in \mathcal{F}_i$. Thus

$$\begin{aligned} 0 &= r_{M_i/X_i}(Z) + r_{M_i/X_i}(A) - r(M_i/X_i) \\ &= r_{M_i}(Z \cup X_i) + r_{M_i}(A \cup X_i) - r(M_i) - r_{M_i}(X_i) \\ &= r_M(Z) + r_M(A) - r(M) - r_{M_i}(X_i) \\ &= \lambda_M(Z) - r_{M_i}(X_i). \end{aligned}$$

Hence $t = \lambda_M(Z) = r_{M_i}(X_i) \leq i < u \leq t$, a contradiction. □

Lemma 3.8. *If $X \subseteq A$ and $\mathrm{cl}_{M_s}(X) \cap \{e_1, e_2, \ldots, e_s\} \neq \emptyset$, then $\sqcap(X, Z) = t$.*

Proof. Because $e_1, e_2, \ldots, e_s$ are clones in M_s, we may assume that $e_1 \in \mathrm{cl}_{M_s}(X)$. Hence $e_1 \in \mathrm{cl}_{M_1}(X)$. As $\mathcal{F}_0$ is the modular cut that generates M_1 from M, it follows that $X \in \mathcal{F}_0$. Thus

$$\begin{aligned} 0 &= r_{M/X}(A - X) + r_{M/X}(Z) - r(M/X) \\ &= r_M(A) + r_M(Z \cup X) - r(M) - r_M(X) \\ &= (r_M(A) + r_M(Z) - r(M)) - (r_M(Z) + r_M(X) - r_M(Z \cup X)) \\ &= \lambda_M(Z) - \sqcap_M(Z, X). \end{aligned}$$

We deduce that $t = \lambda_M(Z) = \sqcap_M(Z, X)$. □

The case that is of most interest to us is the case when $s = t$. The next result captures some key properties in this case. We state the full set of hypotheses.

Theorem 3.9. *Let M be a matroid and (Z, A) be a partition of its ground set for which $\lambda_M(Z) = t > 0$. Let M_t denote the matroid obtained by freely adding the set $\{e_1, e_2, \dots, e_t\}$ into the guts of Z. Then $\{e_1, e_2, \dots, e_t\}$ is an independent set of clones in M_t. Moreover $\mathrm{cl}_{M_t}(A) \cap \mathrm{cl}_{M_t}(Z)$ contains and is spanned by $\{e_1, e_2, \dots, e_t\}$.*

Proof. By Lemmas 3.6 and 3.7, $\{e_1, e_2, \dots, e_t\}$ is a rank-t set of clones in M_t. As $e_i \in \mathrm{cl}_{M_i}(Z) \cap \mathrm{cl}_{M_i}(A \cup \{e_1, e_2, \dots, e_{i-1}\})$ for all i, we see that $e_i \in \mathrm{cl}_{M_i}(Z) \cap \mathrm{cl}_{M_i}(A)$. Hence $\mathrm{cl}_{M_t}(Z) = \mathrm{cl}_M(Z) \cup \{e_1, e_2, \dots, e_t\}$ and $\mathrm{cl}_{M_t}(A) = \mathrm{cl}_M(A) \cup \{e_1, e_2, \dots, e_t\}$. Thus $r(\mathrm{cl}_{M_t}(Z)) = r_M(Z)$ and $r(\mathrm{cl}_{M_t}(A)) = r_M(A)$. Since $\{e_1, e_2, \dots, e_t\} \subseteq \mathrm{cl}_{M_t}(Z) \cap \mathrm{cl}_{M_t}(A)$, we deduce that

$$
\begin{aligned}
t &\leq r_{M_t}(\mathrm{cl}_{M_t}(A) \cap \mathrm{cl}_{M_t}(Z)) \\
&= r_M((\mathrm{cl}_M(A) \cap \mathrm{cl}_M(Z)) \cup \{e_1, e_2, \dots, e_t\}) \\
&= r_M((\mathrm{cl}_M(A) \cap \mathrm{cl}_M(Z)) \\
&\leq r_M((\mathrm{cl}_M(A)) + r_M(\mathrm{cl}_M(Z)) - r(M) \\
&= r_M(A) + r_M(Z) - r(M) \\
&= \lambda_M(Z) = t.
\end{aligned}
$$

Hence $\{e_1, e_2, \dots, e_t\}$ is, indeed, a basis for $M_t|(\mathrm{cl}_{M_t}(A) \cap \mathrm{cl}_{M_t}(Z))$. □

For the next five results, we remain under the hypotheses of Theorem 3.9. Let $\{e_1, e_2, \dots, e_t\} = G$. Then the ground set of the matroid M_t constructed in the last theorem is the disjoint union of Z, A, and G.

Lemma 3.10. *Let $X \subseteq A$. Then $r_{M/Z}(X) = r_{M/G}(X)$, that is,*

$$r_{M_t}(X \cup Z) - r_{M_t}(Z) = r_{M_t}(X \cup G) - r_{M_t}(G).$$

Proof. By Theorem 3.9, G spans $\mathrm{cl}_{M_t}(Z) \cap \mathrm{cl}_{M_t}(A)$. Since $\lambda_M(Z) = t = r_{M_t}(G)$, we deduce that $\lambda_{M_t/G}(Z) = 0$. Thus

$$M_t/G\backslash Z = M_t/G/Z = (M_t/Z)/G = (M_t/Z)\backslash G$$

where the last step holds because $G \subseteq \mathrm{cl}_{M_t}(Z)$. Hence $r_{M_t/G\backslash Z}(X) = r_{M_t/Z\backslash G}(X)$, that is, $r_{M/Z}(X) = r_{M/G}(X)$. □

Lemma 3.11. *Let X and Y be disjoint subsets of A. Then*

(i) $\sqcap_M(X, Y) = \sqcap_{M_t\backslash Z}(X, Y)$*; and*
(ii) $\sqcap_M(X \cup Z, Y) = \sqcap_{M_t\backslash Z}(X \cup G, Y)$*.*

Proof. Part (i) is immediate since M is a restriction of M_t. For (ii), we have

$$\begin{aligned}
\sqcap_M(X\cup Z, Y) &= r_M(X\cup Z) + r_M(Y) - r_M(X\cup Z\cup Y)\\
&= (r_M(X\cup Z) - r_M(Z)) + r_M(Y)\\
&\quad -(r_M(X\cup Z\cup Y) - r_M(Z))\\
&= (r_{M_t}(X\cup G) - r_{M_t}(G)) + r_{M_t}(Y)\\
&\quad -(r_{M_t}(X\cup G\cup Y) - r_{M_t}(G))\\
&= r_{M_t}(X\cup G) + r_{M_t}(Y) - r_{M_t}(X\cup G\cup Y)\\
&= \sqcap_{M_t}(X\cup G, Y)\\
&= \sqcap_{M_t\backslash Z}(X\cup G, Y)
\end{aligned}$$

where the third step follows from two applications of Lemma 3.10. □

Corollary 3.12. *Suppose $F \subseteq A$. Then $F\cup Z$ is a flat of M if and only if $F\cup G$ is a flat of $M_t\backslash Z$.*

Proof. Take e in $A-F$. By Lemma 3.11(ii), $\sqcap_M(F\cup Z,\{e\}) = \sqcap_{M_t\backslash Z}(F\cup G,\{e\})$. Thus $e\in \mathrm{cl}_M(F\cup Z)$ if and only if $e\in \mathrm{cl}_{M_t\backslash Z}(F\cup G)$. The result follows. □

Lemma 3.13. *Let X and Y be disjoint subsets of A. Then*

(i) $\sqcap^*_M(X,Y) = \sqcap^*_{M_t\backslash Z}(X,Y)$; *and*
(ii) $\sqcap^*_M(X\cup Z,Y) = \sqcap^*_{M_t\backslash Z}(X\cup G,Y)$.

Proof. Suppose $X'\in\{X, X\cup G\}$. Then

$$\begin{aligned}
\sqcap^*_{M_t\backslash Z}(X',Y) &= \sqcap_{(M_t\backslash Z)^*}(X',Y)\\
&= \sqcap_{M_t^*/Z}(X',Y)\\
&= r_{M_t^*/Z}(X') + r_{M_t^*/Z}(Y) - r_{M_t^*/Z}(X'\cup Y)\\
&= r_{M_t^*}(X'\cup Z) + r_{M_t^*}(Y\cup Z)\\
&\quad -r_{M_t^*}(X'\cup Y\cup Z) - r_{M_t^*}(Z)\\
&= r(M_t\backslash(X'\cup Z)) + r(M_t\backslash(Y\cup Z))\\
&\quad -r(M_t\backslash(X'\cup Y\cup Z)) - r(M_t\backslash Z). \qquad (3.1)
\end{aligned}$$

Thus, recalling that $E(M_t)$ is the disjoint union of Z, A, and G, we have

$$\begin{aligned}
\sqcap^*_{M_t\backslash Z}(X,Y) &= r_{M_t}((A-X)\cup G) + r_{M_t}((A-Y)\cup G)\\
&\quad -r_{M_t}((A-(X\cup Y))\cup G) - r_{M_t}(A\cup G)\\
&= r_{M_t}((A-X)\cup Z) + r_{M_t}((A-Y)\cup Z)\\
&\quad -r_{M_t}((A-(X\cup Y))\cup Z) - r_{M_t}(A\cup Z) \qquad (3.2)
\end{aligned}$$

where the last step follows by four applications of Lemma 3.10.

For X'' in $\{X, X \cup Z\}$, we have

$$\begin{aligned}\sqcap^*_M(X'', Y) &= r^*_M(X'') + r^*_M(Y) - r^*_M(X'' \cup Y)\\ &= |X''| + r(E(M) - X'') + |Y| + r(E(M) - Y) - |X'' \cup Y|\\ &\quad -r(E(M) - (X'' \cup Y)) - r(M). \qquad (3.3)\end{aligned}$$

Thus

$$\sqcap^*_M(X, Y) = r((A-X)\cup Z) + r((A-Y)\cup Z) - r((A-(X\cup Y))\cup Z) - r(M).$$

Therefore, by (3.2),

$$\sqcap^*(X, Y) = \sqcap^*_{M_t\backslash Z}(X, Y),$$

that is, (i) holds.

Now, by (3.3),

$$\sqcap^*_M(X \cup Z, Y) = r(A - X) + r((A - Y) \cup Z) - r(A - (X \cup Y)) - r(A \cup Z).$$

Moreover, by (3.1),

$$\begin{aligned}\sqcap^*_{M_t\backslash Z}(X \cup G, Y) &= r_{M_t}(A - X) + r_{M_t}((A - Y) \cup G)\\ &\quad -r_{M_t}(A - (X \cup Y)) - r_{M_t}(A \cup G)\\ &= r_M(A - X) + r_M((A - Y) \cup Z)\\ &\quad -r_M(A - (X \cup Y)) - r_M(A \cup Z)\end{aligned}$$

by two applications of Lemma 3.10. We conclude that $\sqcap^*_M(X \cup Z, Y) = \sqcap^*_{M_t\backslash Z}(X \cup G, Y)$, that is (ii) holds. $\square$

Lemma 3.14. *In $M_t\backslash Z$, the set G is an independent, coindependent set of clones of cardinality $\lambda_M(Z)$, and $\lambda_{M_t\backslash Z}(G) = \lambda_M(Z)$.*

Proof. By Theorem 3.9, G is an independent set of clones in M_t of cardinality $\lambda_M(Z)$. Now

$$\begin{aligned}r^*_{M_t\backslash Z}(G) &= r_{M^*_t/Z}(G)\\ &= r_{M^*_t}(G \cup Z) - r_{M^*_t}(Z)\\ &= |G \cup Z| + r_{M_t}(A) - r(M_t)\\ &\quad -(|Z| + r_{M_t}(A \cup G) - r(M_t))\\ &= |G|\end{aligned}$$

where the last step holds because A spans G in M_t. Thus G is coindependent in $M_t\backslash Z$. Finally,

$$\begin{aligned}\lambda_{M_t\backslash Z}(G) &= r_{M_t\backslash Z}(G) + r^*_{M_t\backslash Z}(G) - |G|\\ &= |G|\\ &= \lambda_M(Z)\end{aligned}$$

where the second step follows because G is independent and coindependent in $M_t\backslash Z$. $\square$

We now consider freely adding different elements into the guts of disjoint sets in a matroid.

Lemma 3.15. *Let M be a matroid and suppose $A \subseteq E(M)$. Let $M_{\langle a\rangle}$ be the matroid that is obtained from M by freely adding a into the guts of A. If $Y \subseteq E(M)$, then $M_{\langle a\rangle}/Y$ is obtained from M/Y by freely adding a into the guts of $A - Y$.*

Proof. Say $X \subseteq E(M) - Y$. Then $a \in \mathrm{cl}_{M_{\langle a\rangle}/Y}(X)$ if and only if $a \in \mathrm{cl}_{M_{\langle a\rangle}}(X \cup Y)$. From the definition of $M_{\langle a\rangle}$, the latter holds if and only if $\lambda_{M/(X\cup Y)}(A-(X\cup Y)) = 0$, that is, if and only if $\lambda_{(M/Y)/X}((A-Y)-X) = 0$. But $\lambda_{(M/Y)/X}((A-Y)-X) = 0$ if and only if X is in the modular cut that corresponds to freely adding a into the guts of $A - Y$ in M/Y. □

Lemma 3.16. *Let M be a matroid, let A and B be disjoint subsets of $E(M)$ and let $\{a, b\}$ be disjoint from $E(M)$. Let $M_{\langle a\rangle}$ be the matroid that is obtained from M by freely adding a into the guts of A; let $M_{\langle b\rangle}$ be obtained from M by freely adding b into the guts of B; let $M_{\langle a\rangle\langle b\rangle}$ be the matroid that is obtained from $M_{\langle a\rangle}$ by freely adding b into the guts of B; and let $M_{\langle b\rangle\langle a\rangle}$ be the matroid that is obtained from $M_{\langle b\rangle}$ by freely adding a into the guts of A. Then $M_{\langle a\rangle\langle b\rangle} = M_{\langle b\rangle\langle a\rangle}$.*

Proof. Clearly $M_{\langle a\rangle\langle b\rangle}\backslash b = M_{\langle a\rangle}$ and $M_{\langle b\rangle\langle a\rangle}\backslash a = M_{\langle b\rangle}$. Next we show that

3.16.1. *$M_{\langle a\rangle\langle b\rangle}\backslash a = M_{\langle b\rangle}$ and $M_{\langle b\rangle\langle a\rangle}\backslash b = M_{\langle a\rangle}$.*

Suppose $X \subseteq E(M)$. We prove that $b \in \mathrm{cl}_{M_{\langle a\rangle\langle b\rangle}\backslash a}(X)$ if and only if $b \in \mathrm{cl}_{M_{\langle b\rangle}}(X)$. Since $M_{\langle a\rangle\langle b\rangle}\backslash a, b = M_{\langle b\rangle}\backslash b$, this will prove that $M_{\langle a\rangle\langle b\rangle}\backslash a = M_{\langle b\rangle}$. Say $b \in \mathrm{cl}_{M_{\langle a\rangle\langle b\rangle}\backslash a}(X)$. Then $b \in \mathrm{cl}_{M_{\langle a\rangle\langle b\rangle}}(X)$. Hence $\lambda_{M_{\langle a\rangle}/X}(B - X) = 0$. But $M/X = M_{\langle a\rangle}/X\backslash a$, so $\lambda_{M/X}(B - X) = 0$. Thus $b \in \mathrm{cl}_{M_{\langle b\rangle}}(X)$.

Assume that $b \in \mathrm{cl}_{M_{\langle b\rangle}}(X)$. Then $\lambda_{M/X}(B-X) = 0$. Since $a \in \mathrm{cl}_{M_{\langle a\rangle}}(A)$ and $A \subseteq E(M)-B$, we have $a \in \mathrm{cl}_{M_{\langle a\rangle}/X}((E(M)-B)-X)$. It follows that $\lambda_{M_{\langle a\rangle}/X}(B - X) = 0$. Hence $b \in \mathrm{cl}_{M_{\langle a\rangle\langle b\rangle}}(X)$. Thus 3.16.1 holds.

Assume that $M_{\langle a\rangle\langle b\rangle} \neq M_{\langle b\rangle\langle a\rangle}$, and that, amongst all counterexamples to the lemma, $|E(M)|$ is minimal. Then there is a set Z that is independent in one of $M_{\langle a\rangle\langle b\rangle}$ and $M_{\langle b\rangle\langle a\rangle}$, say the second, and is a circuit in the other, $M_{\langle a\rangle\langle b\rangle}$. By (3.16.1), $\{a, b\} \subseteq Z$. This implies that neither a nor b is a loop of $M_{\langle a\rangle\langle b\rangle}$ or of $M_{\langle b\rangle\langle a\rangle}$. Hence $\lambda_M(A), \lambda_M(B) > 0$.

Let $Z' = Z - \{a, b\}$ and suppose $Z' \neq \emptyset$. In this case, it follows from Lemma 3.15 that the triple $(M/Z', M_{\langle a\rangle\langle b\rangle}/Z', M_{\langle b\rangle\langle a\rangle}/Z')$ also gives a counterexample to the theorem contradicting the minimality of $|E(M)|$. Hence $Z = \{a, b\}$.

Let $C = E(M) - (A \cup B)$. Since $\{a, b\}$ is a circuit in $M_{\langle a\rangle\langle b\rangle}$, we have $b \in \mathrm{cl}_{M_{\langle a\rangle\langle b\rangle}}(\{a\})$. This means that $\{a\}$ is in the modular cut that generates $M_{\langle a\rangle\langle b\rangle}$ from $M_{\langle a\rangle}$. Since $M_{\langle a\rangle\langle b\rangle}$ is obtained from $M_{\langle a\rangle}$ by freely adding b into the guts of B, we have $\lambda_{M_{\langle a\rangle}/a}(B) = 0$. We have observed that $\lambda_M(B) > 0$. We deduce that $(A \cup C, B)$ is a 2-separation in M, so $M_{\langle a\rangle}$ is a parallel connection with basepoint a of matroids with ground sets $A \cup C \cup a$ and $B \cup a$. Hence $a \in \mathrm{cl}_{M_{\langle a\rangle}}(A \cup C)$ and $a \in \mathrm{cl}_{M_{\langle a\rangle}}(B)$.

Since $M_{\langle b\rangle\langle a\rangle}\backslash b = M_{\langle a\rangle}$, we have that $a \in \mathrm{cl}_{M_{\langle b\rangle\langle a\rangle}}(A \cup C)$ and that $a \in \mathrm{cl}_{M_{\langle b\rangle\langle a\rangle}}(B)$. As $M_{\langle b\rangle}$ is obtained from M by freely adding b into the guts of B, and $A\cup C = E(M)-B$, we have that $b \in \mathrm{cl}_{M_{\langle b\rangle}}(B)$ and $b \in \mathrm{cl}_{M_{\langle b\rangle}}(A\cup C)$. But $M_{\langle b\rangle\langle a\rangle}$ is an extension of $M_{\langle b\rangle}$. It follows that $b \in \mathrm{cl}_{M_{\langle b\rangle\langle a\rangle}}(B)$ and $b \in \mathrm{cl}_{M_{\langle b\rangle\langle a\rangle}}(A \cup C)$.

Now $\sqcap_{M_{\langle b\rangle\langle a\rangle}}(A \cup C, B) = 1$ and $\{a, b\} \subseteq \mathrm{cl}_{M_{\langle b\rangle\langle a\rangle}}(A \cup C) \cap \mathrm{cl}_{M_{\langle b\rangle\langle a\rangle}}(B)$. Hence $\{a, b\}$ is dependent in $M_{\langle b\rangle\langle a\rangle}$, contradicting the assumption that this set is independent in $M_{\langle b\rangle\langle a\rangle}$. □

Lemma 3.17. *Let $\{X_1, X_2, \ldots, X_n\}$ be a collection of disjoint sets in a matroid M and let $\{Y_1, Y_2, \ldots, Y_n\}$ be a collection of disjoint sets each of which is disjoint from $E(M)$. Let ϕ be a permutation of $[n]$. Let $M_{\phi(0)} = M$, and, for each i in $[n]$, let $M_{\phi(i)}$ be the matroid that is obtained from $M_{\phi(i-1)}$ by freely adding the elements of Y_i into the guts of X_i. Let $M_\phi = M_{\phi(n)}$. Then the following hold.*

(i) *If ψ is also a permutation of $[n]$, then $M_\psi = M_\phi$.*
(ii) *If $i \in [n]$, then M_ϕ is obtained from $M_\phi\backslash Y_i$ by freely adding the elements of Y_i into the guts of X_i.*

Proof. Part (i) follows from Lemma 3.16 and a routine induction. We omit the details. For (ii), choose a permutation ψ such that $\psi(i) = n$. □

The clonal core of a partitioned matroid. When M is a matroid on a set E, and $\mathcal{X}$ is a partition $\{X_1, X_2, \ldots, X_n\}$ of E, we call the pair $(M, \mathcal{X})$ a *partitioned matroid*. We now describe a construction that builds an associated matroid $\widehat{M}$ from the partitioned matroid $(M, \mathcal{X})$.

(i) Let $\{Y_1, Y_2, \ldots, Y_n\}$ be a collection of disjoint sets each disjoint from E such that $|Y_i| = \lambda_M(X_i)$ for each i in $[n]$.
(ii) Let $M_0 = M$ and, for each i in $[n]$, let M_i be the matroid obtained from M_{i-1} by freely adding the elements of Y_i into the guts of X_i in M_{i-1}.
(iii) Let $\widehat{M} = M\backslash E$.

It follows from Lemma 3.17 that the matroid $\widehat{M}$ does not depend on the ordering of the members of $\mathcal{X}$. We say that $\widehat{M}$ is the *clonal core* of $(M, \mathcal{X})$. Note that there is no assumption here that $\lambda(X_i) > 0$. When $\lambda(X_i) = 0$, we have that $Y_i = \emptyset$. In particular, if some X_i is a separator of M, then the clonal core of M is the same as the clonal core of $M\backslash X_i$. Thus if every X_i is a separator of M, then the clonal core of $(M, \{X_1, X_2, \ldots, X_n\})$ is the empty matroid, $U_{0,0}$.

A major reason for the introduction of the clonal core is to enable us to infer certain connectivity properties of M from the corresponding connectivity properties of $\widehat{M}$. The proof of the next result, though not deep, is long and technical. When $\{W_1, W_2, \ldots, W_n\}$ is a family of subsets of a set S, and J is a non-empty subset of $[n]$, we write W_J for $\cup_{j\in J} W_j$; when J is empty, $W_J = \emptyset$.

Theorem 3.18. *Let M be a matroid whose ground set is partitioned into sets $X_1, X_2, \dots, X_n$. Then the clonal core $\widehat{M}$ of $(M, \{X_1, X_2, \dots, X_n\})$ has ground set that is the disjoint union of $Y_1, Y_2, \dots, Y_n$ and has cardinality $\lambda_M(X_1) + \lambda_M(X_2) + \cdots + \lambda_M(X_n)$. Moreover, in $\widehat{M}$,*

(i) *each Y_i consists of an independent, coindependent set of clones of cardinality $\lambda_M(X_i)$; and*
(ii) *for all non-empty disjoint subsets J and K of $[n]$,*
    (a) $\lambda_M(X_J) = \lambda_{\widehat{M}}(Y_J)$*;*
    (b) $\sqcap_M(X_J, X_K) = \sqcap_{\widehat{M}}(Y_J, Y_K)$*; and*
    (c) $\sqcap^*_M(X_J, X_K) = \sqcap^*_{\widehat{M}}(Y_J, Y_K)$*.*

Proof. Let $\lambda_M(X_i) = t_i$. Construct the matroid M_{t_1} from M by freely adding a t_1-element independent set G_1 of clones to the guts of $(X_1, X_2 \cup X_3 \cup \cdots \cup X_n)$ as in Theorem 3.9. Let $N_1 = M_{t_1} \backslash X_1$. By Lemma 3.14, the set G_1 is an independent, coindependent set of clones in N_1 and $\lambda_{N_1}(G_1) = \lambda_M(X_1)$. Then the ground set of N_1 has a partition into non-empty sets $G_1, X_2, X_3, \dots, X_n$. Moreover, by Lemma 3.11, for all disjoint subsets J and K of $\{2, 3, \dots, n\}$, we have $\sqcap_M(X_J, X_K) = \sqcap_{N_1}(X_J, X_K)$ and $\sqcap_M(X_1 \cup X_J, X_K) = \sqcap_{N_1}(G_1 \cup X_J, X_K)$. Also, by Lemma 3.13, $\sqcap^*_M(X_J, X_K) = \sqcap^*_{N_1}(X_J, X_K)$ and $\sqcap^*_M(X_1 \cup X_J, X_K) = \sqcap^*_{N_1}(G_1 \cup X_J, X_K)$.

Assume that $N_1, N_2, \dots, N_i$ have been defined so that $E(N_i)$ is the disjoint union of $G_1, G_2, \dots, G_i, X_{i+1}, X_{i+2}, \dots, X_n$ where

(1) for $j \le i$, each G_j is an independent, coindependent set of clones of N_i of cardinality $\lambda_M(X_j)$; and
(2) for all disjoint subsets I_1 and I_2 of $\{1, 2, \dots, i\}$ and all disjoint subsets J_1 and J_2 of $\{i+1, i+2, \dots, n\}$,

$$\sqcap_M (X_{I_1} \cup X_{J_1}, X_{I_2} \cup X_{J_2}) = \sqcap_{N_i}(G_{I_1} \cup X_{J_1}, G_{I_2} \cup X_{J_2}) \tag{3.4}$$

and

$$\sqcap^*_M (X_{I_1} \cup X_{J_1}, X_{I_2} \cup X_{J_2}) = \sqcap^*_{N_i}(G_{I_1} \cup X_{J_1}, G_{I_2} \cup X_{J_2}). \tag{3.5}$$

To define N_{i+1} from N_i, first extend the latter by an independent, coindependent set G_{i+1} of clones added into the guts of $(X_{i+1}, E(N_i) - X_{i+1})$ where $|G_{i+1}| = \lambda_M(X_{i+1})$. Let N'_i be the resulting extension and let $N_{i+1} = N'_i \backslash X_{i+1}$. Thus

$$E(N_{i+1}) = G_1 \cup G_2 \cup \cdots \cup G_{i+1} \cup X_{i+2} \cup \cdots \cup X_n.$$

By Lemma 3.14, in N_{i+1}, the set G_{i+1} is independent and coindependent. Moreover, G_{i+1} is a set of clones of cardinality $\lambda_{N_i}(X_{i+1})$. By (3.4), this cardinality is $\lambda_M(X_{i+1})$.

3.18.1. *For each t in $\{1, 2, \dots, i\}$, the set G_t is an independent, coindependent set of clones of cardinality $\lambda_M(X_t)$.*

To see this, first note that $|G_t| = \lambda_M(X_t)$. Moreover, G_t is an independent, coindependent set of clones in N_t. Thus G_t is an independent set in N_{i+1}. Suppose that N_{i+1} has a cyclic flat F for which there are elements x and y of G_t such that $x \in F$ and $y \notin F$. Because x and y are clones in $N_{i+1}\backslash G_{i+1}$, which equals $N_i\backslash X_{i+1}$, the cyclic flat F contains an element of G_{i+1}. Thus F contains G_{i+1}. By Corollary 3.12, $(F - G_{i+1}) \cup X_{i+1}$ is a flat of N_i that contains x but not y. Thus x must be a coloop of $N_i|((F - G_{i+1}) \cup X_{i+1})$. But x is not a coloop of $N_{i+1}|F$, so F contains a circuit C containing x. Then C meets G_{i+1}. Let $C \cap G_{i+1} = \{x_1, x_2, \dots, x_s\}$. Then $x_1 \in \mathrm{cl}_{N_i'}(X_{i+1})$, so there is a circuit C_1 such that $x_1 \in C_1 \subseteq X_{i+1} \cup \{x_1\}$. Thus there is a circuit C' such that $x \in C' \subseteq (C \cup C_1) - \{x_1\}$. Hence $C' \cap G_{i+1} \subseteq \{x_2, x_3, \dots, x_s\}$. By repeatedly eliminating the elements of $C' \cap G_{i+1}$, we obtain the contradiction that x is in a circuit that is contained in $(F - G_i) \cup X_{i+1}$. We conclude that G_t is a set of clones in N_{i+1}.

Finally, from Lemma 3.11(ii), $\lambda_{N_{i+1}}(G_t) = \lambda_{N_i}(G_t) = |G_t|$, so G_t is coindependent in N_{i+1}. Thus 3.18.1 holds.

Now let I_1 and I_2 be disjoint subsets of $\{1, 2, \dots, i\}$ and let J_1 and J_2 be disjoint subsets of $\{i+2, i+3, \dots, n\}$. Then, by (3.4) and Lemma 3.11, we have

$$\begin{aligned} \sqcap_M(X_{I_1 \cup J_1}, X_{I_2 \cup J_2}) &= \sqcap_{N_i}(G_{I_1} \cup X_{J_1}, G_{I_2} \cup X_{J_2}) \\ &= \sqcap_{N_{i+1}}(G_{I_1} \cup X_{J_1}, G_{I_2} \cup X_{J_2}), \end{aligned}$$

and

$$\begin{aligned} \sqcap_M(X_{I_1 \cup J_1 \cup \{i+1\}}, X_{I_2 \cup J_2}) &= \sqcap_{N_i}(G_{I_1} \cup X_{J_1 \cup \{i+1\}}, G_{I_2} \cup X_{J_2}) \\ &= \sqcap_{N_{i+1}}(G_{I_1 \cup \{i+1\}} \cup X_{J_1}, G_{I_2} \cup X_{J_2}). \end{aligned}$$

Likewise, by 3.5 and Lemma 3.13,

$$\sqcap^*_M(X_{I_1 \cup J_1}, X_{I_2 \cup J_2}) = \sqcap^*_{N_{i+1}}(G_{I_1} \cup X_{J_1}, G_{I_2} \cup X_{J_2})$$

and

$$\sqcap_M(X_{I_1 \cup J_1 \cup \{i+1\}}, X_{I_2 \cup J_2}) = \sqcap_{N_{i+1}}(G_{I_1 \cup \{i+1\}} \cup X_{J_1}, G_{I_2} \cup X_{J_2}).$$

The lemma follows by taking $Y_i = G_i$ for all i in $[n]$, noting that we get from Lemma 3.17 that $\widehat{M}$ is the matroid N_n constructed above. □

4. The Behaviour of $(4, c)$-Flexipaths

When we have a matroid M having a path $(L, Q_1, Q_2, \dots, Q_n, R)$ of 4-separations and $t \leq n$, we can consider

$$(L \cup Q_1 \cup Q_2 \cup \cdots \cup Q_j, Q_{j+1}, Q_{j+2}, \dots, Q_{j+t}, Q_{j+t+1} \cup Q_{j+t+2} \cup \cdots \cup Q_n \cup R),$$

which is also a path 4-separations, this one having exactly t internal steps. Moreover, if the original path is a 4-flexipath, so too is the second path.

Now let $(L, Q_1, Q_2, \dots, Q_n, R)$ be a 4-flexipath in a matroid M. Because we are dealing with a flexipath, we may use the idea from the previous paragraph of absorbing internal steps into the end steps to assume that $\lambda(Q_i) = \lambda(Q_j)$ for all i and j. By Lemma 2.3, for distinct i and j, we have $\lambda(Q_i \cup Q_j) \geq \lambda(Q_i)$. If equality holds here, we may replace Q_i and Q_j by a new step, $Q_i \cup Q_j$. By repeating this process, we eventually obtain a $(4, c)$-flexipath for some c in $\{1, 2, 3\}$, that is, $\lambda(Q_i) = c$ for all i, and $\lambda(Q_i \cup Q_j) > c$ for all distinct i and j.

In this section, we shall derive some general properties of a $(4, c)$-flexipath $(L, Q_1, Q_2, \dots, Q_n, R)$. We show in Lemma 4.4 that we may assume that $c \leq 3$ otherwise $n \leq 1$. By Theorem 3.18, there is a matroid $\widehat{M}$ having $cn + 6$ elements whose ground set is the disjoint union of the sets $\widehat{L}, \widehat{Q}_1, \widehat{Q}_2, \dots, \widehat{Q}_n, \widehat{R}$ where, for each i in $[n]$, the set $\widehat{Q}_i$ is a c-element independent, coindependent set of clones, and each of $\widehat{L}$ and $\widehat{R}$ consists of a 3-element independent, coindependent set of clones. Moreover, for all subsets I of $\{1, 2, \dots, n\}$, we have $\lambda_{\widehat{M}}(\widehat{L} \cup \widehat{Q}_I) = 3$ and, for all disjoint subsets I_1 and I_2 of $[n]$,

$$\sqcap_M(Q_{I_1}, Q_{I_2}) = \sqcap_{\widehat{M}}(\widehat{Q}_{I_1}, \widehat{Q}_{I_2})$$

and

$$\sqcap_M(L \cup R \cup Q_{I_1}, Q_{I_2}) = \sqcap_{\widehat{M}}(\widehat{L} \cup \widehat{R} \cup \widehat{Q}_{I_1}, \widehat{Q}_{I_2}).$$

In view of this, as noted in the previous section, we can infer much about the matroid M by focusing on its clonal core $\widehat{M}$.

In the next section, we will focus on $(4, 2)$-flexipaths. Before doing that, we develop some general results for $(4, c)$-flexipaths. In all of the results in this section, $(L, Q_1, Q_2, \dots, Q_n, R)$ is a $(4, c)$-flexipath in a matroid M. The main results of this section, Corollary 4.9 and Theorem 4.13, determine all possible $(4, 1)$-flexipaths and all possible $(4, 3)$-flexipaths, respectively, Each of the latter has at most two internal steps.

Lemma 4.1. *For all i in $[n]$,*

$$\sqcap(L, Q_i) + \sqcap^*(L, Q_i) = c = \sqcap(R, Q_i) + \sqcap^*(R, Q_i).$$

Proof. By symmetry, it suffices to prove the first equality. Using Lemma 2.1, we have

$$\begin{aligned} 3 = \lambda(L \cup Q_i) &= \lambda(L) + \lambda(Q_i) - \sqcap(L, Q_i) - \sqcap^*(L, Q_i) \\ &= 3 + c - \sqcap(L, Q_i) - \sqcap^*(L, Q_i). \end{aligned}$$

□

Lemma 4.2. *For all distinct i and j in $[n]$,*

$$\sqcap(Q_i, Q_j) + \sqcap^*(Q_i, Q_j) \leq c - 1.$$

Proof. Using Lemma 2.1, we have

$$\begin{aligned} c + 1 \leq \lambda(Q_i \cup Q_j) &= \lambda(Q_i) + \lambda(Q_j) - \sqcap(Q_i, Q_j) - \sqcap^*(Q_i, Q_j) \\ &= c + c - \sqcap(Q_i, Q_j) - \sqcap^*(Q_i, Q_j). \end{aligned}$$

The result follows immediately. □

In each of the remaining proofs in this section, by relying on Theorem 3.18, we shall argue in the clonal core of $(M, \{L, Q_1, Q_2, \dots, Q_n, R\})$ to obtain the result for $(M, \{L, Q_1, Q_2, \dots, Q_n, R\})$ itself. When we do this, to simplify the notation, we will denote this clonal core by $(M, \{L, Q_1, Q_2, \dots, Q_n, R\})$ rather than by $(\widehat{M}, \{\widehat{L}, \widehat{Q_1}, \widehat{Q_2}, \dots, \widehat{Q_n}, \widehat{R}\})$.

Lemma 4.3. *For all i in $[n]$,*

$$\sqcap(L, Q_i) = \sqcap(R, Q_i).$$

Proof. Let (L, Q, R) be a path of 4-separations. Then, by Lemma 2.2(ii),

$$\begin{aligned} \sqcap(L, Q) &= \sqcap(R, Q) + \sqcap(R \cup Q, L) - \sqcap(L \cup Q, R) \\ &= \sqcap(R, Q) + \lambda(L) - \lambda(R) \\ &= \sqcap(R, Q). \end{aligned}$$

In particular, the lemma holds when $n = 1$.

Assume $n \geq 2$. Because we are dealing with a flexipath, we may assume that $i = 1$. Then $(L, Q_1, Q_2 \cup \cdots \cup Q_n \cup R)$ is a path of 4-separations, so

$$\sqcap(L, Q_1) = \sqcap(Q_1, Q_2 \cup Q_3 \cup \cdots \cup Q_n \cup R) \geq \sqcap(Q_1, R),$$

where the inequality follows by the monotonicity of $\sqcap$ in each argument. By symmetry, $\sqcap(R, Q_1) \geq \sqcap(Q_1, L)$ so $\sqcap(L, Q_1) = \sqcap(R, Q_1)$. □

Lemma 4.4. *If $n \geq 2$, then $\sqcap(L, R) + \sqcap^*(L, R) \leq 5 - c$.*

Proof. We have, by Lemma 2.1, that

$$\sqcap(L, R) + \sqcap^*(L, R) = \lambda(L) + \lambda(R) - \lambda(L \cup R) = 3 + 3 - \lambda(L \cup R).$$

As $\lambda(L \cup R) = \lambda(Q_1 \cup Q_2 \cup \cdots \cup Q_n) \geq c + 1$, the lemma follows. □

Lemma 4.5. *If $n \geq 3$, then $c \leq 2$.*

Proof. Because $(L, Q_1, Q_2, \dots, Q_n, R)$ is a $(4, c)$-flexipath, so is $(L \cup Q_1, Q_2, \dots, Q_n, R)$. Thus, by Lemma 4.4, $\sqcap(L \cup Q_1, R) + \sqcap^*(L \cup Q_1, R) \leq 5 - c$. Therefore, by Lemma 4.1 and monotonicity,

$$c = \sqcap(Q_1, R) + \sqcap^*(Q_1, R) \leq 5 - c,$$

and the lemma follows. □

Lemma 4.6. *If $c \geq 4$, then $n \leq 1$.*

Proof. Assume $n \geq 2$. By Lemma 4.5, $n = 2$. By Lemma 4.4,

$$0 \leq \sqcap(L, R) + \sqcap^*(L, R) \leq 5 - c \leq 1.$$

Thus $c \in \{4, 5\}$.

Suppose $\sqcap(L, Q_i) = 3$. Since we are operating in the clonal core, this means that $L \subseteq \mathrm{cl}(Q_i)$. By Lemma 4.3, $R \subseteq \mathrm{cl}(Q_i)$, so $L \cup R \subseteq \mathrm{cl}(Q_i)$ and $r(L \cup R) \leq r(Q_i) = c$. Thus

$$6 - \sqcap(L, R) \leq c.$$

This contradicts Lemma 4.4. Thus

$$\sqcap(L, Q_i) \leq 2.$$

By duality, $\sqcap^*(L, Q_i) \leq 2$. By Lemma 4.1,

$$c = \sqcap(L, Q_i) + \sqcap^*(L, Q_i) \leq 4,$$

so

$$c = 4,$$

and, for each i in $\{1, 2\}$.

$$\sqcap(L, Q_i) = 2 = \sqcap^*(L, Q_i).$$

We deduce that, for each N in $\{M, M^*\}$ and each i in $\{1, 2\}$,

$$r_N(L \cup Q_i) = 5 = r_N(R \cup Q_i).$$

Thus, by the submodularity of r_N,

$$r_N(L \cup R \cup Q_i) + r_N(Q_i) \leq r_N(L \cup Q_i) + r_N(R \cup Q_i) = 10.$$

Hence $r_N(L \cup R \cup Q_i) \leq 6$. Therefore, by submodularity again,

$$12 \geq r_N(L \cup R \cup Q_1) + r_N(L \cup R \cup Q_2) \geq r_N(L \cup R) + r(N).$$

Taking each N in $\{M, M^*\}$, we have

$$24 \geq r_M(L \cup R) + r(M) + r_{M^*}(L \cup R) + r(M^*).$$

But $r(M) + r(M^*) = |E(M)| = 14$. Thus $10 \geq r_M(L \cup R) + r_{M^*}(L \cup R)$, so

$$10 \geq r(L) + r(R) + r^*(L) + r^*(R) - \sqcap(L, R) - \sqcap^*(L, R).$$

Hence $\sqcap(L, R) + \sqcap^*(L, R) \geq 2$, which contradicts Lemma 4.4. □

The rest of this section is concerned with determining all possible $(4, 1)$- and $(4, 3)$-flexipaths beginning with the former. For such a flexipath, $\lambda(Q_i) = 1$ for each i, so since we are operating in the clonal core, we will take $Q_i = \{e_i\}$.

Lemma 4.7. *Let $(L, e_1, e_2, \ldots, e_n, R)$ be a $(4, 1)$-flexipath with $\sqcap(L, R) = 2$ and $n \geq 1$. Suppose $\sqcap(L, e_i) = 1$ for each i in $\{1, 2, \ldots, t\}$ and $\sqcap(L, e_j) = 0$ for each j in $\{t + 1, t + 2, \ldots, n\}$. Then $\min\{t, n - t\} = 1$.*

Proof. By Lemma 4.1, $\sqcap^*(L, e_i) = 0$ for each i in $\{1, 2, \ldots, t\}$ and $\sqcap(L, e_j) = 1$ for each j in $\{t + 1, t + 2, \ldots, n\}$. Suppose $n - t = 0$. Then

$$3 = \lambda(L \cup e_1 \cup \cdots \cup e_n) = r(L \cup e_1 \cup \cdots \cup e_n) + r(R) - r(M) \leq 3 + 3 - 4,$$

a contradiction. Thus $n - t > 0$. By duality, $t > 0$.

Assume $t, n - t \geq 2$. By moving to the clonal core of the $(4, 1)$-flexipath $(L \cup e_1 \cup \cdots \cup e_{t-2}, e_{t-1}, e_t, e_{t+1}, e_{t+2}, e_{t+3} \cup \cdots \cup e_n \cup R)$, we may assume that $t = n - t = 2$. Since $\lambda(\{e_i, e_j\}) = 2$ for $i \neq j$, we deduce that $r(\{e_i, e_j\}) = 2 = r^*(\{e_i, e_j\})$. Hence

$$\begin{aligned} 2 + r^*(L \cup R) &= r^*(\{e_3, e_4\}) + r^*(L \cup R \cup \{e_3, e_4\}) \\ &\leq r^*(L \cup \{e_3, e_4\}) + r^*(R \cup \{e_3, e_4\}) \\ &= 3 + 3. \end{aligned}$$

Thus $r^*(L \cup R) \leq 4$, so $\sqcap^*(L, R) \geq 2$. But $\sqcap(L, R) = 2$ and, by Lemma 4.4, $\sqcap(L, R) + \sqcap^*(L, R) \leq 4$, so $\sqcap^*(L, R) = 2$. This means that we can make inferences about M^* from what we determine about M.

We have $4 = r^*(L \cup R \cup \{e_3, e_4\}) = |L \cup R \cup \{e_3, e_4\}| + r(\{e_1, e_2\}) - r(M)$, so $r(M) = 6$. Dually, $r^*(M) = 6$, so $|E(M)| = 12$, a contradiction. □

We remind the reader that each of the matroids M that arises in our lemmas is the clonal core of a $(4, c)$-flexipath $(L, Q_1, Q_2, \ldots, Q_n, R)$.

Lemma 4.8. *Let $(L, e_1, e_2, \ldots, e_n, R)$ be a $(4, 1)$-flexipath in a matroid M with $\sqcap(L, R) = 3$. Then $r(M) = 3$ and $M|\{e_1, e_2, \ldots, e_n\}$ can be any n-element simple matroid of rank at most three.*

Proof. First we show that

4.8.1. *$e_i \in \text{cl}(L)$ for all i in $[n]$.*

We observe that it suffices to show that $\sqcap(L, e_1) = 1$. Assume that $\sqcap(L, e_1) = 0$. Then $e_1 \notin \text{cl}(L)$. Consider the $(4, 1)$-flexipath $(L, e_1, \{e_2, e_3, \ldots, e_n\} \cup R)$ rewriting this as (L, e_1, R'). As $r(L) = 3$, we have $3 \geq \sqcap(L, R') \geq \sqcap(L, R) = 3$ so $\sqcap(L, R') = 3$. We now move to the clonal core of $(M, \{L, e_1, R'\})$ denoting this clonal core by $(\widehat{M}, \{L, e_1, \widehat{R'}\})$. Then $(L, e_1, \widehat{R'})$ is a $(4, 1)$-flexipath and $e_1 \notin \text{cl}_{\widehat{M}}(L)$. But $\text{cl}_{\widehat{M}}(L) = \text{cl}_{\widehat{M}}(L \cup \widehat{R'})$, so $e_1 \notin \text{cl}_{\widehat{M}}(L \cup \widehat{R'})$. This is a contradiction as it means e_1 is a coloop of $\widehat{M}$, so $\lambda_{\widehat{M}}(\{e_1\}) = 0$. Thus 4.8.1 holds.

By 4.8.1, it follows that $r(M) = 3$. Clearly $r(M|\{e_1, e_2, \ldots, e_n\}) \leq 3$. Now, let N be any simple matroid of rank at most three with ground set $\{e_1, e_2, \ldots, e_n\}$ and take M_0 to be a copy of $U_{3,6}$ with ground set $\{f_1, f_2, \ldots, f_6\}$ where $\{e_1, e_2, \ldots, e_n\} \cap \{f_1, f_2, \ldots, f_6\} = \emptyset$. Then, in the truncation to rank three of the direct sum of M_0 and N, we see that $(\{f_1, f_2, f_3\}, e_1, e_2, \ldots, e_n, \{f_4, f_5, f_6\})$ is a $(4, 1)$-flexipath. □

Extending the last two lemmas, we get the following characterization, up to duality, of the clonal cores of all $(4, 1)$-flexipaths.

Corollary 4.9. *Let $(L, e_1, e_2, \ldots, e_n, R)$ be a $(4, 1)$-flexipath in a matroid M with $\sqcap(L, R) \geq \sqcap^*(L, R)$. Then one of the following holds.*

(i) *$\sqcap(L, R) = 3$ and $r(M) = 3$, while $M|\{e_1, e_2, \ldots, e_n\}$ is any n-element simple matroid of rank at most three, and $\{e_1, e_2, \ldots, e_n\} \subseteq \text{cl}(L) \cap \text{cl}(R)$.*

(ii) $\sqcap(L,R) = 2$ *and* $r(M) = 4$, *where* $n \geq 2$ *and, for some relabelling of* $\{e_1, e_2, \ldots, e_n\}$, *the matroid* $M|\{e_1, e_2, \ldots, e_{n-1}\}$ *is simple and uniform of rank at most two where* $\{e_1, e_2, \ldots, e_{n-1}\} = \mathrm{cl}(L) \cap \mathrm{cl}(R)$ *and* $\{e_n\} = \mathrm{cl}^*(L) \cap \mathrm{cl}^*(R)$.

Proof. By Lemma 4.4, $\sqcap(L,R) + \sqcap^*(L,R) \leq 4$ provided $n \geq 2$. In Lemmas 4.7 and 4.8, we treated the cases where $\sqcap(L,R) \in \{2,3\}$. Now each e_i is in exactly one of $\mathrm{cl}(L) \cap \mathrm{cl}(R)$ and $\mathrm{cl}^*(L) \cap \mathrm{cl}^*(R)$. Since $\sqcap(L,R) \geq \sqcap^*(L,R)$, we see that $|\mathrm{cl}^*(L) \cap \mathrm{cl}^*(R)| \leq 1$. If $\sqcap(L,R) = 3$, then (i) holds. If $\sqcap(L,R) = 2$, then $r(M) \geq 4$ and (ii) holds. □

Lemma 4.10. *In a* $(4,3)$*-flexipath with* $n = 2$,

$$(\sqcap(L,R), \sqcap^*(L,R)) = (\sqcap^*(Q_1,Q_2) + 6 - r(M), \sqcap(Q_1,Q_2) + 6 - r^*(M)).$$

Proof. By duality, it suffices to prove that the first coordinates are equal. We have

$$\begin{aligned}
\sqcap^*(Q_1,Q_2) &= r^*(Q_1) + r^*(Q_2) - r^*(Q_1 \cup Q_2) \\
&= 3 + 3 - r^*(Q_1 \cup Q_2) \\
&= r(L) + r(R) - (|Q_1 \cup Q_2| + r(L \cup R) - r(M)) \\
&= \sqcap(L,R) + r(M) - 6.
\end{aligned}$$

□

Lemma 4.11. *In a* $(4,3)$*-flexipath with* $n = 2$, *if* $\sqcap(L,Q_1) = 2$, *then* $r(M) \leq 5$.

Proof. By Lemma 4.3, $\sqcap(R,Q_1) = 2$, so

$$\begin{aligned}
4 + 4 &= r(L \cup Q_1) + r(R \cup Q_1) \\
&\geq r(Q_1) + r(L \cup R \cup Q_1)) \\
&= 3 + r(M)
\end{aligned}$$

as Q_1 is independent and Q_2 is coindependent. Thus $r(M) \leq 5$. □

Lemma 4.12. *In a* $(4,3)$*-flexipath with* $n = 2$, *for some* N *in* $\{M, M^*\}$ *and each* i *in* $\{1,2\}$,

$$(\sqcap_N(L,Q_i), \sqcap^*_N(L,Q_i)) = (2,1).$$

Proof. By Lemma 4.1, for each N in $\{M, M^*\}$, we have $\sqcap_N(L,Q_i) + \sqcap^*_N(L,Q_i) = 3$. If $(\sqcap_N(L,Q_1), \sqcap^*_N(L,Q_1)) = (2,1)$ and $(\sqcap_N(L,Q_2), \sqcap^*_N(L,Q_2)) = (1,2)$, then $(\sqcap_N(R,Q_2), \sqcap^*_N(R,Q_2)) = (1,2)$. Thus

$$3 = \lambda_N(L \cup Q_1) = r_N(L \cup Q_1) + r_N(R \cup Q_2) - r(N) = 4 + 5 - r(N),$$

so $r(N) = 6$. As $|E(M)| = 12$, it follows that $r(M) = r^*(M) = 6$. But, by Lemma 4.11 and its dual, $r(M) \leq 5$ and $r^*(M) \leq 5$, a contradiction. We deduce that $(\sqcap_N(L,Q_1), \sqcap^*_N(L,Q_1)) = (\sqcap_N(L,Q_2), \sqcap^*_N(L,Q_2))$ and the lemma follows. □

The next theorem determines, up to duality, the possible clonal cores of all $(4,3)$-flexipaths with at least two internal steps.

Theorem 4.13. *Consider a $(4,3)$-flexipath with $n \geq 2$ and $\sqcap(L,R) \geq \sqcap^*(L,R)$. Then $n = 2$ and one of the following three possibilities arises:*

(i) $(\sqcap(L,R), \sqcap^*(L,R)) = (2,0)$ *and* $(\sqcap(Q_1,Q_2), \sqcap^*(Q_1,Q_2)) = (1,1)$*;*
(ii) $(\sqcap(L,R), \sqcap^*(L,R)) = (1,0) = (\sqcap(Q_1,Q_2), \sqcap^*(Q_1,Q_2))$*; or*
(iii) $(\sqcap(L,R), \sqcap^*(L,R)) = (1,1)$ *and* $(\sqcap(Q_1,Q_2), \sqcap^*(Q_1,Q_2)) = (2,0)$*.*

Proof. By Lemma 4.5, since we are dealing with a $(4,3)$-flexipath, $n \leq 2$, so $n = 2$. By Lemmas 4.12, for some N in $\{M, M^*\}$, we have $\sqcap_N(L,Q_1) = 2$ and $\sqcap_N(R,Q_2) = 2$. Thus $r_N(L \cup Q_1) = 4 = r_N(R \cup Q_2)$. Hence

$$3 = \lambda_N(L \cup Q_1) = r_N(L \cup Q_1) + r_N(R \cup Q_2) - r(N) = 4 + 4 - r(N),$$

so $r(N) = 5$. As $|E(N)| = 12$, we see that $r^*(N) = 7$. It follows by Lemma 4.10 that

$$(\sqcap_N(L,R), \sqcap_N^*(L,R)) = (\sqcap_N^*(Q_1,Q_2) + 1, \sqcap_N(Q_1,Q_2) - 1), \tag{4.1}$$

so

$$\sqcap_N(L,R) + \sqcap_N^*(L,R) = \sqcap_N(Q_1,Q_2) + \sqcap_N^*(Q_1,Q_2).$$

By Lemma 4.4, $\sqcap_N(L,R) + \sqcap_N^*(L,R) \leq 2$. It follows by (4.1) that $1 \leq \sqcap_N(L,R) \leq 2$ and $1 \geq \sqcap_N^*(L,R) \geq 0$. As $\sqcap_M(L,R) \geq \sqcap_M^*(L,R)$, we deduce that

$$(\sqcap_M(L,R), \sqcap_M^*(L,R)) \in \{(2,0),(1,0),(1,1)\}.$$

Moreover, by (4.1) again, we get $(\sqcap_M(Q_1,Q_2), \sqcap_M^*(Q_1,Q_2))$ for each of the three cases. □

To see that each of the matroids in Theorem 4.13 can actually arise, we shall describe a construction of M^* in each case. Let B be a basis $\{b_1, b_2, \ldots, b_7\}$ of $V(7,\mathbb{R})$. To the planes spanned by $\{b_1,b_2,b_3\}$ and $\{b_4,b_5,b_6\}$, freely add the sets $\{\alpha_1,\alpha_2,\alpha_3\}$ and $\{\alpha_4,\alpha_5,\alpha_6\}$, respectively. For M^* corresponding to (i) of the theorem, freely add $\{\beta_1,\beta_2,\beta_3\}$ and $\{\beta_4,\beta_5,\beta_6\}$ to the planes spanned by $\{b_1,b_4,b_7\}$ and $\{b_2,b_5,b_7\}$, respectively. Then delete B, letting $(L,R) = (\{\alpha_1,\alpha_2,\alpha_3\},\{\alpha_4,\alpha_5,\alpha_6\})$, and $(Q_1,Q_2) = (\{\beta_1,\beta_2,\beta_3\},\{\beta_4,\beta_5,\beta_6\})$.

Instead, to get M^* corresponding to (ii) of the theorem, keep L and R unchanged, and let $(Q_1,Q_2) = (\{\gamma_1,\gamma_2,\gamma_3\},\{\gamma_4,\gamma_5,\gamma_6\})$ where $\{\gamma_1,\gamma_2,\gamma_3\}$ is freely added to the plane spanned by $\{b_1,b_4,b_1+b_4+b_7\}$, while $\{\gamma_4,\gamma_5,\gamma_6\}$ is freely added to the plane spanned by $\{b_2,b_5,b_2+b_5+b_7\}$. In this case, $E(M^*) = \{\alpha_1,\alpha_2,\ldots,\alpha_6,\gamma_1,\gamma_2,\ldots,\gamma_6\}$.

Finally, to get M^* corresponding to (iii) of the theorem, we can modify (i) by interchanging L with Q_1 and interchanging R with Q_2. This switch does indeed produce an example because, in (i), we had $(\sqcap(Q_1,Q_2), \sqcap^*(Q_1,Q_2)) = (1,1)$ and $\sqcap(L,Q_1) = \sqcap(R,Q_1) = 2 = \sqcap(L,Q_2) = \sqcap(R,Q_2)$.

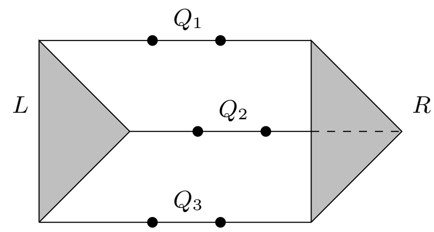

FIGURE 3. A prism-like flexipath (L, Q_1, Q_2, Q_3, R).

5. TYPES OF $(4, 2)$-FLEXIPATHS

The purpose of this section is to prove the main result of the paper, Theorem 5.15, which describes all possible $(4, 2)$-flexipaths. Let $\mathbf{Q}$ be a $(4, 2)$-flexipath $(L, Q_1, Q_2, \ldots, Q_n, R)$ in a matroid M. In the introduction, we identified four special types of $(4, 2)$-flexipaths, namely, spike-reminiscent, paddle-reminiscent, squashed, and stretched. Moreover, we noted that $\mathbf{Q}$ is spike-reminiscent in M if and only if it is paddle-reminiscent in M^*; and $\mathbf{Q}$ is squashed in M if and only if $\mathbf{Q}$ is stretched in M^*. Each of the remaining seven types of $(4, 2)$-flexipaths has exactly three internal steps.

The flexipath $\mathbf{Q}$ is *relaxed-spike-reminiscent* if all of the following hold:

(i) $n = 3$;
(ii) $\sqcap(L, R) = 0$ and $\sqcap^*(L, R) = 2$;
(iii) $\sqcap(Q_i, Q_j) = 1$ and $\sqcap^*(Q_i, Q_j) = 0$ for all distinct i and j in $[n]$; and
(iv) $\sqcap(Q_i, L) = \sqcap(Q_i, R) = 1 = \sqcap^*(Q_i, L) = \sqcap^*(Q_i, R)$ for all i in $[n]$.

The flexipath $\mathbf{Q}$ is *relaxed-paddle-reminiscent* if all of the following hold:

(i) $n = 3$;
(ii) $\sqcap(L, R) = 2$ and $\sqcap^*(L, R) = 0$;
(iii) $\sqcap(Q_i, Q_j) = 0$ and $\sqcap^*(Q_i, Q_j) = 1$ for all distinct i and j in $[n]$; and
(iv) $\sqcap(Q_i, L) = \sqcap(Q_i, R) = 1 = \sqcap^*(Q_i, L) = \sqcap^*(Q_i, R)$ for all i in $[n]$.

Note that $\mathbf{Q}$ is relaxed-spike-reminiscent in M if and only if it is relaxed-paddle-reminiscent in M^*.

The flexipath $\mathbf{Q}$ is *prism-like* if all of the following hold:

(i) $n = 3$;
(ii) $\sqcap(Q_i, Q_j) = \sqcap^*(Q_i, Q_j) = 0$ for all distinct i and j in $[n]$;
(iii) $\sqcap(L, R) = \sqcap^*(L, R) = 0$; and
(iv) $\sqcap(Q_i, L) = \sqcap(Q_i, R) = 1 = \sqcap^*(Q_i, L) = \sqcap^*(Q_i, R)$ for all i in $[n]$.

Observe that $\mathbf{Q}$ is prism-like in M if and only if $\mathbf{Q}$ is prism-like in M^*. A diagram representing a rank-6 matroid with a prism-like flexipath is shown in Fig 3.

The flexipath $\mathbf{Q}$ is *tightened-prism-like* if all of the following hold.

(i) $n = 3$;
(ii) $\sqcap(Q_i, Q_j) = \sqcap^*(Q_i, Q_j) = 0$ for all distinct i and j in $\{1, 2, 3\}$;
(iii) $\sqcap(L, R) = 0$ and $\sqcap^*(L, R) = 1$; and

(iv) $\sqcap(Q_i, L) = \sqcap(Q_i, R) = 1 = \sqcap^*(Q_i, L) = \sqcap^*(Q_i, R)$ for all i in $\{1, 2, 3\}$.

Note that we have not formally named what $\mathbf{Q}$ is in M^* when $\mathbf{Q}$ is tightened-prism-like in M.

The flexipath $\mathbf{Q}$ is *doubly-tightened-prism-like* if all of the following hold.

(i) $n = 3$;
(ii) $\sqcap(Q_i, Q_j) = \sqcap^*(Q_i, Q_j) = 0$ for all distinct i and j in $\{1, 2, 3\}$;
(iii) $\sqcap(L, R) = 1 = \sqcap^*(L, R)$; and
(iv) $\sqcap(Q_i, L) = \sqcap(Q_i, R) = 1 = \sqcap^*(Q_i, L) = \sqcap^*(Q_i, R)$ for all i in $\{1, 2, 3\}$.

We see that $\mathbf{Q}$ is doubly-tightened-prism-like in M if and only if $\mathbf{Q}$ is doubly-tightened-prism-like in M^*.

The flexipath $\mathbf{Q}$ is *Vámos-inspired* if, in either M or M^*, all of the following hold.

(i) $n = 3$;
(ii) $\sqcap(L, R) = 0$ and $\sqcap^*(L, R) = 1$;
(iii) $\sqcap(Q_i, L) = \sqcap(Q_i, R) = 1 = \sqcap^*(Q_i, L) = \sqcap^*(Q_i, R)$ for all i in $\{1, 2, 3\}$;
(iv) $\sqcap^*(Q_i, Q_j) = 0$ for all distinct i and j; and
(iv) after a possible permutation of $\{1, 2, 3\}$,

$$\sqcap(Q_1, Q_2) = 0 = \sqcap(Q_1, Q_3) \text{ and } \sqcap (Q_2, Q_3) = 1.$$

Note that, by definition, $\mathbf{Q}$ is Vámos-inspired in M if and only if $\mathbf{Q}$ is Vámos-inspired in M^*.

The flexipath $\mathbf{Q}$ is *nasty* if all of the following hold.

(i) $n = 3$;
(ii) $\sqcap(L, R) = 1 = \sqcap^*(L, R)$;
(iii) $\sqcap(Q_i, L) = \sqcap(Q_i, R) = 1 = \sqcap^*(Q_i, L) = \sqcap^*(Q_i, R)$ for all i in $\{1, 2, 3\}$; and
(iv) after a possible permutation of $\{1, 2, 3\}$,

$$\begin{bmatrix} \sqcap(Q_1, Q_2) & \sqcap^*(Q_1, Q_2) \\ \sqcap(Q_1, Q_3) & \sqcap^*(Q_1, Q_3) \\ \sqcap(Q_2, Q_3) & \sqcap^*(Q_2, Q_3) \end{bmatrix} \in \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

These three types are called, respectively, *mixed nasty*, *plane nasty*, and *dual-plane nasty*. Clearly, $\mathbf{Q}$ is plane-nasty in M if and only if $\mathbf{Q}$ is dual-plane nasty in M^*; and $\mathbf{Q}$ is mixed nasty in M if and only if $\mathbf{Q}$ is mixed nasty in M^*.

We say that Q_i is a *specially placed step* in a $(4, 2)$-flexipath $(L, Q_1, Q_2, \dots, Q_n, R)$ in M if either

(S1) $\sqcap(L, R) = 2$ and $\sqcap(L, Q_i) = 2 = \sqcap(R, Q_i)$; or
(S2) $\sqcap^*(L, R) = 2$ and $\sqcap^*(L, Q_i) = 2 = \sqcap^*(R, Q_i)$.

Evidently, Q_i is a specially placed step of type (S2) in M if and only if Q_i is a specially placed step of type (S1) in M^*. Specially placed steps are not particularly problematic for, as we now show, there is at most one of them. In this and the remaining results in this section, $(L, Q_1, Q_2, \dots, Q_n, R)$ is a $(4,2)$-flexipath $\mathbf{Q}$.

Lemma 5.1. *$\mathbf{Q}$ has at most one specially placed step.*

Proof. Assume that Q_1 and Q_2 are both specially placed elements of type (S1). For the rest of the argument, we will again be operating in the clonal core. There, since $\sqcap(L, Q_i) = 2$ for each i in $\{1,2\}$, we deduce that $Q_1 \cup Q_2 \subseteq \mathrm{cl}(L)$. By symmetry, $Q_1 \cup Q_2 \subseteq \mathrm{cl}(R)$. Hence $Q_1 \cup Q_2 \subseteq \mathrm{cl}(L) \cap \mathrm{cl}(R)$. Thus

$$\lambda(Q_1 \cup Q_2) \le r(Q_1 \cup Q_2) \le r(\mathrm{cl}(L) \cap \mathrm{cl}(R)) \le \sqcap(L, R) = 2,$$

a contradiction.

By duality, $\mathbf{Q}$ has at most one specially placed step of type (S2). Now suppose that Q_1 is specially placed of type (S1), and Q_2 is specially placed of type (S2). Then $\sqcap(L, R) = 2 = \sqcap^*(L, R)$, so $\sqcap(L, R) + \sqcap^*(L, R) = 4$, a contradiction to Lemma 4.4. □

Lemma 5.2. *If $\sqcap(L, R) = 3$, then $\sqcap(L, Q_i) = 2$ for all i.*

Proof. We argue in the clonal core. If $n = 1$, then $r(L)n = r(M) = 3$ and $\sqcap(L, Q_1) = r(Q_1) = 2$. Next assume that $n = 2$. We have

$$3 \le \lambda(Q_1 \cup Q_2) = r(Q_1 \cup Q_2) + r(L \cup R) - r(M).$$

But $r(L \cup R) = 3$ since $\sqcap(L, R) = 3$. Thus

$$3 \le r(M) \le r(Q_1 \cup Q_2) \le 4.$$

Suppose $r(M) = 3$. Then $\sqcap(L, Q_i) = 2$ for all i. Thus we may assume that $r(M) = 4$. Then

$$3 = \lambda(L \cup Q_1) = r(L \cup Q_1) + r(R \cup Q_2) - r(M).$$

Thus $r(L \cup Q_1) + r(R \cup Q_2) = 7$. Hence we may assume that $r(L \cup Q_1) = 3$. Then $r(L \cup Q_1 \cup R) = 3$, so $\lambda(Q_2) = 1$, a contradiction. We conclude that the result holds for $n = 2$.

Assume the result holds for $n < k$ and let $n = k \ge 3$. Consider the path $(L, Q_1, Q_2, Q_3 \cup Q_4 \cup \dots \cup Q_k \cup R)$ of 4-separations. By applying the result for $n = 2$ to the clonal core of $(M, \{L, Q_1, Q_2, Q_3 \cup Q_4 \cup \dots \cup Q_k \cup R\})$, we deduce that $\sqcap(L, Q_1) = 2$ and the lemma follows by induction. □

Lemma 5.3. *Let $n \ge 2$. Assume that $\mathbf{Q}$ has no specially placed steps of type* (S1). *If $\sqcap(L, Q_i) = 2$ for some i in $[n]$, then $\sqcap(L, R) = 3$ and $\sqcap(L, Q_j) = 2 = \sqcap(R, Q_j)$ for all j in $[n]$.*

Proof. Again we argue in the clonal core. We may assume that $i = 1$. Assume first that $n = 2$. Then $r(L \cup Q_1) = 3$ and, by Lemma 4.3, $\sqcap(R, Q_1) = 2$. Thus $\sqcap(L, R) \ge 2$. If $\sqcap(L, R) = 2$, then Q_1 is a specially placed step of type (S1), a contradiction. Thus $\sqcap(L, R) = 3$. Hence $r(L \cup R) = 3 = r(L \cup R \cup Q_1)$, so $r(M) = 3$ and $\sqcap(L, Q_2) = \sqcap(R, Q_2) = 2$.

We now know the result holds for $n = 2$. Assume it holds for $n < k$ and let $n = k \geq 3$. Then, by considering the path $(L, Q_1, Q_2, Q_3 \cup Q_4 \cup \cdots \cup Q_k \cup R)$ of 4-separations and applying the induction assumption to the clonal core of $(M, \{L, Q_1, Q_2, Q_3 \cup Q_4 \cup \cdots \cup Q_k \cup R\})$, we deduce that $\sqcap(L, Q_2) = 2$. Because we are dealing with a $(4, 2)$-flexipath, we get that $\sqcap(L, Q_j) = 2$ for all j in $\{1, 2, \ldots, k\}$. Then $r(L \cup Q_1 \cup Q_2 \cup \cdots \cup Q_k) = 3$. Thus $r(M) = 3$ and $\sqcap(L, R) = 3$. We conclude, by induction, that the lemma holds. □

Lemma 5.4. *Assume that the* $(4, 2)$*-flexipath* $\mathbf{Q}$ *has at least two internal steps and has no specially placed steps of type* (S1)*. If* $\sqcap(L, Q_i) = 2$ *for some* i *in* $[n]$*, then* $\mathbf{Q}$ *is a squashed* $(4, 2)$*-flexipath.*

Proof. By Lemma 5.3, $\sqcap(L, R) = 3$ and $\sqcap(L, Q_j) = 2 = \sqcap(R, Q_j)$ for all j in $[n]$. By Lemmas 4.4 and 4.1, $\sqcap^*(L, R) = 0$, and $\sqcap^*(L, Q_j) = 0 = \sqcap^*(R, Q_j)$ for all j in $[n]$. Finally, working in the clonal core, we have $r(L) = 3 = r(L \cup Q_j)$ and $r(Q_j) = 2$ for all j. Thus $\sqcap(Q_g, Q_h) \geq 1$ for all distinct g and h. Thus, by Lemma 4.2, $\sqcap(Q_g, Q_h) = 1$ and $\sqcap^*(Q_g, Q_h) = 0$ for all distinct g and h. Hence $\mathbf{Q}$ is a squashed $(4, 2)$-flexipath. □

The dual of the last lemma is the following.

Lemma 5.5. *Assume that the* $(4, 2)$*-flexipath* $\mathbf{Q}$ *has at least two internal steps and has no specially placed steps of type* (S2)*. If* $\sqcap(L, Q_i) = 0$ *for some* i *in* $[n]$*, then* $\mathbf{Q}$ *is a stretched* $(4, 2)$*-flexipath.*

Lemma 5.6. *Assume that the* $(4, 2)$*-flexipath* $\mathbf{Q}$ *has at least two internal steps and has no specially placed steps. If* $\mathbf{Q}$ *is neither a squashed nor a stretched* $(4, 2)$*-flexipath, then, for all* i *in* $[n]$*,*

$$\sqcap(L, Q_i) = \sqcap^*(L, Q_i) = 1 = \sqcap(R, Q_i) = \sqcap^*(R, Q_i).$$

Proof. By Lemmas 5.4 and 5.5, $\sqcap(L, Q_i) = 1$, for all i in $[n]$. Thus, by Lemma 4.1, $\sqcap^*(L, Q_i) = 1$ for all i. Moreover, by Lemma 4.3, $\sqcap(R, Q_i) = 1 = \sqcap^*(R, Q_i)$ for all i. □

Recall that, for a non-empty subset J of $[n]$, we are abbreviating $\cup_{j \in J} Q_j$ as Q_J. When J is empty, so is Q_J.

Lemma 5.7. *Let* $\mathbf{Q}$ *be a* $(4, 2)$*-flexipath* $(L, Q_1, Q_2, \ldots, Q_n, R)$ *in a matroid* M *where* $n \geq 2$*. Assume that* $\mathbf{Q}$ *is neither squashed nor stretched and has no specially placed steps. Then*

(i) *for all* $J \subseteq [n] - \{i\}$*,*

$$\sqcap(L \cup Q_J, Q_i) = \sqcap(Q_i, Q_J \cup R) = 1 = \sqcap^*(L \cup Q_J, Q_i) = \sqcap^*(Q_i, Q_J \cup R);$$

(ii) $r(L \cup Q_J) = r(L) + \sum_{j \in J} r(Q_j) - |J|$ *for all* $J \subseteq [n]$*;*

(iii) $r(M) = r(L) + r(Q_1) + r(Q_2) + \cdots + r(Q_n) + r(R) - n - 3.$

Proof. By Lemma 5.6, for all i in $[n]$, we have

$$\sqcap(L, Q_i) = \sqcap(Q_i, R) = 1 = \sqcap^*(L, Q_i) = \sqcap^*(Q_i, R).$$

To prove (i), we may assume that $i = 1$ and $J = \{2, 3, \dots, j\}$. Then $(L \cup Q_J, Q_1, Q_{j+1}, \dots, Q_n, R)$ is a path of 4-separations and $\sqcap(R, Q_1) = 1$, so, by Lemma 4.3, $\sqcap(L \cup Q_J, Q_1) = 1$. Thus (i) holds.

By (i), we have $r(L \cup Q_1) = r(L) + r(Q_1) - 1$ and

$$r(L \cup Q_1 \cup Q_2 \cup \cdots \cup Q_j) = r(L \cup Q_1 \cup Q_2 \cup \cdots \cup Q_{j-1}) + r(Q_j) - 1,$$

so $r(L \cup Q_J) = r(L) + r(Q_1) + r(Q_2) + \cdots + r(Q_j) - j$, so (ii) holds. In particular,

$$r(L \cup Q_1 \cup Q_2 \cup \cdots \cup Q_n) = r(L) + r(Q_1) + r(Q_2) + \cdots + r(Q_n) - n.$$

As $\sqcap(L \cup Q_1 \cup Q_2 \cup \cdots \cup Q_n, R) = 3$, we deduce that

$$r(M) = r(L) + r(Q_1) + r(Q_2) + \cdots + r(Q_n) + r(R) - n - 3,$$

so (iii) holds. □

Lemma 5.8. *In the clonal core of M, for distinct i and j,*

(i) $\sqcap(Q_i, Q_j) = 1$ *if and only if* $Q_i \cup Q_j$ *is a circuit;*
(ii) $\sqcap(Q_i, Q_j) = 0$ *if and only if* $Q_i \cup Q_j$ *is independent;*
(iii) $\sqcap^*(Q_i, Q_j) = 1$ *if and only if* $Q_i \cup Q_j$ *is a cocircuit; and*
(iv) $\sqcap^*(Q_i, Q_j) = 0$ *if and only if* $Q_i \cup Q_j$ *is coindependent.*

Proof. By duality, it suffices to prove (i) and (ii). We have

$$\begin{aligned} \sqcap(Q_i, Q_j) &= r(Q_i) + r(Q_j) - r(Q_i \cup Q_j) \\ &= 4 - r(Q_i \cup Q_j). \end{aligned}$$

Thus $r(Q_i \cup Q_j) = 4 - \sqcap(Q_i, Q_j)$. Because the elements of Q_j are clones, parts (i) and (ii) follow immediately. □

Lemma 5.9. *Let* $\mathbf{Q}$ *be a* $(4, 2)$*-flexipath* $(L, Q_1, Q_2, \dots, Q_n, R)$ *in a matroid. Assume that* $\mathbf{Q}$ *is neither squashed nor stretched and has no specially placed steps. If* $\sqcap(L, R) = 0$*, then, in the clonal core* M *of the matroid,*

(i) $\sqcap(L, Q_i) = \sqcap^*(L, Q_i) = 1 = \sqcap(R, Q_i) = \sqcap^*(R, Q_i)$ *for all* i*;*
(ii) $n = 3$*;*
(iii) $r(M) = 6 = r^*(M)$*;*
(iv) $\sqcap^*(Q_i, Q_j) = 0$ *for all distinct* i *and* j*;*
(v) $\sqcap^*(L, R) = 6 - r(Q_1 \cup Q_2 \cup Q_3)$*;*
(vi) $r(Q_1 \cup Q_2 \cup Q_3) \in \{4, 5, 6\}$*;*
(vii) *if* $r(Q_1 \cup Q_2 \cup Q_3) = 6$*, then* $\mathbf{Q}$ *is prism-like;*
(viii) *if* $r(Q_1 \cup Q_2 \cup Q_3) = 4$*, then* $\sqcap(Q_i, Q_j) = 1$ *for all distinct* i *and* j*, and* $\mathbf{Q}$ *is relaxed-spike-reminiscent; and*
(ix) *if* $r(Q_1 \cup Q_2 \cup Q_3) = 5$*, then either*
  (a) $\sqcap(Q_i, Q_j) = 0$ *for all distinct* i *and* j *in* $\{1, 2, 3\}$*, and* $\mathbf{Q}$ *is tightened-prism-like; or*
  (b) *after a possible permutation of* $\{1, 2, 3\}$*,*

$$\sqcap(Q_1, Q_2) = 0 = \sqcap(Q_1, Q_3) \text{ and } \sqcap(Q_2, Q_3) = 1,$$

  and $\mathbf{Q}$ *is Vámos-inspired.*

Proof. Suppose $n = 1$. Then, working in the clonal core, $3 = \lambda(L \cup Q_1) = r(L \cup Q_1) + r(R) - r(M)$. Since $r(R) = 3$, we have $r(L \cup Q_1) = r(M)$. But $\sqcap(L, R) = 0$, we see that $r(M) \geq 6$, while $r(L \cup Q_1) \leq 5$, a contradiction. Hence $n \geq 2$. Part (i) is immediate from Lemma 5.6. Then, as $\sqcap(L, Q_i) = 1 = \sqcap(R, Q_i)$, Lemma 2.2 gives that

$$\begin{aligned} 2 \geq \sqcap(Q_i, L \cup R) + \sqcap(L, R) &= \sqcap(Q_i \cup L, R) + \sqcap(Q_i, L) \\ &\geq \sqcap(Q_i, R) + \sqcap(Q_i, L) = 2. \end{aligned}$$

Thus $\sqcap(Q_i, L \cup R) = 2$. Hence $L \cup R$ spans M so $r(M) = r(L \cup R) = 6$. By Lemma 5.7(iii), $6 = r(L) + 2n - n$, so $n = 3$ and (ii) holds. Moreover, for each distinct i and j, we see that $Q_i \cup Q_j$ is coindependent. Thus, by Lemma 5.8(iv), $\sqcap^*(Q_i, Q_j) = 0$, that is, (iv) holds.

Since $r(M) = 6$ and

$$|E(M)| = 3 + 2 + 2 + 2 + 3 = 12,$$

we see that $r^*(M) = 6$, that is, (iii) holds. Now, by Lemma 2.1,

$$\begin{aligned} \sqcap^*(L, R) &= \lambda(L) + \lambda(R) - \sqcap(L, R) - \lambda(L \cup R) \\ &= 3 + 3 - 0 - (r(L \cup R) + r(Q_1 \cup Q_2 \cup Q_3) - r(M)) \\ &= r(M) - r(Q_1 \cup Q_2 \cup Q_3) \\ &= 6 - r(Q_1 \cup Q_2 \cup Q_3). \end{aligned} \tag{5.1}$$

Thus (v) holds.

As $\lambda(Q_1 \cup Q_2) \geq 3$, we see that $r(Q_1 \cup Q_2) \geq 3$. Suppose $r(Q_1 \cup Q_2 \cup Q_3) = 3$. Then

$$3 = \lambda(R) = r(R) + r(L \cup Q_1 \cup Q_2 \cup Q_3) - r(M).$$

Thus $r(L \cup Q_1 \cup Q_2 \cup Q_3) = r(M) = 6$. As $r(L) = 3 = r(Q_1 \cup Q_2 \cup Q_3)$, it follows that $\sqcap(L, Q_1 \cup Q_2 \cup Q_3) = 0$, so $\sqcap(L, Q_1) = 0$, a contradiction. Hence $r(Q_1 \cup Q_2 \cup Q_3) \geq 4$, so (vi) holds.

Now suppose that $r(Q_1 \cup Q_2 \cup Q_3) = 6$. Then $\sqcap(Q_i, Q_j) = 0$ for all distinct i and j. It follows that $\mathbf{Q}$ is prism-like, so (vii) holds.

Next suppose that $r(Q_1 \cup Q_2 \cup Q_3) = 4$. Then $\sqcap^*(L, R) = 2$, so $r^*(L \cup R) = 4$. We now show that

5.9.1. $\sqcap(Q_i, Q_j) = 1$ *for all distinct* i *and* j.

We have that

$$\begin{aligned} r(Q_1 \cup Q_2) &= |Q_1 \cup Q_2| + r^*(L \cup Q_3 \cup R) - r^*(M) \\ &= r^*(L \cup Q_3 \cup R) - 2. \end{aligned}$$

Now

$$2 \geq \sqcap^*(Q_3, L \cup R) \geq \sqcap^*(Q_3, L) = 1$$

where the last inequality follows by Lemma 4.1. If $\sqcap^*(Q_3, L \cup R) = 2$, then $r^*(L \cup Q_3 \cup R) = r^*(L \cup R) = 4$, so

$$\begin{aligned} \lambda(Q_1 \cup Q_2) &= r^*(Q_1 \cup Q_2) + r^*(L \cup R \cup Q_3) - r^*(M) \\ &\leq 4 + 4 - 6 = 2, \end{aligned}$$

a contradiction. Thus $\sqcap^*(Q_3, L \cup R) = 1$, so $r^*(L \cup Q_3 \cup R) = 5$ and $\sqcap(Q_1, Q_2) = 1$. We conclude, by symmetry, that 5.9.1 holds, so (viii) holds.

Finally suppose that $r(Q_1 \cup Q_2 \cup Q_3) = 5$. Then, by (5.1), $\sqcap^*(L, R) = 1$, so $r^*(L \cup R) = 5$. If $r^*(L \cup R \cup Q_1) = 5$ and $r^*(L \cup R \cup Q_2) = 5$, then $r^*(L \cup R \cup Q_1 \cup Q_2) = 5$. But this gives a contradiction as $r^*(Q_3) = 2$ and $\lambda(Q_3) = 2$. Thus, by potentially taking a permutation of $\{1, 2, 3\}$, we may assume that

(a) $r^*(L \cup R \cup Q_i) = 6$ for all i; or
(b) $r^*(L \cup R \cup Q_1) = 5$ and $r^*(L \cup R \cup Q_2) = 6 = r^*(L \cup R \cup Q_3)$.

In case (a), we have, using the formula for the rank function in the dual of a matroid,

$$\begin{aligned} 6 &= r^*(L \cup R \cup Q_3) \\ &= |L \cup R \cup Q_3| + r(Q_1 \cup Q_2) - r(M) \\ &= 8 + r(Q_1 \cup Q_2) - 6. \end{aligned}$$

Hence $r(Q_1 \cup Q_2) = 4$. Similarly, $\sqcap(Q_i, Q_j) = 0$ for all distinct i and j. Thus $\mathbf{Q}$ is tightened-prism-like.

In case (b), $\sqcap(Q_1, Q_2) = 0 = \sqcap(Q_1, Q_3)$ and $\sqcap(Q_2, Q_3) = 1$. Thus $\mathbf{Q}$ is Vámos-inspired. □

Following Lemma 5.12, we provide specific examples of matroids that satisfy (viii), (ix)(a), and (ix)(b) of the last lemma.

Lemma 5.10. *Let $(L, Q_1, Q_2, \ldots, Q_n, R)$ be a $(4, 2)$-flexipath $\mathbf{Q}$ with no specially placed steps. Assume that $\mathbf{Q}$ is neither squashed nor stretched. Suppose $n \geq 2$ and $n \neq 3$. If $\sqcap(L, R) = 2$, then*

(i) $\sqcap^*(L, R) = 1$;
(ii) *for all i in $[n]$ and all $J \subseteq [n] - \{i\}$,*

$$\sqcap(L, Q_i) = \sqcap(L \cup Q_J, Q_i) = 1 = \sqcap^*(L, Q_i) = \sqcap^*(L \cup Q_J, Q_i);$$

(iii) $r(L \cup Q_J) = r(L) + \sum_{j \in J} r(Q_j) - |J|$ *for all* $J \subseteq [n]$;
(iv) $r(M) = r(L) + \sum_{i=1}^{n} r(Q_i) + r(R) - n - 3$;
(v) $r(Q_i \cup Q_j) = r(Q_i) + r(Q_j)$, *for all distinct i and j in $[n]$;*
(vi) $r(Q_J) = \sum_{j \in J} r(Q_j) - |J| + 2$ *for all $J \subseteq [n]$ such that $|J| \geq 2$; and*
(vii) $r(L \cup R \cup Q_J) = r(L) + r(R) + \sum_{j \in J} r(Q_j) - |J| - 2$ *for all $J \subseteq [n]$ such that $2 \leq |J| \leq n - 1$.*

Proof. By Lemma 4.4, $\sqcap(L, R) + \sqcap^*(L, R) \leq 3$. As $\sqcap(L, R) = 2$, we deduce that $\sqcap^*(L, R) \leq 1$. If $\sqcap^*(L, R) = 0$, then, by Lemma 5.9, and duality, $n = 3$, a contradiction. Thus $\sqcap^*(L, R) = 1$, so (i) holds. Parts (ii), (iii), and (iv) repeat parts (i), (ii), and (iii) of Lemma 5.7.

For (v) and (vi), since $\sqcap(L, R) = 2$, we have $r(L \cup R) = r(L) + r(R) - 2$. As $r(L \cup Q_3 \cup Q_4 \cup \cdots \cup Q_n) = r(L) + r(Q_3) + r(Q_4) + \cdots + r(Q_n) - (n - 2)$, we see that

$$\begin{aligned} r(L \cup R \cup Q_3 \cup Q_4 \cup \cdots \cup Q_n) &= r(L) + r(R) + \sum_{i=3}^{n} r(Q_i) - (n-2) \\ &\quad - \sqcap (R, L \cup Q_3 \cup Q_4 \cup \cdots \cup Q_n) \\ &\leq r(L) + r(R) + \sum_{i=3}^{n} r(Q_i) - n, \end{aligned}$$

where the last step follows because

$$\sqcap(R, L \cup Q_3 \cup Q_4 \cup \cdots \cup Q_n) \geq \sqcap(R, L) = 2.$$

Thus

$$\begin{aligned} 3 \leq \lambda(Q_1 \cup Q_2) &= r(Q_1 \cup Q_2) + r(L \cup R \cup Q_3 \cup Q_4 \cup \cdots \cup Q_n) - r(M) \\ &\leq r(Q_1 \cup Q_2) + \sum_{i=3}^{n} r(Q_i) - n + r(L) + r(R) \\ &\quad - \sum_{i=1}^{n} r(Q_i) + n + 3 - r(L) - r(R). \end{aligned}$$

Hence $r(Q_1) + r(Q_2) \leq r(Q_1 \cup Q_2)$ so $\sqcap(Q_1, Q_2) = 0$. Thus (v) holds, so (vi) holds for $|J| = 2$.

Now

$$\begin{aligned} \sqcap(L, Q_1 \cup Q_2) &= r(L) + r(Q_1 \cup Q_2) - r(L \cup Q_1 \cup Q_2) \\ &= r(L) + r(Q_1) + r(Q_2) - r(L) - r(Q_1) - r(Q_2) + 2 \\ &= 2. \end{aligned}$$

Thus, for all subsets J of $[n]$ with $|J| \geq 2$.

$$2 \leq \sqcap(L, Q_J). \tag{5.2}$$

Since $\sqcap$ is monotonic, for a proper subset J of $[n]$,

$$3 = \sqcap(L \cup Q_{[n]-J}, Q_J \cup R) \geq \sqcap(L, Q_J \cup R) \geq \sqcap(L, R) = 2.$$

If $\sqcap(L, Q_J \cup R) = 3$, then, by Lemma 5.2, $\sqcap(L, Q_i) = 2$ for all i in $[n] - J$. But $\sqcap(L, Q_j) = 1$ for all j in $[n]$, a contradiction. Hence $\sqcap(L, Q_J \cup R) = 2$ for all proper subsets J of $[n]$. Combining this with (5.2), we get that

$$2 \leq \sqcap(L, Q_J) \leq \sqcap(L, Q_J \cup R) = 2 \tag{5.3}$$

provided $2 \leq |J| \leq n - 1$. Thus, for such J,

$$\begin{aligned} r(Q_J) &= r(L \cup Q_J) - r(L) + \sqcap(L, Q_J) \\ &= r(L) + \sum_{j \in J} r(Q_j) - |J| - r(L) + 2. \end{aligned}$$

We have

$$r(Q_{[n]-\{1\}}) + r(Q_{[n]-\{n\}}) \geq r(Q_{[n]}) + r(Q_{[n]-\{1,n\}}),$$

so

$$\sum_{i=2}^{n} r(Q_i) - n + 3 + \sum_{i=1}^{n-1} r(Q_i) - n + 3 \geq r(Q_{[n]}) + \sum_{i=2}^{n-1} r(Q_i) - (n-2) + 2.$$

Hence

$$r(Q_{[n]}) \leq \sum_{i=1}^{n} r(Q_i) - n + 2. \tag{5.4}$$

Also, as $\sqcap(L,R) + \sqcap^*(L,R) \leq 3$, it follows by Lemma 2.1 that

$$\begin{aligned} 3 &\leq \lambda(L \cup R) \\ &= r(Q_{[n]}) + r(L) + r(R) - 2 - r(M) \\ &= r(Q_{[n]}) + r(L) + r(R) - 2 - r(L) \\ &\quad - \sum_{i=1}^{n} r(Q_i) - r(L) + n + 3. \end{aligned}$$

Thus

$$\sum_{i=1}^{n} r(Q_i) - n + 2 \leq r(Q_{[n]}). \tag{5.5}$$

Combining 5.4 and 5.5, we get

$$r(Q_{[n]}) = \sum_{i=1}^{n} r(Q_i) - n + 2.$$

Hence, for all $J \subseteq [n]$ such that $|J| \geq 2$, we have

$$r(Q_J) = \sum_{j \in J} r(Q_j) - |J| + 2,$$

that is, (vi) holds.

By (5.3), $\sqcap(L, Q_J \cup R) = 2$ for all J with $2 \leq |J| \leq n-1$, we have

$$\begin{aligned} r(L \cup Q_J \cup R) &= r(L) + r(Q_J \cup R) - 2 \\ &= r(L) + r(R) + \sum_{j \in J} r(Q_j) - |J| - 2. \end{aligned}$$

We conclude that (vii) holds. □

Next, having dealt with the case when $\sqcap(L,R) = 0$ in Lemma 5.9, we consider the case when $\sqcap(L,R) = 1$.

Lemma 5.11. *Let* $\mathbf{Q}$ *be a* $(4,2)$*-flexipath* (L, Q_1, Q_2, Q_3, R) *for which* $\sqcap(L,R) = 1$*. Then, in the clonal core of* M*,*

$$r(M) = 6 = r^*(M),$$

and the following hold.

(i) *If* $\sqcap^*(L,R) = 1$, *then* $r(Q_1 \cup Q_2 \cup Q_3) = 5$ *and, after a possible permutation of* $\{1,2,3\}$,

$$\begin{bmatrix} r(Q_1 \cup Q_2) & r^*(Q_1 \cup Q_2) \\ r(Q_1 \cup Q_3) & r^*(Q_1 \cup Q_3) \\ r(Q_2 \cup Q_3) & r^*(Q_2 \cup Q_3) \end{bmatrix} \in \left\{ \begin{bmatrix} 4 & 4 \\ 4 & 3 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{bmatrix} \right\}.$$

(ii) *If* $\sqcap^*(L,R) = 2$, *then* $r(Q_1 \cup Q_2 \cup Q_3) = 4$. *Moreover, for all distinct* i *and* j,

$$\sqcap(Q_i, Q_j) = 1 \text{ and } \sqcap^*(Q_i, Q_j) = 0,$$

so $r(Q_i \cup Q_j) = 3$ *and* $r^*(Q_i \cup Q_j) = 4$. *In particular,* $\mathbf{Q}$ *is spike-reminiscent.*

Proof. We assume that we are operating in the clonal core, which we label as M. Thus $r(L) = 3 = r(R)$ and $r(Q_i) = 2$ for all i. By Lemma 5.7(ii), $r(L \cup Q_1 \cup Q_2) = 5$ and $r(R \cup Q_3) = 4$. Thus

$$3 = \lambda(L \cup Q_1 \cup Q_2) = r(L \cup Q_1 \cup Q_2) + r(R \cup Q_3) - r(M).$$

Thus $r(M) = 6$, so $r^*(M) = 6$.

Next observe that, from the formula for the rank in the dual, we have

$$\begin{aligned} r(Q_1 \cup Q_2 \cup Q_3) &= |Q_1 \cup Q_2 \cup Q_3| + r^*(L \cup R) - r^*(M) \\ &= 6 + (r^*(L) + r^*(R) - \sqcap^*(L,R)) - r^*(M) \\ &= 6 - \sqcap^*(L,R). \end{aligned} \tag{5.6}$$

When $\sqcap^*(L,R) = 0$, by duality, we can deduce the structure of M from Lemma 5.9. Thus, we may assume that $\sqcap^*(L,R) \geq 1$. By Lemma 4.4, $\sqcap^*(L,R) \leq 2$. Hence $r(Q_1 \cup Q_2 \cup Q_3) \geq 4$.

Next we show that

5.11.1. $r(Q_i \cup Q_j) + r^*(Q_i \cup Q_j) \geq 7$ *for all distinct* i *and* j.

This follows immediately since

$$3 \leq \lambda(Q_i \cup Q_j) = r(Q_i \cup Q_j) + r^*(Q_i \cup Q_j) - |Q_i \cup Q_j|.$$

5.11.2. *If* $\sqcap^*(L,R) = 1$, *then at most one of* $r(Q_1 \cup Q_2)$, $r(Q_1 \cup Q_3)$, *and* $r(Q_2 \cup Q_3)$ *is* 3.

To see this, let $\{i,j,k\} = \{1,2,3\}$. Then, by 5.6, $r(Q_1 \cup Q_2 \cup Q_3) = 5$, so

$$r(Q_i \cup Q_j) + (Q_i \cup Q_k) \geq r(Q_1 \cup Q_2 \cup Q_3) + r(Q_i) = 5 + 2 = 7.$$

Thus (5.11.2) holds.

By symmetry,

$$\begin{bmatrix} r(Q_1 \cup Q_2) & r^*(Q_1 \cup Q_2) \\ r(Q_1 \cup Q_3) & r^*(Q_1 \cup Q_3) \\ r(Q_2 \cup Q_3) & r^*(Q_2 \cup Q_3) \end{bmatrix} \in \left\{ \begin{bmatrix} 4 & 4 \\ 4 & 3 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{bmatrix} \right\}.$$

Thus (i) holds.

5.11.3. *If $\sqcap^*(L,R) = 2$, then $r(Q_i \cup Q_j) = 3$ and $r^*(Q_i \cup Q_j) = 4$ for all distinct i and j.*

To see this, observe that $r^*(L \cup R) = 4$ as $\sqcap^*(L,R) = 2$. Now, using the formula for the rank function of the dual, we have

$$\begin{aligned} r(Q_1 \cup Q_2) &= |Q_1 \cup Q_2| + r^*(L \cup R \cup Q_3) - r^*(M) \\ &= r^*(L \cup R \cup Q_3) - 2. \end{aligned}$$

Since $r(Q_1 \cup Q_2) \geq 3$, we deduce that $r^*(L \cup R \cup Q_3) \geq 5$. But, by Lemma 5.7(i),

$$\sqcap^*(Q_3, L \cup R) \geq \sqcap^*(Q_3, L) = 1.$$

Thus $r^*(L \cup R \cup Q_3) \leq 5$ so $r^*(L \cup R \cup Q_3) = 5$ and $r(Q_1 \cup Q_2) = 3$. It follows by symmetry that $r(Q_i \cup Q_j) = 3$ for all distinct i and j. By 5.11.1, we deduce that $r^*(Q_i \cup Q_j) = 4$ for all distinct i and j. Thus $\mathbf{Q}$ is spike-reminiscent, so (ii) holds. $\square$

Combining Theorem 3.18 with the last lemma gives the following.

Lemma 5.12. *Let $\mathbf{Q}$ be a $(4,2)$-flexipath (L, Q_1, Q_2, Q_3, R) in a matroid M.*

(i) *If $\sqcap(L,R) = 1 = \sqcap^*(L,R)$, then, after a possible permutation of $\{1,2,3\}$,*

$$\begin{bmatrix} \sqcap(Q_1,Q_2) & \sqcap^*(Q_1,Q_2) \\ \sqcap(Q_1,Q_3) & \sqcap^*(Q_1,Q_3) \\ \sqcap(Q_2,Q_3) & \sqcap^*(Q_2,Q_3) \end{bmatrix} \in \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

In particular, in M or M^*, the flexipath $\mathbf{Q}$ is nasty or is doubly-tightened-prism-like.

(ii) *If $\sqcap(L,R) = 1$ and $\sqcap^*(L,R) = 2$, then $\mathbf{Q}$ is spike-reminiscent.*

(iii) *If $\sqcap(L,R) = 2$ and $\sqcap^*(L,R) = 1$, then $\mathbf{Q}$ is paddle-reminiscent.*

Next we provide examples of matroids satisfying (viii), (ix)(a) and (ix)(b) of Lemma 5.9. We also provide examples of a doubly-tightened-prism-like $(4,2)$-flexipath and of one of the types of nasty $(4,2)$-flexipaths. To explain this, we consider the operation of tightening a basis. Following Ferroni and Vecchi [3], we call a basis B in a matroid M a *free basis* if $0 < r(M) < |E(M)|$ and $B \cup \{e\}$ is a circuit for all e in $E(M) - B$. Equivalently, B is a free basis of M if it is not the unique basis of M and every fundamental circuit with respect to B is spanning. As is well known (see, for example, [5, Exercise 1.5.14]), a matroid M is a relaxation of another matroid N if and only if M has a free basis B, in which case, B is a circuit-hyperplane of N. We call N a *tightening* of M. Formally, $E(N) = E(M)$ and $\mathcal{B}(N) = \mathcal{B}(M) - \{B\}$.

For a matroid satisfying Lemma 5.9(viii), begin with a rank-7 free spike whose legs are $\{x_i, y_i\}$ for all i in $\{1,2,\dots,7\}$. Add elements α_1, α_2, and α_3 freely to the plane spanned by $\{x_1, y_1, x_2, y_2\}$. Then add elements β_1, β_2, and β_3 freely to the plane spanned by $\{x_6, y_6, x_7, y_7\}$. Let $Q_i = \{x_{i+2}, y_{i+2}\}$ for each i in $\{1,2,3\}$. Now truncate this matroid to rank 6, and delete

$\{x_1, y_1, x_2, y_2, x_6, y_6, x_7, y_7\}$. Let $L = \{\alpha_1, \alpha_2, \alpha_3\}$ and $R = \{\beta_1, \beta_2, \beta_3\}$. In the matroid M that we now have, $L \cup R$ is a circuit-hyperplane. Moreover, (L, Q_1, Q_2, Q_3, R) is spike-reminiscent in M. In M, relax the circuit-hyperplane $L \cup R$ to get a rank-6 matroid M_8 with a $(4, 2)$-flexipath (L, Q_1, Q_2, Q_3, R) in which $\sqcap(L, R) = 0$ and $r(Q_1 \cup Q_2 \cup Q_3) = 4$. It is not difficult to check that M_8 satisfies Lemma 5.9(viii). Indeed, (L, Q_1, Q_2, Q_3, R) is relaxed-spike-reminiscent in M_8.

To give examples of a tightened-prism-like and doubly-tightened-prism-like flexipaths, we begin by giving an example of a prism-like matroid. Begin with a 6-element independent set $\{b_1, b_2, \dots, b_6\}$. Now, for each i in $\{1, 2, 3\}$, freely add two points, x_i and y_i, on the line spanned by $\{b_i, b_{i+3}\}$, and let $Q_i = \{x_i, y_i\}$. Now freely add points α_1, α_2, and α_3 to the plane spanned by $\{b_1, b_2, b_3\}$. Similarly, freely add points β_1, β_2, and β_3 to the plane spanned by $\{b_4, b_5, b_6\}$. Now delete $\{b_1, b_2, \dots, b_6\}$, and let $L = \{\alpha_1, \alpha_2, \alpha_3\}$ and $R = \{\beta_1, \beta_2, \beta_3\}$. In this rank-6 matroid M, we have a $(4, 2)$-flexipath (L, Q_1, Q_2, Q_3, R) that is prism-like. Moreover, in M, the set $\{x_1, y_1, x_2, y_2, x_3, y_3\}$ is a free basis B. Let N be the matroid that is obtained by tightening B. In N, one can easily check that (L, Q_1, Q_2, Q_3, R) is a tightened-prism-like flexipath, that is, Lemma 5.9(ix)(a) holds. In N, we see that $\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}$ is a free basis B_N. Let P be the matroid obtained from N by tightening B_N. In that case, (L, Q_1, Q_2, Q_3, R) is a doubly-tightened-prism-like flexipath in P.

To describe a matroid satisfying Lemma 5.9(ix)(b), begin with a Vámos matroid V with ground set $\{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$ where $\{a_1, a_2, d_1, d_2\}$ is a basis and the only non-spanning circuits are the circuit-hyperplanes $\{a_1, a_2, b_1, b_2\}$, $\{a_1, a_2, c_1, c_2\}$, $\{b_1, b_2, c_1, c_2\}$, $\{b_1, b_2, d_1, d_2\}$, and $\{c_1, c_2, d_1, d_2\}$. Let $A = \{a_1, a_2\}$ and $D = \{d_1, d_2\}$. Take the direct sum of V and $U_{2,2}$ where the latter has ground set $\{a, d\}$. Now freely add points α_1, α_2, α_3 and α_4 to the plane spanned by $A \cup \{a\}$. Similarly, freely add points δ_1, δ_2, δ_3 and δ_4 to the plane spanned by $D \cup \{d\}$. On the line spanned by α_4 and δ_4, freely add points β_1 and γ_1. Let $Q_1 = \{\beta_1, \gamma_1\}$. Delete $\{a_1, a_2, d_1, d_2, a, d, \alpha_4, \delta_4\}$ to give a matroid M_9. Let $Q_2 = \{b_1, b_2\}$ and $Q_3 = \{c_1, c_2\}$. Then $\sqcap(Q_1, Q_2) = 0 = \sqcap(Q_1, Q_3)$ and $\sqcap(Q_2, Q_3) = 1$, while $\sqcap^*(Q_i, Q_j) = 0$ for all distinct i and j. Let $L = \{\alpha_1, \alpha_2, \alpha_3\}$ and $R = \{\delta_1, \delta_2, \delta_3\}$. In M_9, we now have that (L, Q_1, Q_2, Q_3, R) is a $(4, 2)$-flexipath that satisfies Lemma 5.9(ix)(b), that is, (L, Q_1, Q_2, Q_3, R) is Vámos-inspired.

We can modify the last example to get an example of one of the types of nasty $(4, 2)$-flexipaths. In the matroid M_9, we have that $\sqcap(L, R) = 0$ and $\sqcap^*(L, R) = 1$. In this matroid, we see that $L \cup R$ is a free basis. Tightening this basis gives a matroid N_9 in which $\sqcap(L, R) = 1$ and $\sqcap^*(L, R) = 1$. Moreover, in N_9, we have that $\sqcap(Q_1, Q_2) = 0 = \sqcap(Q_1, Q_3)$ and $\sqcap(Q_2, Q_3) = 1$, while $\sqcap^*(Q_i, Q_j) = 0$ for all distinct i and j, and $r(Q_1 \cup Q_2 \cup Q_3) = 5$. Thus, in N_9, we see that (L, Q_1, Q_2, Q_3, R) is an example of the second type of nasty $(4, 2)$-flexipath. By dualizing, we get an example of the third type of nasty $(4,2)$-flexipath. We do not give an example coming from a matroid of the first type of nasty $(4,2)$-flexipath. We note, however, that, because we can determine the ranks of all subsets of $\{L, R, Q_1, Q_2, Q_3\}$, we can routinely check that if X and Y are such subsets, then $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$. The reader familiar with polymatroids will see that the verification that the submodular inequality holds for each such pair $\{X, Y\}$ establishes the existence of the corresponding polymatroid; from this, one immediately gets the existence of the first type of nasty $(4,2)$-flexipath.

Lemma 5.13. *Let* $\mathbf{Q}$ *be a* $(4,2)$*-flexipath* $(L, Q_1, Q_2, \dots, Q_n, R)$ *where* $\sqcap(L, R) = 1 = \sqcap^*(L, R)$*, and* $\sqcap(L, Q_i) = 1$ *for all* i *in* $[n]$*. Then* $\mathbf{Q}$ *has at most three internal steps.*

Proof. We assume that $n \geq 4$. We operate in the clonal core, calling this M. Then $r(L) = 3$ and $r(L \cup Q_i) = 4$ for all i. Moreover, by Lemma 5.7(i), for all distinct i, j, and k, we have $r(L \cup Q_i \cup Q_j) = 5$ and $r(L \cup Q_i \cup Q_j \cup Q_k) = 6$.

Next we show the following.

5.13.1. *If* $\sqcap(L, R \cup Q_1) = 2 = \sqcap(L, R \cup Q_2)$*, then* $r(L \cup R \cup Q_1 \cup Q_2) \leq 5$*.*

We have

$$\begin{aligned} r(L \cup R \cup Q_1 \cup Q_2) &\leq r(L \cup R \cup Q_1) + r(L \cup R \cup Q_2) - r(L \cup R) \\ &= 5 + 5 - 5 = 5. \end{aligned}$$

Thus 5.13.1 holds.

First suppose that $n = 4$. Then, by Lemma 5.7,

$$r(L \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4) = r(M) = 7.$$

Next we show the following.

5.13.2. *For* $\{g, h, i, j\} = \{1, 2, 3, 4\}$*, if* $\sqcap(Q_g, Q_h) = 0$*, then* $\sqcap^*(Q_i, Q_j) = 1$*.*

To see this, observe that $r(Q_g \cup Q_h) = 4$ as $\sqcap(Q_g, Q_h) = 0$. Now $r(L \cup Q_g \cup Q_h) = 5 = r(R \cup Q_g \cup Q_h)$. Thus

$$\begin{aligned} r(L \cup R \cup Q_g \cup Q_h) &\leq r(L \cup Q_g \cup Q_h) + r(R \cup Q_g \cup Q_h) - r(Q_g \cup Q_h) \\ &= 5 + 5 - 4 \\ &= r(M) - 1. \end{aligned}$$

Hence $Q_i \cup Q_j$ contains a cocircuit of M. Because each of Q_i and Q_j consists of a clonal pair of elements, if $\mathrm{cl}(L \cup R \cup Q_g \cup Q_h)$ meets Q_i, then it contains Q_i. In that case, $\lambda(Q_j) \leq 1$, a contradiction. We conclude that $Q_i \cup Q_j$ is a cocircuit of M. Thus, by Lemma 5.8(iii), $\sqcap^*(Q_i, Q_j) = 1$. Hence 5.13.2 holds.

5.13.3. *If* $\sqcap(L, R \cup Q_1) = 2$*, then* $\sqcap^*(Q_i, Q_j) = 1$ *for all distinct* i *and* j *in* $\{2, 3, 4\}$*.*

By Lemma 5.8(iii) again, $\sqcap^*(Q_i, Q_j) = 1$ if and only if $Q_i \cup Q_j$ is a cocircuit of M. Since $\sqcap(L, R \cup Q_1) = 2$, we have

$$\begin{aligned} r(L \cup R \cup Q_1) &= r(L) + r(R \cup Q_1) - 2 \\ &= 3 + 4 - 2 = 5. \end{aligned}$$

Then, for i in $\{2, 3, 4\}$, as $\sqcap(Q_i, L \cup R \cup Q_1) \geq \sqcap(Q_i, L) = 1$, it follows that $r(L \cup R \cup Q_1 \cup Q_i) \leq 6$. Thus $Q_j \cup Q_k$ contains a cocircuit where $\{i, j, k\} = \{2, 3, 4\}$.

Continuing with the proof of 5.13.3, we now show that

5.13.4. $r(L \cup R \cup Q_1 \cup Q_i) = 6$ *for all* i *in* $\{2, 3, 4\}$.

Assume that $r(L \cup R \cup Q_1 \cup Q_i) = 5$ for some i in $\{2, 3, 4\}$. Then $r(L \cup R \cup Q_1 \cup Q_i \cup Q_j) \leq 6$ for $\{j, k\} = \{2, 3, 4\} - \{i\}$, so $\lambda(Q_k) \leq 1$, a contradiction. Thus 5.13.4 holds.

It follows as above that, for $\{j, k\} = \{2, 3, 4\} - \{i\}$, since each of Q_j and Q_k consists of a clonal pair of elements, $L \cup R \cup Q_1 \cup Q_i$ is a hyperplane, so $Q_j \cup Q_k$ is a cocircuit. Hence $\sqcap^*(Q_j, Q_k) = 1$. Thus 5.13.3 holds.

Next we show the following.

5.13.5. *If* $\sqcap(L, R \cup Q_1) = 2$, *then* $\sqcap(L, R \cup Q_i) = 1$ *for all distinct* i *in* $\{2, 3, 4\}$.

As $\sqcap(L, R \cup Q_2) = 2$, by 5.13.1, $r(L \cup R \cup Q_1 \cup Q_2) \leq 5$. But this contradicts 5.13.4. Thus 5.13.5 holds.

Now $\sqcap(L, R \cup Q_i) \geq \sqcap(L, R) = 1$. Moreover,

$$\sqcap (L, R \cup Q_i) + \sqcap^*(L, R \cup Q_i) \leq 3 \tag{5.7}$$

since

$$\begin{aligned} \sqcap(L, R \cup Q_i) + \sqcap^*(L, R \cup Q_i) &= \lambda(L) + \lambda(R \cup Q_i) - \lambda(L \cup R \cup Q_i) \\ &\leq 3 + 3 - 3 = 3. \end{aligned}$$

By 5.13.5, duality, and (5.7), we may assume the following without loss of generality.

5.13.6. *If* $\sqcap^*(L, R \cup Q_i) \neq 1$, *then* $i = 1$ *and* $\sqcap^*(L, R \cup Q_i) = 2$. *If* $\sqcap(L, R \cup Q_j) \neq 1$, *then* $j = 2$ *and* $\sqcap(L, R \cup Q_j) = 2$.

Next we show the following.

5.13.7. *If* $\sqcap^*(L, R \cup Q_1) = 2$, *then* $\sqcap(Q_i, Q_j) = 1$ *and* $\sqcap^*(Q_i, Q_j) = 0$ *for all distinct* i *and* j *in* $\{2, 3, 4\}$.

We have $\sqcap(L, R \cup Q_1) \geq \sqcap(L, R) = 1$. By (5.7), $\sqcap(L, R \cup Q_1) \leq 3 - \sqcap^*(L, R \cup Q_1) = 1$. Thus $\sqcap(L, R \cup Q_1) = 1$. Hence, for the $(4, 2)$-flexipath $(L, Q_2, Q_3, Q_4, R \cup Q_1)$, we have $\sqcap^*(L, R \cup Q_1) = 2$ and $\sqcap(L, R \cup Q_1) = 1$. Thus, it follows by Lemma 5.11(ii) that $\sqcap(Q_i, Q_j) = 1$ and $\sqcap^*(Q_i, Q_j) = 0$ for all distinct i and j in $\{2, 3, 4\}$, that is, 5.13.7 holds.

By 5.13.7 and duality, we immediately obtain the following.

5.13.8. *If* $\sqcap(L, R \cup Q_2) = 2$*, then* $\sqcap^*(Q_i, Q_j) = 1$ *and* $\sqcap(Q_i, Q_j) = 0$ *for all distinct* i *and* j *in* $\{1, 3, 4\}$*.*

By considering $\sqcap(Q_3, Q_4)$ and $\sqcap^*(Q_3, Q_4)$, we get on combining 5.13.7 and 5.13.8 that we may assume, using duality, that

5.13.9. $\sqcap(L, R \cup Q_i) = 1$ *for all* i *in* $\{1, 2, 3, 4\}$*, and* $\sqcap^*(L, R \cup Q_j) = 1$ *for all* j *in* $\{2, 3, 4\}$*. Moreover,* $\sqcap^*(L, R \cup Q_1) \in \{1, 2\}$*.*

For each j in $\{2, 3, 4\}$, consider the path $(L, Q_g, Q_h, Q_i, Q_j \cup R)$, which we relabel as (L, Q_g, Q_h, Q_i, R'). Then $\sqcap(L, R') = 1 = \sqcap^*(L, R')$. We take the clonal core of $(M, (L, Q_g, Q_h, Q_i, R'))$. It is a rank-6 matroid M'. By Lemma 5.11, for each j in $\{2, 3, 4\}$, there are distinct elements s and t in $\{1, 2, 3, 4\} - \{j\}$ such that $\sqcap(Q_s, Q_t) = 0 = \sqcap^*(Q_s, Q_t)$. Then, by 5.13.2, $\sqcap(Q_p, Q_q) = 1 = \sqcap^*(Q_p, Q_q)$ where $\{p, q\} = \{1, 2, 3, 4\} - \{s, t\}$. This contradicts Lemma 4.2. We conclude that $\mathbf{Q}$ does not have exactly four internal steps.

We now consider $(L, Q_1, Q_2, \dots, Q_n, R)$ where $n \geq 4$ and $\sqcap(L, R) = 1 = \sqcap^*(L, R)$. We prove by induction on n that such a path of 4-separations does not exist. We proved this above for $n = 4$. Assume it is true when the path has fewer than n internal steps and suppose that it has exactly n internal steps where $n \geq 5$. We continue to operate in the clonal core and to label this clonal core as M. As $\sqcap(L, Q_i) = 1$ for all i, it follows that

$$r(M) = n + 3. \tag{5.8}$$

5.13.10. *If* $\sqcap(L, R \cup Q_1) = 2$*, then* $\sqcap(L, R \cup Q_i) = 1$ *for all* i *in* $\{2, 3, \dots, n\}$*.*

Assume that $\sqcap(L, R \cup Q_2) = 2$. Then, by 5.13.1,

$$r(L \cup R \cup Q_1 \cup Q_2) \leq 5. \tag{5.9}$$

As $\sqcap(L, R \cup Q_1) = 2$, by considering the path $(L, Q_2, Q_3, \dots, Q_n, R \cup Q_1)$ of 4-separations, which has at least four internal steps, we deduce by Lemma 5.10 that

$$r(Q_3 \cup Q_4 \cup \cdots \cup Q_n) = n. \tag{5.10}$$

Then, by (5.9), (5.10), and (5.8),

$$\begin{aligned} 3 &\leq \lambda(Q_3 \cup Q_4 \cup \cdots \cup Q_n) \\ &\leq n + 5 - (n + 3) = 2, \end{aligned}$$

a contradiction. Thus 5.13.10 holds.

By 5.13.10, duality, and symmetry, we may assume that $\sqcap(L, R \cup Q_n) = 1 = \sqcap^*(L, R \cup Q_n)$. Then the path $(L, Q_1, Q_2, \dots, Q_{n-1}, R \cup Q_n)$ is a path of 4-separations that violates the induction assumption. The lemma now follows by induction. □

Lemma 5.14. *Let* $\mathbf{Q}$ *be a* $(4, 2)$*-flexipath* $(L, Q_1, Q_2, \dots, Q_n, R)$ *in a matroid* M*, where* $n \geq 2$ *but* $n \neq 3$*. Assume that* $\mathbf{Q}$ *is neither squashed nor stretched and has no specially placed steps. Then exactly one of the following holds for all distinct* i *and* j *in* $[n]$*.*

(i) $\sqcap(Q_i, Q_j) = 0$ *and* $\sqcap^*(Q_i, Q_j) = 1$.
(ii) $\sqcap(Q_i, Q_j) = 1$ *and* $\sqcap^*(Q_i, Q_j) = 0$.
(iii) $n = 2$ *and* $\sqcap(Q_i, Q_j) = 0 = \sqcap^*(Q_i, Q_j)$, *while* $\sqcap(L, R) = 1 = \sqcap^*(L, R)$.

Proof. By Lemma 4.2, for a given pair i, j, we must either have one of the outcomes described in the lemma, or

$$\sqcap(Q_i, Q_j) = 0 = \sqcap^*(Q_i, Q_j). \tag{5.11}$$

It remains to prove that we have the same outcome for all such pairs and that, when (5.11) arises, $n = 2$. By Lemmas 5.9 and 5.10, since $n \neq 3$,

(a) $\sqcap(L, R) = 2$ and $\sqcap^*(L, R) = 1$; or
(b) $\sqcap^*(L, R) = 2$ and $\sqcap(L, R) = 1$; or
(c) $n = 2$ and $\sqcap(L, R) = 1 = \sqcap^*(L, R)$.

Suppose that (c) holds. Then, in the clonal core, which we write as M, we have $r(L \cup R) = 5$, so $r(M) \geq 5$. Now, by Lemma 5.7, $r(L \cup Q_1 \cup Q_2) = 5$ and

$$3 = \lambda(R) = r(L \cup Q_1 \cup Q_2) + r(R) - r(M).$$

Thus $r(M) = r(L \cup Q_1 \cup Q_2) = 5 = r(L \cup R)$. Hence $Q_1 \cup Q_2$ is not a cocircuit of M. By Lemma 5.8, we deduce that $\sqcap^*(Q_1, Q_2) = 0$. By duality, $\sqcap(Q_1, Q_2) = 0$.

By duality, we may now assume that (a) holds. We also assume that we are operating in the clonal core, where, as usual, we relabel this as M. Then, by Lemma 5.10(v), $\sqcap(Q_i, Q_j) = 0$ for all distinct i and j in $[n]$. Now, fix i and j, and let $J = [n] - \{i, j\}$. Then, by Lemma 5.10(vi) and (iv),

$$\begin{aligned} r(M) - r(L \cup R \cup Q_J) &= r(L) + \sum_{h=1}^{n} r(Q_h) + r(R) - n - 3 \\ &\quad -(r(L) + \sum_{h \in J} r(Q_h) + r(R) - (n-2) - 2 \\ &= r(Q_i) + r(Q_j) - 3 \\ &= 2 + 2 - 3 = 1. \end{aligned}$$

We deduce that $Q_i \cup Q_j$ contains a cocircuit of M. As each of Q_i and Q_j consists of a pair of clones, $Q_i \cup Q_j$ is a cocircuit of M. Then, by Lemma 5.8(iii), $\sqcap^*(Q_i, Q_j) = 1$. We conclude that, when (a) holds, so does (i). By duality, when (b) holds, so does (ii). Thus, (5.11) never arises. □

Theorem 5.15. *Let* $\mathbf{Q}$ *be a* $(4, 2)$*-flexipath* $(L, Q_1, Q_2, \ldots, Q_n, R)$ *in a matroid* M, *where* $n \geq 2$. *Then the following hold.*

(i) *If* $\mathbf{Q}$ *has no specially placed steps, then either*
  (a) $\mathbf{Q}$ *is squashed, stretched, paddle-reminiscent, or spike-reminiscent; or*
  (b) $n = 3$ *and, in either* M *or* M^*, *the* $(4, 2)$*-flexipath* $\mathbf{Q}$ *is prism-like, tightened-prism-like, doubly-tightened-prism-like, relaxed-spike-reminiscent, Vámos-inspired, or nasty; or*

(c) *$n = 2$ and $\sqcap(Q_i, Q_j) = 0 = \sqcap^*(Q_i, Q_j) = 0$, while $\sqcap(L, R) = 1 = \sqcap^*(L, R)$.*

(ii) *If Q_n is a specially placed step of type* (S1)*, and $n \geq 3$, then $(L, Q_1, \ldots, Q_{n-1}, Q_n \cup R)$ is paddle-reminiscent or relaxed-paddle-reminiscent.*

(iii) *If Q_n is a specially placed step of type* (S2)*, and $n \geq 3$, then $(L, Q_1, \ldots, Q_{n-1}, Q_n \cup R)$ is spike-reminiscent or relaxed-spike-reminiscent.*

Proof. Suppose that $\mathbf{Q}$ has no specially placed steps and that $\mathbf{Q}$ is not squashed or stretched. Then, by Lemma 5.6, for all i in $[n]$,

$$\sqcap (L, Q_i) = \sqcap^*(L, Q_i) = 1 = \sqcap(R, Q_i) = \sqcap^*(R, Q_i). \tag{5.12}$$

Suppose that $n \neq 3$. Then, by Lemma 5.9 and its dual, $\sqcap(L, R) \neq 0$ and $\sqcap^*(L, R) \neq 0$. Thus $\sqcap(L, R) \geq 1$ and $\sqcap^*(L, R) \geq 1$. By Lemma 4.4,

$$\sqcap(L, R) + \sqcap^*(L, R) \leq 3.$$

If $\sqcap(L, R) = 1 = \sqcap^*(L, R)$, then, as $n \neq 3$, by Lemma 5.13, we get that $n = 2$. We deduce that either

(a) $\sqcap(L, R) = 2$ and $\sqcap^*(L, R) = 1$; or
(b) $\sqcap^*(L, R) = 2$ and $\sqcap(L, R) = 1$; or
(c) $n = 2$ and $\sqcap(L, R) = 1 = \sqcap^*(L, R)$.

Suppose that (c) holds. Then, by Lemma 5.14, $\sqcap(Q_1, Q_2) = 0 = \sqcap^*(Q_1, Q_2)$. If (a) holds, then, by Lemma 5.10(v), $\sqcap(Q_i, Q_j) = 0$ for all distinct i and j. Thus, by Lemma 5.14, $\sqcap^*(Q_i, Q_j) = 1$ for all distinct i and j. We deduce that $\mathbf{Q}$ is paddle-reminiscent. By duality, if (b) holds, then $\mathbf{Q}$ is spike-reminiscent.

Now let $n = 3$ and assume that $\mathbf{Q}$ is neither paddle-reminiscent nor spike-reminiscent. By Lemma 4.4, $\sqcap(L, R) + \sqcap^*(L, R) \leq 3$. By duality, we may assume that $\sqcap(L, R) \leq \sqcap^*(L, R)$. If $\sqcap(L, R) = 0$, then the possibilities for $\mathbf{Q}$ are identified in Lemma 5.9, namely, $\mathbf{Q}$ is relaxed-spike-reminiscent, tightened-prism-like, or Vámos-inspired. We may now assume that $\sqcap(L, R) = 1$. Then, by Lemma 5.11, $\sqcap^*(L, R) = 1$ and the possibilities for $\mathbf{Q}$ are identified in (i) of that lemma. In particular, $\mathbf{Q}$ is doubly-tightened-prism-like or is nasty.

By duality, it only remains to prove (ii). Assume Q_n is a specially placed step of type (S1) and that $n \geq 3$. Then $(L, Q_1, Q_2, \ldots, Q_{n-1}, Q_n \cup R\}$ is a $(4, 2)$-flexipath $\mathbf{Q}'$. Suppose $\mathbf{Q}'$ has a specially placed element Q_i. Assume first that Q_i is of type (S1). Then $\sqcap(L, Q_i) = 2$, so, by Lemma 4.3, Q_i is specially placed in $\mathbf{Q}$. Thus $\mathbf{Q}$ has two specially placed elements, a contradiction to Lemma 5.1. Thus Q_i is specially placed of type (S2). Then $\sqcap^*(L, Q_i) = 2$, so, again, Q_i is specially placed in $\mathbf{Q}$, a contradiction. We conclude that $\mathbf{Q}'$ has no specially placed steps.

We now argue in the clonal core. Because Q_n is a specially placed step of type (S1), $\sqcap(L, R) = 2$, so $r(L \cup R) = 4$. Also $\sqcap(R, Q_n) = 2$, so $r(Q_n \cup R) = 3$

and $r(L \cup Q_n \cup R) = 4$. Thus

$$\sqcap(L, Q_n \cup R) = r(L) + r(Q_n \cup R) - r(L \cup Q_n \cup R) = 3 + 3 - 4 = 2.$$

Hence $\mathbf{Q}'$ is neither squashed nor stretched. By Lemma 5.6, $\sqcap(L, Q_i) = 1 = \sqcap^*(L, Q_i)$ for all i in $[n-1]$. If $\sqcap^*(L, Q_n \cup R) = 0$, then, by Lemma 5.9, $n-1 = 3$, so $n = 4$ and $\mathbf{Q}'$ is relaxed-paddle-reminiscent. If $\sqcap^*(L, Q_n \cup R) = 1$, then, by Lemma 5.11, $\mathbf{Q}'$ is paddle-reminiscent. Thus (ii) holds. □

The complexity of the last result can be simplified by classifying the numerous outcomes by a more succinct list of defining characteristics.

Corollary 5.16. *Let* $\mathbf{Q}$ *be a* $(4, 2)$*-flexipath* $(L, Q_1, Q_2, \ldots, Q_n, R)$ *in a matroid, where* $n \geq 2$ *and* $\mathbf{Q}$ *has no specially placed steps. For* $\sqcap(L, R) \leq \sqcap^*(L, R)$*, the following outcomes are possible.*

(i) *If* $(\sqcap(L, R), \sqcap^*(L, R)) = (0, 0)$*, then* $\mathbf{Q}$ *is prism-like.*
(ii) *If* $(\sqcap(L, R), \sqcap^*(L, R)) = (0, 1)$*, then* $n = 3$ *and*
  (a) $\sqcap(Q_i, Q_j) = 0$ *for all distinct* i *and* j*, and* $\mathbf{Q}$ *is tightened-prism-like; or*
  (b) $\sqcap(Q_i, Q_j) = 1$ *for exactly one distinct pair* $\{i, j\}$*, and* $\mathbf{Q}$ *is Vámos-inspired.*
(iii) *If* $(\sqcap(L, R), \sqcap^*(L, R)) = (0, 2)$*, then* $\mathbf{Q}$ *is relaxed-spike-reminiscent.*
(iv) *If* $(\sqcap(L, R), \sqcap^*(L, R)) = (0, 3)$*, then* $\mathbf{Q}$ *is stretched.*
(v) *If* $(\sqcap(L, R), \sqcap^*(L, R)) = (1, 1)$*, then* $n \in \{2, 3\}$*.*
(vi) *If* $(\sqcap(L, R), \sqcap^*(L, R)) = (1, 1)$ *and* $n = 2$*, then* $\sqcap(Q_1, Q_2) = 0 = \sqcap^*(Q_1, Q_2)$*.*
(vii) *If* $(\sqcap(L, R), \sqcap^*(L, R)) = (1, 1)$ *and* $n = 3$*, then*
  (a) $\sqcap(Q_i, Q_j) = 0 = \sqcap^*(Q_i, Q_j)$ *for all distinct* i *and* j*, and* $\mathbf{Q}$ *is doubly-tightened-prism-like; or*
  (b) *the multiset of pairs* $\{(\sqcap(Q_i, Q_j), \sqcap^*(Q_i, Q_j)); i \neq j\}$ *contains*
    (1) *both* $(0, 1)$ *and* $(1, 0)$ *and* $\mathbf{Q}$ *is mixed nasty; or*
    (2) $(1, 0)$ *but not* $(0, 1)$ *and* $\mathbf{Q}$ *is plane nasty; or*
    (3) $(0, 1)$ *but not* $(1, 0)$ *and* $\mathbf{Q}$ *is dual-plane nasty.*
(viii) *If* $(\sqcap(L, R), \sqcap^*(L, R)) = (1, 2)$*, then* $\mathbf{Q}$ *is spike-reminiscent.*

To see an example satisfying (vi), we can modify a prism-like matroid as follows. Take a 6-element independent set $\{b_1, b_2, \ldots, b_6\}$. Add b'_1, b'_2, and b'_3 freely on the flat spanned by $\{b_1, b_2, b_3\}$ and add b'_4, b'_5, and b'_6 freely on the flat spanned by $\{b_4, b_5, b_6\}$. Add a point c freely on the line spanned by $\{b_3, b_6\}$. Add points c_1 and c_4 freely on the line spanned by $\{b_1, b_4\}$. Add points c_2 and c_5 freely on the line spanned by $\{b_2, b_5\}$. Contract c and delete $\{b_1, b_2, \ldots, b_6\}$ to get a rank-5 matroid M. Let $(L, R) = (\{b'_1, b'_2, b'_3\}, \{b'_4, b'_5, b'_6\})$ and $(Q_1, Q_2) = \{c_1, c_4\}, \{c_2, c_5\})$. Then $\sqcap(L, R) = 1$ so $r(L \cup R) = 5 = r(M)$. Also $Q_1 \cup Q_2$ is neither a circuit nor a cocircuit so $\sqcap(Q_1, Q_2) = 0 = \sqcap^*(Q_1, Q_2)$. Finally, $r^*(L \cup R) = |L \cup R| + r(Q_1 \cup Q_2) - r(M) = 6 + 4 - 5 = 5$. It follows that $\sqcap^*(L, R) = 1$.

We conclude by noting that Theorem 1.1 follows from Theorem 5.15.

Proof of Theorem 1.1. By Lemma 5.1, when we absorb any specially placed steps of $\mathbf{Q}$ into its right end, we get a $(4, 2)$-flexipath $\mathbf{Q}'$ with at least four internal steps none of which is specially placed. The theorem now follows immediately from Theorem 5.15(i). □

References

[1] Brylawski, T.H., Modular constructions for combinatorial geometries, *Trans. Amer. Math. Soc.* **203** (1975), 1–44.

[2] Ding, G,, Oporowski, B., Oxley J., and Vertigan, D., Unavoidable minors of large 3-connected matroids, *J. Combin. Theory Ser. B* **71**(1997), 244–293.

[3] Ferroni, L .and Vecchi, L., Matroid relaxations and Kazhdan-Lusztig non-degeneracy, *Algbr. Comb.*, **5** (2022), 745–769.

[4] Geelen, J., Gerards, B., and Whittle, G., Matroid T-connectivity, *SIAM J. Disc. Math.*, **20** (2006), 588–596.

[5] Oxley, J., *Matroid Theory,* Second Edition, Oxford University Press, New York, 2011.

[6] Oxley, J., Semple, C., and Whittle, G., The structure of the 3-separations of 3-connected matroids, *J. Combin. Theory Ser. B* **92** (2004), 257–293.

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