Unavoidable cycle-contraction minors of large 2-connected graphs

Wayne Ge James Oxley

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Abstract

It is well known that every sufficiently large connected graph has, as an induced subgraph, K_n, $K_{1,n}$, or an n-vertex path. A 2023 paper of Allred, Ding, and Oporowski identified the unavoidable induced subgraphs of sufficiently large 2-connected graphs. In this paper, we establish a dual version of this theorem by focusing on the minors obtained by contracting cycles, the dual operation of deleting vertices.

1 Introduction

For graph and matroid terminology not explicitly defined here, we follow [2] and [5]. In particular, we allow graphs to have loops and parallel edges; a graph is *simple* if it has neither; a graph is *nontrivial* if it has more than one vertex.

For an integer k exceeding one, a graph G is *k-connected* if $|V(G)| \geq k$ and, whenever u and v are distinct vertices of G, there are at least k pairwise internally disjoint uv-paths. In particular, for $n \geq 2$, the *bond graph* B_n that consists of two vertices joined by n parallel edges is 2-connected. Observe that our definition of a k-connected graph is broader than that of many authors who require that $|V(G)| \geq k+1$ for such a graph G. A graph H with at least two vertices is *k-edge-connected* if $H \setminus Z$ is connected for all subsets Z of $E(H)$ with $|Z| < k$. By convention, a single-vertex graph is neither 2-connected nor 2-edge-connected.

A well-known theorem shows that every sufficiently large graph has, as an induced subgraph, K_n, $K_{1,n}$, or an n-vertex path. Allred, Ding, and Oporowski [1] identified the unavoidable induced subgraphs of sufficiently large 2-connected graphs. The goal of this paper is to prove a dual result.

Let G be a graph and H be an induced subgraph of G. Evidently H can be obtained from G by a sequence of operations each consisting of deleting a bond from the current graph or deleting an isolated vertex from the current graph. When G is a plane graph having G^* as its planar dual, the planar dual H^* of H is obtained from G^* by a sequence of operations each consisting of contracting a cycle from the current graph. Thus the dual operation of deleting a bond from a graph is contracting a cycle.

A graph K is a *cycle-contraction minor* or *cc-minor* of a graph J if there is a sequence $J_0, J_1, \ldots, J_m$ of graphs such that $(J_0, J_m) = (J, K)$ and, for each $i \in [m]$, there is a cycle C_{i-1} of J_{i-1} such that $J_i = J_{i-1}/C_{i-1}$. In this paper, we determine a list of loopless 2-connected graphs such that every sufficiently large 2-connected graph has a member of the list as a cc-minor. Unless otherwise stated, each cycle contraction we perform is accompanied by the contraction of any loops it creates.

In the next section, we shall prove the following result, which links induced subgraphs and cc-minors via duality.

Lemma 1.1. *Let G be a loopless 2-connected plane graph. A graph H is a 2-connected induced subgraph of G if and only if H^* is a 2-connected cc-minor of G^*.*

The statements of both our main result and of the theorem of Allred, Ding, and Oporowski [1] will rely on Tutte's tree-decomposition result for 2-connected graphs, which we shall introduce next.

Let G_1 and G_2 be graphs such that $V(G_1) \cap V(G_2) = \{u, v\}$ and $E(G_1) \cap E(G_2) = \{e\}$ where e is neither a loop nor a cut edge of G_1 or G_2. The graph $G_1 \cup G_2$ is the *parallel connection* of G_1 and G_2 with *basepoint* e. The graph obtained from $G_1 \cup G_2$ by deleting e is the 2-*sum*, $G_1 \bigoplus_2 G_2$, of G_1 and G_2 with *basepoint* e.

A *graph-labelled tree* is a tree T with vertex set $\{G_1, G_2, \ldots, G_k\}$ for some positive integer k such that

(i) each G_i is a graph;

(ii) if G_{j_1} and G_{j_2} are joined by an edge e of T, then $E(G_{j_1}) \cap E(G_{j_2}) = \{e\}$ and e is neither a loop nor a cut edge of G_{j_1} or G_{j_2}; and

(iii) if G_{j_1} and G_{j_2} are non-adjacent, then $E(G_{j_1}) \cap E(G_{j_2})$ is empty.

We call $G_1, G_2, \ldots, G_k$ the *vertex labels* of T; for each h in $[k]$, the edges in $E(G_h) \cap E(T)$ are the *basepoints* of G_h.

Let e be an edge of a graph-labelled tree T and suppose e joins the vertices H_1 and H_2. If we contract e from T and relabel by $H_1 \bigoplus_2 H_2$ the vertex that results by identifying the endpoints of e leaving all other edge and vertex labels unchanged, then we get a new graph-labelled tree, T/e.

A *tree-decomposition* of a loopless 2-connected graph G is a graph-labelled tree T such that if $V(T) = \{G_1, G_2, \ldots, G_k\}$ and $E(T) = \{e_1, e_2, \ldots, e_{k-1}\}$, then

(i) $E(G) = (E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)) - \{e_1, e_2, \ldots, e_{k-1}\}$;

(ii) $|E(G_i)| \geq 3$ for all i unless $|E(G)| < 3$, in which case $k = 1$ and $G_1 = G$; and

(iii) G is the graph that labels the single vertex of $T/e_1, e_2, \ldots, e_{k-1}$.

Tutte [9] proved that every 2-connected graph has a tree decomposition in which the vertex labels are restricted.

Theorem 1.2. *Let G be a loopless 2-connected graph. Then G has a tree decomposition in which every vertex label is a simple 3-connected graph, a copy of K_3, or a copy of B_3. Moreover, each vertex label is isomorphic to a minor of G.*

Next we use tree decompositions to state Allred, Ding, Oporowski's [1] identification of the unavoidable 2-connected induced subgraphs of large simple 2-connected graphs.

Theorem 1.3. *Let k be an integer exceeding two. Then there is an integer $f(k)$ such that every simple 2-connected graph with at least $f(k)$ vertices has, as an induced subgraph, one of*

(i) K_k;

(ii) a subdivision of $K_{2,k}$;

(iii) a graph that is obtained from a subdivision of $K_{2,k}$ by adding an edge joining the two degree-k vertices; or

(iv) a k-vertex graph whose tree decomposition is a path P each vertex of which is labelled by

(a) a copy of K_4 in which the basepoints form a matching; or

(b) a copy of K_3 or B_3

where neither end of P is labelled by B_3 and no two consecutive vertices of P are labelled by B_3.

The following theorem presents the main result of this paper. A graph H is a *parallel extension* of a graph G if $|E(H)| > |E(G)|$ and H can be obtained from G by, for each edge e in G, adding a possibly empty set of edges parallel to e. Similarly, a graph J is a *parallel-path extension* of a graph K if J can be obtained from K by, for each edge e in K, deleting e and adding a non-empty set of internally disjoint paths, each of which contains at least one edge and connects the ends of e.

Theorem 1.4. *Let r be a positive integer. There is an integer $g(r)$ such that every loopless 2-connected graph G with $|E(G)| \geq g(r)$ has, as a cc-minor, a parallel-path extension of a graph whose tree decomposition is a path on at least r vertices in which each vertex is labelled by a copy of K_3 or B_3, or by a copy of K_4 in which the basepoints form a matching.*

We aim to extend Theorem 1.3 to regular matroids using Seymour's decomposition theorem [7]. Not only is Theorem 1.4 the dual of Theorem 1.3 but it is also a crucial step in this extension.

2 Preliminaries

This section presents some definitions along with a proof of Lemma 1.1. Specifically, we define a class of graphs called r-templates, whose tree decompositions exhibit a particular structure. Additionally, we motivate the study of cc-minors, showing how such minors relate to induced subgraphs and minors.

2.1 r-templates

Figure 1 provides an example of a 2-connected graph G with a tree decomposition, as described by Tutte in Theorem 1.2. Although, for an arbitrary 2-connected graph G, such a graph-labelled tree may exhibit an arbitrary structure, we introduce a class of 2-connected graphs whose tree decompositions have specific, notable features.

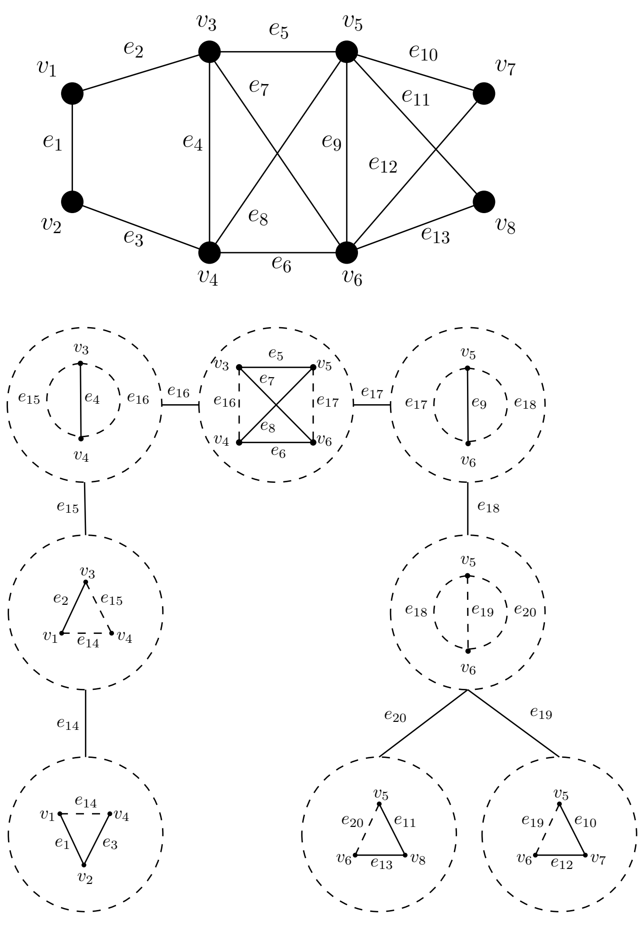

Figure 1: A 2-connected graph G and its tree-decomposition

A *fan graph* F_n is a simple graph that is obtained from an n-vertex path $v_1 v_2 \dots v_n$ by joining each v_i to a new vertex u. We call each uv_i edge a *spoke*. In particular, each edge joining u with v_1 or v_n is an *outer spoke*. For positive integers $t_1, t_2, \dots, t_n$, we obtain a *fan-type graph* $F_{t_1,t_2,\dots,t_n}$ by replacing each spoke uv_i of F_n by t_i parallel edges. In Figure 2, we show three different examples of fan-type graphs. Note that bond graphs are considered to be fan-type graphs, since they arise from single-vertex paths.

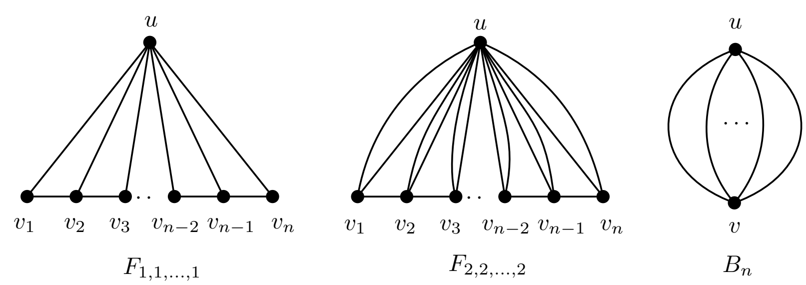

Figure 2: Three examples of fan-type graphs

An r-*template* is a 2-connected graph G that can be obtained from an r-vertex path P_r by using the following operations.

(i) Label each vertex of P_r by K_4, K_3, or B_3. Such vertex labels are called *parts*.

(ii) For each part, pick one basepoint for each of its adjacent parts. Moreover, if a part that is labelled by K_4 is adjacent to two other parts, then we always pick two non-adjacent edges in that K_4.

(iii) Apply 2-sums across the specified basepoints.

Figure 3 shows the possible parts for templates, and Figure 4 shows the construction process of a sample 6-template. The following are some special examples of templates.

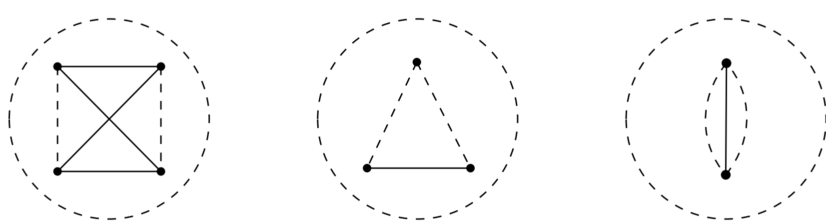

Figure 3: Possible parts of templates

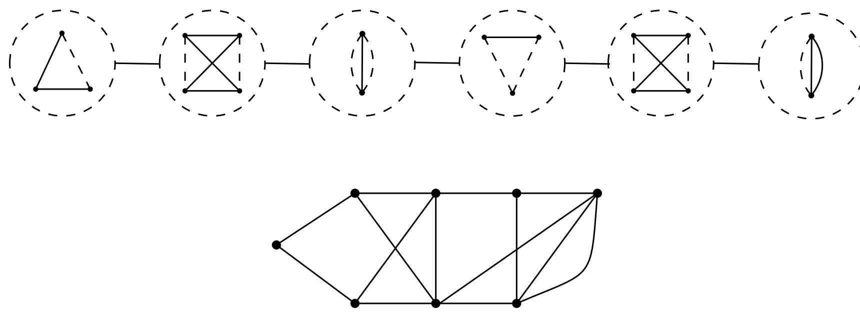

Figure 4: A sample 6-template

(i) A fan graph F_n is a $(2n-3)$-template in which the parts alternate between K_3 and B_3, beginning and ending with K_3.

(ii) A fan-type graph $F_{t_1,t_2,\ldots,t_n}$ other than B_2 is a $((\sum_{i=1}^{n} t_i)+n-3)$-template.

(iii) For $n \geq 3$, a bond graph B_n is an $(n-2)$-template for which every part is a B_3.

(iv) For $n \geq 3$, a cycle C_n is an $(n-2)$-template for which every part is a K_3.

The main result of this paper asserts that the unavoidable cc-minors of a sufficiently large 2-connected, loopless graph are parallel-path extensions of large templates. Before presenting the proofs, we provide motivation for studying cc-minors in the context of graph theory.

2.2 cc-minors of graphs

Graph relations, including induced subgraphs and minors, have garnered significant attention in various research areas. However, the study of cc-minors in graphs remains relatively unexplored. Although this relation is novel in certain respects, it has a strong connection to other graph relations. Here, we demonstrate how cc-minors relate to induced subgraphs through duality.

Proof of Lemma 1.1. First observe that the following statements are equivalent for a plane graph H.

(i) H is a 2-connected graph.

(ii) $M(H)$ is a 2-connected matroid with at least two elements.

(iii) $M^*(H)$ is a 2-connected matroid with at least two elements.

(iv) $M(H^*)$ is a 2-connected matroid with at least two elements.

(v) H^* is a 2-connected graph.

Assume that H is a 2-connected induced subgraph of G. Then H can be obtained from G by consecutively deleting some vertices $v_1, v_2, \dots, v_m$ in that order. For each i, the set of edges of $G - \{v_1, v_2, \dots, v_{i-1}\}$ meeting v_i is a disjoint union of a collection $B_1, B_2, \dots, B_h$ of bonds in $G - \{v_1, v_2, \dots, v_{i-1}\}$. As $B_1, B_2, \dots, B_h$ are cycles in the dual of $G - \{v_1, v_2, \dots, v_{i-1}\}$, we see that H^* is a cc-minor of G^*.

Conversely, assume that H^* is a cc-minor of G^*. Let Z be the set of edges of G^* such that $H^* = G^*/Z$. By the construction of H^*, every edge of Z is in a cycle of G^* that is contained in Z. Since H^* is 2-connected, for each cycle C in the plane graph G^* such that $C \subseteq Z$, either all of the edges in the interior of C are in Z or all of the edges in the exterior of C are in Z, but not both. In the first case, we color the faces in the interior of C red. In the second case, we color the faces in the exterior of C red. In particular, if $F \subseteq Z$ and F is a cycle bounding a face of G^*, then that face is colored red. Now, in G, consider the set R of vertices that correspond to the red faces of G^*. Then deleting the vertices of R from G gives the dual of the graph H^*. □

Rather than defining a graph minor through the local operations of deletion and contraction, it can also be characterized by its global structure.

Proposition 2.1. *A graph G has a graph H as a minor if and only if G has a set $\{G_v : v \in V(H)\}$ of disjoint connected subgraphs and a set $\{f_e : e \in E(H)\}$ of distinct edges that is disjoint from $\cup_{v \in V(H)} E(G_v)$ such that, for every edge $e \in E(H)$ having ends u and v, the ends of f_e are contained in G_u and G_v, respectively.*

We can characterize a cc-minor similarly.

Proposition 2.2. *A loopless graph G contains a graph H as a cc-minor if and only if G has*

(i) a collection $\{G_v : v \in V(H)\}$ of disjoint subgraphs such that

  (a) each G_v is either a 2-edge-connected subgraph of G or a single-vertex subgraph of G, and

  (b) $\bigcup_{v \in V(H)} V(G_v) = V(G)$;

  and

(ii) a set $\{f_e : e \in E(H)\}$ of distinct edges in G that is the complement of $\bigcup_{v \in V(H)} E(G_v)$ such that, for every edge $e \in E(H)$ with endpoints u and v, the endpoints of f_e lie in G_u and G_v, respectively.

The proof of Proposition 2.2 is deferred to Section 4.

3 Internally disjoint XY-paths

Let G be a graph and X, Y be two disjoint set of vertices of G. An *XY-path* P is an xy-path such that there are vertices x and y for which $V(P) \cap X = \{x\}$ and

$V(P) \cap Y = \{y\}$. When H and K are disjiont subgraphs of G, a $V(H)V(K)$-path will sometimes be called an *HK-path*. Two XY-paths P_1 and P_2 are *internally disjoint* if $(V(P_1) \cap V(P_2)) - X - Y = \emptyset$.

Lemma 3.1. *Let G be a graph, and let X and Y be two disjoint sets of vertices in G. For a cycle C in G such that $V(C) \cap X \neq \emptyset$ and $V(C) \cap Y = \emptyset$, define $X' = V(G[X \cup V(C)]/C)$. If there are k internally disjoint XY-paths in G, then there are at least k internally disjoint $X'Y$-paths in G/C.*

Proof. Let $P_1, P_2, \ldots, P_k$ be k internally disjoint XY-paths in G. Since contracting C can be achieved by repeatedly contracting single edges of C, it suffices to establish the following assertion.

3.1.1. *If $G' = G/e$ where e joins u and v, with $\{u, v\} \cap X \neq \emptyset$ and $\{u, v\} \cap Y = \emptyset$, then G' contains k internally disjoint $X'Y$-paths, where $X' = V(G[X \cup \{u, v\}]/e)$.*

If $\{u, v\} \subseteq X$ or if $u \in X$ and $v \notin V(P_i)$ for all $i \in [k]$, then $P_1, P_2, \ldots, P_k$ are internally disjoint $X'Y$-paths in G'. Now, assume $u \in X$ and $v \in V\left(\bigcup_{i\in[k]} P_i\right) - X$. Since $P_1, P_2, \ldots, P_k$ are internally disjoint and $v \notin Y$, there is exactly one path, say P_1, containing v. Let y denote the unique vertex in $Y \cap V(P_1)$, and let P_1' be the yv-subpath of P_1. Then $P_1', P_2, \ldots, P_k$ form k internally disjoint $X'Y$-paths in G'. This completes the proof of 3.1.1, and the lemma follows. □

Corollary 3.2. *For a positive integer k, let G be a k-edge-connected graph, and let H be a cc-minor of G with at least two vertices. Then H is k-edge-connected.*

Proof. It suffices to show that if e is an edge of a graph G with at least three vertices, then G/e is k-edge-connected whenever G is k-edge-connected. This is an immediate consequence of the fact that every bond of G/e is also a bond of G. □

Another way to prove Corollary 3.2 is using matroid duality and the fact that the bonds of G are the circuits of $M^*(G)$.

4 cc-minors of 2-connected graphs

In this section, we determine the unavoidable cc-minors in 2-connected graphs.

Lemma 4.1. *Let G be a graph. Then the following hold.*

(i) If G is 2-edge-connected and C is a cycle of G, then G/C is either a single-vertex graph or is 2-edge-connected.

(ii) If G is not 2-edge-connected and C is a cycle of G, then G/C is not 2-edge-connected.

(iii) If G is 2-edge-connected, then $G/E(G)$, which is isomorphic to K_1, is a cc-minor of G.

(iv) If $|V(G)| \geq 2$ and K_1 is a cc-minor of G, then G is 2-edge-connected.

Proof. If G is 2-edge-connected, then it has no cut edge. By Corollary 3.2, each cc-minor of G also has no cut edge, thus proving (i). Define $G_0 = G$. If G_i contains an edge e, then G_i has a cycle C_i that contains e. Let $G_{i+1} = G_i/C_i$. This process generates a sequence $G_0, G_1, \ldots, G_q$ such that G_q is connected and $E(G_q) = \emptyset$. Hence $G_q \cong K_1$, proving (iii). If G is not 2-edge-connected, then either G is a single-vertex graph and all the cycles of G are loops, or G has a cut edge e. In either case, if C is a cycle of G, then G/C is not 2-edge-connected, which proves (ii). In particular, if G has a cut edge, then every cc-minor of G will also have a cut edge, and thus it can never be isomorphic to K_1, proving (iv). □

Using Lemma 4.1, we define the operation of *contracting a 2-edge-connected subgraph F of G* as performing a sequence of cycle contractions equivalent to contracting all of the edges in F.

Corollary 4.2. *If F is a 2-edge-connected subgraph of G, then $G/E(F)$ is a cc-minor of G.*

Next we prove a characterization of cc-minors.

Proof of Proposition 2.2. If (i) and (ii) hold, then, by Lemma 4.1(iii), G has H as a cc-minor. Note that $H = G/\big(\cup_{v \in V(H)} E(G_v)\big)$. To prove the converse, suppose that $H = G/\{e_1, e_2, \ldots, e_k\}$. Let J be the subgraph of G induced by the set $\{e_1, e_2, \ldots, e_k\}$ of edges. It suffices to show that each component of J is 2-edge-connected. Since none of $e_1, e_2, \ldots, e_k$ is a loop, each component of J has at least two vertices. Moreover, each component has K_1 as a cc-minor and hence, by Lemma 4.1(iv), is 2-edge-connected. □

The next lemma identifies the unavoidable cc-minors in 2-connected graphs when preserving a specified edge. Let G_1 and G_2 be graphs such that $V(G_1) \cap V(G_2) = \{u, v\}$ and $E(G_1) \cap E(G_2) = \{e\}$, where e is neither a loop nor a cut edge in G_1 or G_2. The graph $G_1 \cup G_2$ is the *parallel connection* of G_1 and G_2 with basepoint e. More generally, let $G_1, G_2, \ldots, G_n$ be a collection of graphs such that, for all distinct i and j in $[n]$, we have $V(G_i) \cap V(G_j) = \{u, v\}$ and $E(G_i) \cap E(G_j) = \{e\}$, where e is neither a loop nor a cut edge in any G_k. The union $G_1 \cup G_2 \cup \cdots \cup G_n$ is called the *parallel connection* of $G_1, G_2, \ldots, G_n$ with basepoint e. If v is a vertex of degree two in a graph G and v does not meet a loop of G, then, by *suppressing* v, we mean deleting v and adding an edge between its two neighbors.

Lemma 4.3. *Let G be a 2-edge-connected graph, and let e be a nonloop edge of G. Then G has a cc-minor H that is the parallel connection, with basepoint e, of a collection of cycles containing e.*

Proof. Let $G_0 = G$ and let $e = uv$. If G_i has a cycle C_i that does not contain both u and v, define $G_{i+1} = G_i/C_i$. Repeating this process generates a sequence

$G_0, G_1, \ldots, G_k$ of graphs such that u and v remain distinct vertices in G_k, and every cycle in G_k contains both u and v. We will show that G_k is a parallel connection, with basepoint e, of a collection of cycles containing e.

Let w be a vertex of G_k that is not in $\{u, v\}$. We first prove that $d(w) = 2$. By Corollary 3.2, G_k is 2-edge-connected, so $d(w) \geq 2$. Now, w does not meet a loop of G_k. Let g be an edge meeting w. Since g is not a cut edge of G_k, there is a cycle C in G_k containing g, and C must contain both u and v. Suppose $d(w) \geq 3$. Then there is an edge f that meets w but is not in C. Let x be the other endpoint of f. By the choice of G_k, we see that $x \notin V(C)$. As G_k is 2-edge-connected, $G_k - f$ is connected. Choose P as a shortest path in $G_k - f$ from x to a vertex in C. Let P^+ be the path in G_k that consists of P and the edge f. Note that $E(P^+) \cap E(C) = \emptyset$ and $V(P^+) \cap V(C) = \{w, y\}$ for some $y \in V(C)$. Since P^+ cannot contain both u and v, there is a cycle in G_k that does not contain both u and v, a contradiction. Hence $d(w) = 2$.

Let A be the set of degree-2 vertices in G_k. If we suppress $A - \{u, v\}$, then for some $n \geq 2$, the resulting graph will be a bond graph B_n with vertex set $\{u, v\}$. Hence, G_k is the parallel connection, with basepoint e, of a collection of $n-1$ cycles containing e. □

Combining Corollary 3.2 and Lemma 4.3, we obtain the following result.

Corollary 4.4. *Let G be a 3-edge-connected graph, and let e be an edge of G. Then G has a cc-minor H that is a bond graph B_n containing e, for some $n \geq 3$.*

5 Classes closed under cc-minors

Let $\mathcal{F}_1$ be the class of loopless connected graphs. For each positive integer $k > 1$, let $\mathcal{F}_k$ be the class consisting of all loopless k-edge-connected graphs along with the single-vertex graph K_1. The next proposition follows immediately by combining Corollary 3.2 and Lemma 4.1(iii).

Proposition 5.1. *For every positive integer k, the class $\mathcal{F}_k$ is closed under cycle contraction.*

Clearly, $\mathcal{F}_{k+1} \subseteq \mathcal{F}_k$for each positive integer k. Now we provide a forbidden cc-minor characterization of $\mathcal{F}_k$ for each k in $\{1, 2, 3\}$.

Theorem 5.2. *The following statements hold for a loopless graph G.*

(i) G is in $\mathcal{F}_1$ if and only if G does not have a forest with at least two components as a cc-minor.

(ii) G is in $\mathcal{F}_2$ if and only if G is in $\mathcal{F}_1$ and G does not have a tree with at least one edge as a cc-minor.

(iii) G is in $\mathcal{F}_3$ if and only if G is in $\mathcal{F}_2$ and G does not have a cycle as a cc-minor.

Proof. First we observe the following.

(a) A forest with at least two components is not in $\mathcal{F}_1$.

(b) A tree with at least one edge is not in $\mathcal{F}_2$.

(c) A cycle is not in $\mathcal{F}_3$.

Since each $\mathcal{F}_k$ is closed under cycle contractions, to prove the theorem, it remains to show that the graphs in (a)-(c) are the only obstructions to membership of $\mathcal{F}_k$ for k in $\{1, 2, 3\}$.

First, suppose G is not connected. Let F be a cc-minor of G obtained by repeatedly contracting cycles until no cycles remain. Evidently, F is a forest that has the same number of components as G. Thus, F is a forest with at least two components, which confirms (i).

Now, suppose $G \in \mathcal{F}_1 - \mathcal{F}_2$. Clearly, G has a cut edge e. Let H be a cc-minor of G obtained by repeatedly contracting cycles until no cycles remain. Evidently, H is a tree that contains e, which confirms (ii).

Finally, suppose that $G \in \mathcal{F}_2 - \mathcal{F}_3$. Since G is not 3-edge-connected, G has a bond $\{e, f\}$. Let J be a cc-minor of G obtained by repeatedly contracting cycles that contain neither e nor f until no such cycles remain. By Corollary 3.2, J is 2-edge-connected. Let C be a cycle of J that contains e or f. Since $\{e, f\}$ is a bond of J, it is a cocircuit of $M(J)$. Thus, C contains both e and f. Therefore, by construction of J, every cycle in J contains both e and f. Suppose C_1 and C_2 are two distinct cycles of J. Then $C_1 \Delta C_2$, which equals $(C_1 \cup C_2) - (C_1 \cap C_2)$, is a non-empty disjoint union of circuits of $M(J)$. However, $\{e, f\} \not\subseteq C_1 \Delta C_2$, a contradiction. Therefore, J is a cycle, which confirms (iii). □

The next theorem characterizes $\mathcal{F}_k$ for all $k > 3$.

Theorem 5.3. *Let k be an integer exceeding three. A loopless graph G is in $\mathcal{F}_k$ if and only if G is in $\mathcal{F}_{k-1}$ and G does not have a cc-minor isomorphic to B_{k-1}.*

Proof. Clearly $\mathcal{F}_{k-1} \subseteq \mathcal{F}_k$ and $B_{k-1} \notin \mathcal{F}_k$. To prove the converse, suppose that G belongs to $\mathcal{F}_{k-1} - \mathcal{F}_k$. Let $\{x_1, x_2, \dots, x_{k-1}\}$ be a bond of G, and let H be a cc-minor of G obtained by repeatedly contracting cycles that do not contain any edge in $\{x_1, x_2, \dots, x_{k-1}\}$ until no such cycles remain. Evidently, $H \setminus \{x_1, x_2, \dots, x_{k-1}\}$ consists of two components, T_1 and T_2, each of which is a tree. Since H is $(k-1)$-edge-connected, we have $d_H(v) \geq k - 1$ for all $v \in V(H)$. Let l be a leaf of T_i for some $i \in \{1, 2\}$. In H, the leaf l is incident with at least $k - 2$ edges from $\{x_1, x_2, \dots, x_{k-1}\}$. However, for $k > 3$, we have $2(k-2) > k-1$. Therefore, T_i has at most one leaf for each $i \in \{1, 2\}$. Thus, both T_1 and T_2 must be single-vertex graphs, and we conclude that $H \cong B_{k-1}$, which confirms the theorem. □

6 cc-minors of 3-connected graphs

In this section, we determine the unavoidable cc-minors of 3-connected graphs. By the *simplification* of a graph G, we mean a simple graph that is obtained from G by deleting all the loops and deleting all but one edge from each maximal set of parallel edges.

Theorem 6.1. *Let G be a simple 3-connected graph, and let e and f be two distinct edges in G. Then G has a cc-minor H containing e and f such that one of the following holds.*

(i) For some $n \geq 3$, the graph H is isomorphic to a bond graph B_n containing e and f; or

(ii) H is isomorphic to a fan-type graph of which e and f are distinct outer spokes that are not parallel; or

(iii) the simplification of H has e and f as non-adjacent edges and is isomorphic to K_4. Moreover, if an edge g of H is not parallel to e or f, then g is not parallel to any edge in $E(H)$.

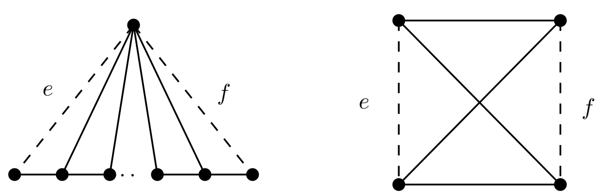

Figure 5: Possible simplifications of H that fall under cases (ii) and (iii).

Proof. Let e have ends x_1 and x_2, and let f have ends y_1 and y_2. First, we prove the following.

6.1.1. *If e and f are adjacent, then G has a cc-minor H such that H is isomorphic to B_n, for some $n \geq 3$, and H has e and f as distinct edges.*

We may assume that $x_1 = y_1$ and $x_2 \neq y_2$. Since G is 3-connected, $G - x_1$ is 2-connected and hence 2-edge-connected. By Lemma 4.1, contracting $E(G - x_1)$ forms a cc-minor that is isomorphic to a bond graph that contains e and f. Moreover, by Lemma 3.2, H has at least three edges. Hence, 6.1.1 holds.

Now we assume e and f are not adjacent. For two distinct vertices u and v, a Θ-*graph on* (u, v) consists of three internally disjoint uv-paths. For each $(x, y) \in \{x_1, x_2\} \times \{y_1, y_2\}$, a Θ-graph on (x, y) that contains both e and f is classified as *type-A* if one of its xy-paths contains both e and f. It is *type-B* if e and f belong to two different xy-paths of the Θ-graph. Examples of these types

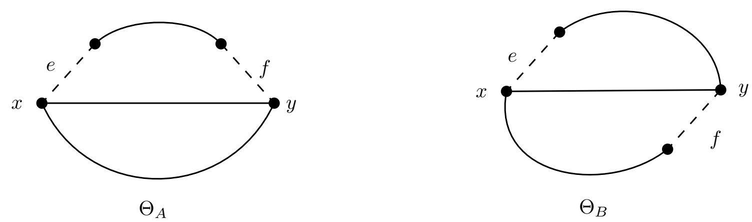

Figure 6: A type-A Θ-graph Θ_A and a type-B Θ-graph Θ_B

of graphs are shown in Figure 6, where e and f represent single edges, while the other lines in the diagram correspond to paths.

Next we prove the following.

6.1.2. *For each pair $(x, y) \in \{x_1, x_2\} \times \{y_1, y_2\}$, there is a Θ-graph on (x, y) that contains e and f.*

Without loss of generality, assume $x = x_1$ and $y = y_1$. By Menger's Theorem, there are three internally disjoint x_1y_1-paths that form a Θ-graph on (x, y). Choose Θ_0 to be a Θ-graph on (x, y) that contains the maximal number of members of $\{e, f\}$. We may assume that $e \notin E(\Theta_0)$. Since $G - x_1$ is connected, there is an x_2y_1-path in $G - x_1$ whose vertices and edges, in order, are $v_1e_1v_2 \dots e_{k-1}v_k$, where $v_1 = x_2$ and $v_k = y_1$. In G, we adjoin the edge e to the beginning of this path to form a path P. Let i be the smallest index such that $p_i \in V(\Theta_0)$.

Suppose $p_i \neq y_1$. Then p_i lies on an x_1y_1-path Q in Θ_0. Note that the x_1p_i-subpath Q' of Q does not use the edge f. Replacing Q' by the x_1p_i-subpath of P, we obtain a Θ-graph that violates the choice of Θ_0. Thus $p_i = y_1$. Choose an x_1y_1-path R in Θ_0 such that $f \notin E(R)$, and let Θ_1 be the Θ-graph obtained by replacing R with P. Then Θ_1 violates the choice of Θ_0. The contradiction completes the proof of 6.1.2.

Next we show that if G contains a type-A Θ-graph, then the theorem holds.

6.1.3. *If G has a cc-minor G' that contains a type-A Θ-graph Θ_A as a subgraph, then G has a cc-minor H containing e and f such that H is either isomorphic to a fan-type graph, with e and f as distinct non-parallel outer spokes, or H is isomorphic to B_n for some $n \geq 3$.*

By Lemma 3.2, every cc-minor of G is 3-edge-connected. Thus, to prove 6.1.3, it suffices to show the following.

6.1.4. *If a 3-edge-connected graph G' contains a type-A Θ-graph Θ_A as a subgraph, then G' has a cc-minor H containing e and f such that H is either isomorphic to a fan-type graph, with e and f as distinct non-parallel outer spokes, or H is isomorphic to B_n for some $n \geq 3$.*

Observe that Θ_A has a cycle C that does not contain e or f. Contracting C in G' has the effect of identifying all the vertices in $V(C)$ as a single vertex c and then deleting all the edges in $E(C)$. Since e and f are contained in a cycle of G/C, there is a maximal 2-connected subgraph L of G'/C containing both e and f.

Since G'/C has no cut edges, every edge of G'/C is in a 2-connected subgraph of G'/C. Now assume we contract the edges of all of the maximal 2-connected subgraphs of G'/C except L. The resulting graph is isomorphic to L, so we continue referring to it as L. Let $L_0 = L$. Assume that, for some $i \geq 0$, the graphs $L_0, L_1, \ldots, L_i$ have been constructed and let $L_i^- = L_i - c$. If L_i^- contains a cycle C_i, let $L_{i+1} = L_i/C_i$. This process produces a sequence $L_0, L_1, \ldots, L_s$ of graphs such that L_s^- is a tree T. Let P be the x_2y_2-path in T. Note that T may be a single vertex if x_2 and y_2 have been identified.

For $j \geq s$, if $L_j^- \neq P$, then there is a leaf $l_j \in V(L_j^-) - V(P)$. By Corollary 3.2, $d(l_j) \geq 3$, so there is a cycle O_j consisting of two edges with ends c and l_j. Let $L_{j+1} = L_j/O_j$. This results in a sequence $L_s, L_{s+1}, \ldots, L_{s+t}$ such that $L_{s+t} - c = P$. In L_{s+t}, since each vertex of P has degree at least three, there is at least one cp-edge for each $p \in V(P)$. Thus, L_{s+t} is a fan-type graph with e and f as two outer spokes. Moreover, e and f are not parallel unless L_{s+t} is a bond graph with at least three edges. Hence 6.1.4 holds, so 6.1.3 holds.

In view of 6.1.3, we may now assume that

6.1.5. *G has no cc-minor that contains a type-A Θ-graph.*

By 6.1.2, G has a type-B Θ-graph Θ_B as a subgraph. Without loss of generality, suppose Θ_B is on (x_1, y_1). Throughout the following argument, for any type-B Θ-graph on (x_1, y_1), we denote the x_2y_1-path as P_1, the x_1y_1-path as P_2, and the x_1y_2-path as P_3, as shown in Figure 7.

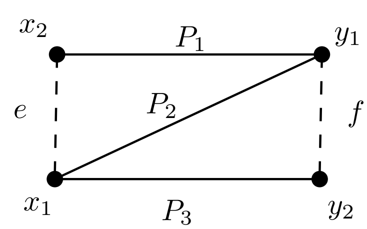

Figure 7: Θ_B

A graph is a *four-path connector* if it consists of the edges e and f, along with four internally disjoint paths, P_1, P_2, P_3, and P_4, that connect the vertex pairs $\{x_2, y_1\}$, $\{x_1, y_1\}$, $\{x_1, y_2\}$, and $\{x_2, y_2\}$, respectively. Moreover, each path P_i contains at least one edge for every $i \in \{1, 2, 3, 4\}$. Throughout the remainder of this proof, in any four-path connector, the labels of these four paths will be consistent with Figure 8. Next, we show the following.

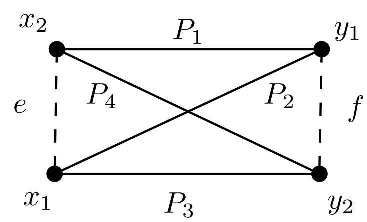

Figure 8: A four-path connector

6.1.6. *The graph G has a cc-minor G_s that contains a four-path connector K as a subgraph.*

Let $(G_0, \Theta_0) = (G, \Theta_B)$. If G_i has a path Q_i connecting x_2 to a vertex v_i on P_1 such that all vertices of Q_i, except for its two ends, are not in Θ_i, then let C_i be the cycle formed by Q_i and the x_2v_i-subpath of P_1. Define $(G_{i+1}, \Theta_{i+1}) = (G_i/C_i, \Theta_i/(E(\Theta_i)\cap E(C_i)))$. This process produces a sequence $(G_0, \Theta_0), (G_1, \Theta_1), \ldots, (G_t, \Theta_t)$ such that, in G_t, no path satisfies the condition that defines Q_i. Note that, in this process, Q_i is never an x_2y_1-path; otherwise, we would obtain a type-A Θ-graph on (x_2, y_1) having as its paths, Q_i, P_1, and the path with edge set $\{e, f\} \cup E(P_3)$, which contradicts 6.1.5. Thus, in G_t, the path P_1 retains at least one edge.

Similarly, for every $i \geq t$, if G_i has a path Q_i connecting y_2 to a vertex v_i on P_3 such that all vertices of Q_i, except for its two ends, are not in Θ_i, we let C_i be the cycle formed by Q_i and the y_2v_i-subpath of P_3. As above, we define $(G_{i+1}, \Theta_{i+1}) = (G_i/C_i, \Theta_i/(E(\Theta_i) \cap E(C_i)))$ and we generate a sequence $(G_t, \Theta_t), (G_1, \Theta_{t+1}), \ldots, (G_s, \Theta_s)$ such that, in G_s, no path satisfies the above conditions. For the same reason as above, Q_i is never a y_2x_1-path, so P_3 has at least one edge in G_s.

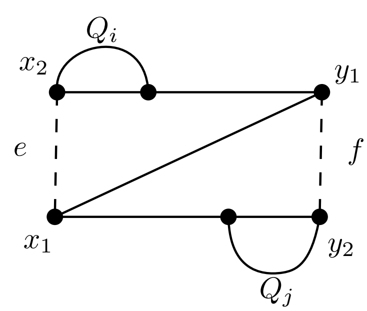

Figure 9: Paths similar to Q_i or Q_j will not appear in G_s.

Let $\mathcal{P}_x$ be the collection of paths in G_s that start at x_2, end at a vertex in $V(\Theta_s)$, and are *internally disjoint from* Θ_s, meaning they are vertex-disjoint from

Θ_s except at their endpoints. Similarly, let $\mathcal{P}_y$ be the collection of paths that start at y_2, end at a vertex in $V(\Theta_s)$, and are internally disjoint from Θ_s.

If there is a path $P \in \mathcal{P}_x$ that ends at a vertex, say w, in $V(P_2) - \{x_1\}$, then there is a type-A Θ-graph on (x_2, y_1) (see Figure 10), which contradicts 6.1.5. Similarly, we may assume that $\mathcal{P}_y$ does not contain any path that ends at a vertex in $V(P_2) - \{y_1\}$.

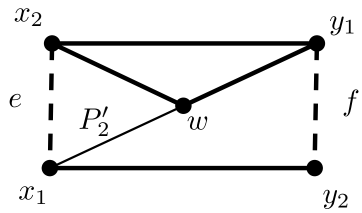

Figure 10: Deleting the edges of P_2' results in a type-A Θ-graph.

Note that G_s is obtained from G by repeatedly contracting cycles containing exactly one of $\{x_2, y_2\}$. Because G is 3-connected, by Lemma 3.1, there are at least three internally disjoint x_2y_2-paths in G_s. However, if all paths in $\mathcal{P}_x$ end at x_1, then every x_2y_2-path in G_s contains either x_1 or the neighbor of x_2 on P_1, contradicting the existence of three internally disjoint x_2y_2-paths. Therefore, by the choice of G_s, we may assume $\mathcal{P}_x$ contains a path P_x that ends at a vertex in $V(P_3) - \{x_1\}$. By symmetry, we may also assume $\mathcal{P}_y$ contains a path P_y that ends at a vertex in $V(P_1) - \{x_1\}$. If P_x and P_y are internally disjoint, then there is a type-A Θ-graph on (x_2, y_2) (see Figure 11), which contradicts 6.1.5.

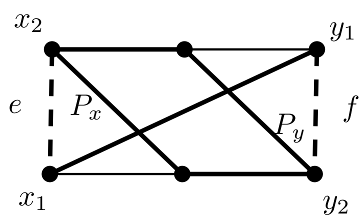

Figure 11: The thickened paths form a type-A Θ-graph.

We may now assume that P_x and P_y are not internally disjoint. Therefore, there is an x_2y_2-path P_4 in G_s that is internally disjoint from Θ_s. Let K be the subgraph of G_s consisting of Θ_s and P_4. Hence 6.1.6 holds.

A four-path connector F in a graph J is *spanning* if $V(F) = V(J)$.

6.1.7. *The graph G_s has a cc-minor G_s' that contains a spanning four-path connector K'.*

In order to prove 6.1.7, it suffices to prove the following.

6.1.8. *If D is a cc-minor of G that contains a non-spanning four-path connector F, then D has a cc-minor D' containing a four-path connector F' such that $|V(D) - V(F)| > |V(D') - V(F')|$.*

Assume that this fails. By Menger's Theorem and Lemma 3.1, we know that, for each $v \in V(D) - V(F)$, there are at least three vF-paths that are disjoint, except for all having v as their first vertex. Thus, we may now assume that there are three internally disjoint vF-paths with distinct endpoints a, b, and c in $V(F)$. Next, we show the following.

6.1.9. *None of the paths P_1, P_2, P_3, P_4 contains more than one of a, b, and c.*

Suppose that $\{a, b\} \subseteq V(P_i)$ for some $i \in \{1, 2, 3, 4\}$. Without loss of generality, suppose that $\{a, b\} \subseteq V(P_1)$. Observe that if $\{a, b\} = \{x_2, y_1\}$, then D contains a type-A Θ-graph, which contradicts 6.1.5. Hence, we may assume that at least one member of $\{a, b\}$ is an internal vertex of P_1. Let C_v be the cycle formed by the va-path, the ab-subpath of P_1, and the bv-path (see the left graph in Figure 12). Then the graphs $D' = D/C_v$ and $F' = F/(E(F) \cap E(C_v))$ satisfy 6.1.8, a contradiction. Thus, 6.1.9 holds.

By 6.1.9, we observe that at least one of a, b, and c is not in $\{x_1, x_2, y_1, y_2\}$. Without loss of generality, we assume that $a \in V(P_1) - \{x_2, y_1\}$. Moreover, by 6.1.9, we know that at least one of b and c belongs to $V(F) - (V(P_1) \cup V(P_3))$. By symmetry, we assume $b \in V(P_2) - \{x_1, y_1\}$. Let C_v be the cycle formed by the va-path, the ay_1-subpath of P_1, the y_1b-subpath of P_2, and the bv-path (see the right graph in Figure 12). Then the graphs $D' = D/C_v$ and $F' = F/(E(F) \cap E(C_v))$ satisfy 6.1.8, a contradiction. Thus 6.1.8 holds, and 6.1.7 follows immediately.

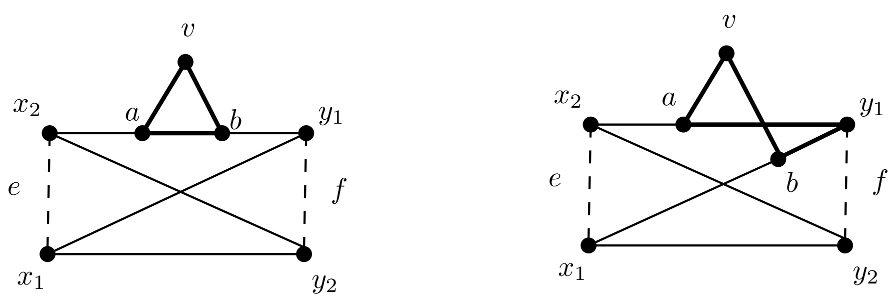

Figure 12: Contracting the thickened cycles absorbs v into F

Next, we prove the following.

6.1.10. *If W is a cc-minor of G that contains a spanning four-path connector R and there is a vertex $u \in V(W) - \{x_1, x_2, y_1, y_2\}$, then W has a cc-minor W' containing a spanning four-path connector R' such that $|V(W) - \{x_1, x_2, y_1, y_2\}| > |V(W') - \{x_1, x_2, y_1, y_2\}|$.*

Without loss of generality, suppose $u \in V(P_1) - \{x_2, y_1\}$. By Corollary 3.2, the graph W is 3-edge-connected, and $d_W(u) \geq 3$. Therefore, u is incident to an edge $uw \notin E(R)$. If u and w lie on two non-adjacent paths of R (say, P_1 and P_3 under our assumption), then W contains a type-A Θ-graph, as shown in Figure 13, a contradiction to 6.1.5.

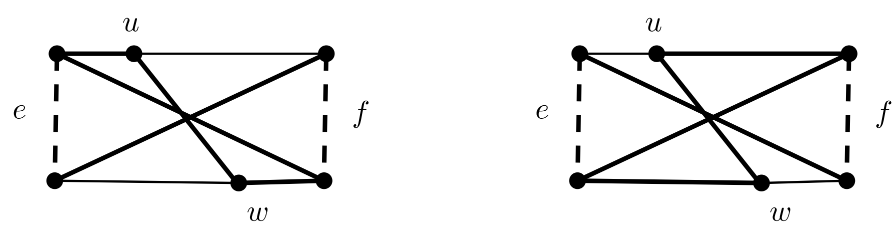

Figure 13: The thickened paths, together with the edges e and f, form type-A Θ-graphs in each case.

We may now assume that u and w either belong to the same path or to two adjacent paths among $\{P_1, P_2, P_3, P_4\}$.

(a) Suppose that u and w belong to the same path, as shown in the left graph of Figure 14. Let C_u be the cycle formed by the edge uw and the wu-subpath of P_1.

(b) Suppose that u and w belong to two adjacent paths, as shown in the right graph of Figure 14. Let C_u be the cycle formed by the edge uw, the wz-subpath of P_1, and the zu-subpath of P_2.

In each case, let $W' = W/C_u$ and $R' = R/(E(C_u) \cap E(R))$. Then R' is a spanning four-path connector of W', so 6.1.10 holds.

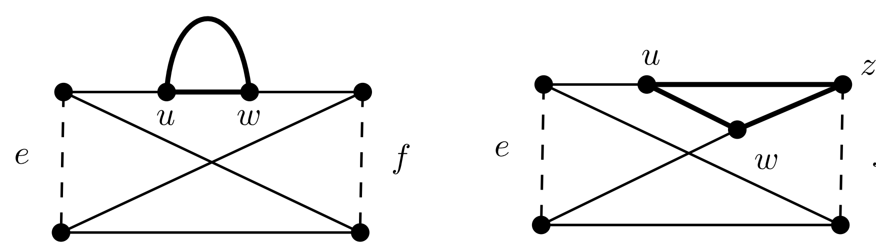

Figure 14: Two cases of a uw-edge

Applying 6.1.10 inductively on G'_s and K', we conclude that G'_s has a cc-minor H containing a four-path connector. Moreover, $V(H) = \{x_1, x_2, y_1, y_2\}$. It is not difficult to see that the edges e and f are non-adjacent in H and that the simplification of H is isomorphic to K_4. Suppose that there is a pair of parallel edges g and h in $E(H)$ that are not parallel to e or f. Because the simplification of H is isomorphic to K_4, there is a 4-cycle U that contains e, f and g. However, the graph with edge set $U \cup h$ is a type-A Θ-graph, a contradiction to 6.1.5. Hence, Theorem 6.1 holds. □

7 cc-minors of large 3-connected graphs

In this section, we determine the unavoidable cc-minors of sufficiently large 3-connected graphs.

Theorem 7.1. *For every integer $t \geq 3$, there is a function $f_{7.1}(t)$ such that if a simple 3-connected graph G has more than $f_{7.1}(t)$ edges, then G has a fan-type graph $F_{t_1,t_2,\ldots,t_n}$ as a cc-minor such that $\sum_{i=1}^{n} t_i \geq t$.*

Before beginning the proof of Theorem 7.1, we present three lemmas. The first is a Ramsey-type result for 3-connected graphs; the second is a result for weighted trees, where we use the latter as auxiliary graphs in our analysis.

Let k be an integer exceeding two. Figure 15 shows three families of graphs that we now describe. The *k-rung ladder* L_k has vertices $v_1, v_2, \ldots, v_k, u_1, u_2, \ldots, u_k$, where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_k$ form paths in the listed order, and v_i is adjacent to u_i for each $i \in \{1, 2, \ldots, k\}$. The graph V_k is obtained from L_k by adding an edge between v_1 and v_k and contracting the edges joining u_1 to v_1 and u_k to v_k. The k-spoke wheel is denoted by W_k. Oporowski, Oxley, and Thomas [4] characterized the unavoidable structures of large 3-connected graphs as follows.

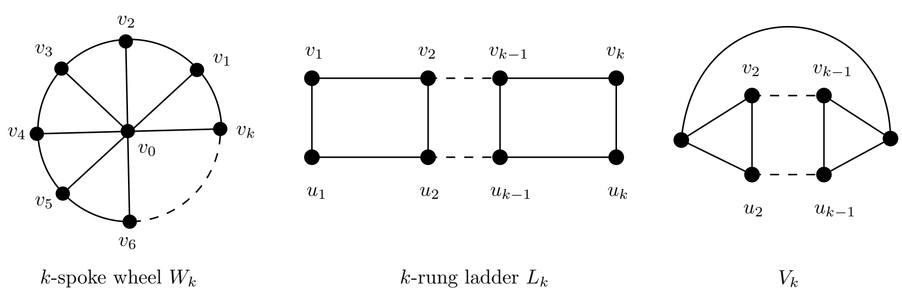

Figure 15: Some important graphs for Lemma 7.2

Lemma 7.2. *For every integer $k \geq 3$, there is a function $f_{7.2}(k)$ such that every 3-connected graph with at least $f_{7.2}(k)$ vertices contains a subgraph isomorphic to a subdivision of one of W_k, V_k, and $K_{3,k}$.*

A *weighted tree* is a tree T together with a weight function w such that each vertex v is assigned a non-negative integer-valued weight $w(v)$, and $w(A) = \sum_{v \in A} w(v)$ for each $A \subseteq V(T)$.

In the next lemma, we use the notion of the *center* of a graph. This is the set of vertices with the smallest maximum distance to other vertices. It is well known that, in a tree, the center consists of either a single vertex or two adjacent vertices of the tree.

Lemma 7.3. *For every integer $t > 1$, there is a function $f_{7.3}(t)$ such that every weighted tree T with $w(V(T)) > f_{7.3}(t)$ contains one of the following:*

(i) a vertex v such that $d(v) > t$;

(ii) a path P such that $|V(P)| > t$; or

(iii) a path P such that $w(V(P)) > t$.

Proof. We prove that $f_{7.3}(t) = \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor + 1} t^i$ satisfies the condition. Let T be a tree. We may assume that both the maximum degree and the number of vertices in a longest path do not exceed t otherwise (i) or (ii) holds. By grouping the vertices of T based on their distance from a center vertex of T, we see that $|V(T)| \leq \sum_{j=0}^{\lfloor \frac{t}{2} \rfloor} t^j$. However, since $w(T) \geq 1 + \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor + 1} t^i$, there must be a vertex v such that $w(v) > t$. Therefore, each path P containing v satisfies $w(V(P)) > t$, so (iii) holds. □

We are now ready to prove the main result of this section.

Proof of Theorem 7.1. Let H be a subgraph of G. An edge e of $E(G) - E(H)$ is an *H-bridge* if e is incident with at least one vertex of H. We call each connected component of $G - V(H)$ an *H-island*. Note that an H-bridge is either

(i) an edge having both vertices in $V(H)$, or

(ii) an edge having one vertex in $V(H)$ and one vertex in an H-island.

We first prove the following.

7.3.1. *If G has a cycle C and a C-island I such that there are at least $f_{7.3}(t)$ C-bridges between C and I, then G has a fan-type graph $F_{t_1,t_2,\ldots,t_n}$ as a cc-minor where $\sum_{i=1}^{n} t_i \geq t$.*

First, observe that $G - V(I)$ does not have a cut edge. Thus we can contract all of the edges of $G - V(I)$ by successively contracting a sequence of cycles. The resulting graph G' is obtained from $G[V(C) \cup V(I)]$ by contracting C. We denote by c the vertex that results by identifying all of the vertices of C. Note that $d_{G'}(c) \geq f_{7.3}(t)$ and the neighbors of c in G' are contained in $V(I)$. Let $G_0 = G'$. If $G_i - c$ has a cycle C_i, define $G_{i+1} = G_i / C_i$. This process results in a sequence $G_0, G_1, \ldots, G_s$ such that $G_s - c$ is a tree T.

Now, define a weight function w on $V(T)$ by, for each vertex v of T, letting $w(v)$ be the number of edges joining c and v. Clearly, $w(V(T)) = d_{G'}(c) \geq f_{7.3}(t)$. By Lemma 7.3, T has a subgraph T' that is one of the following:

(i) a vertex v such that $d(v) > t$;

(ii) a path P such that $|V(P)| > t$; or

(iii) a path P such that $w(V(P)) > t$.

Let $T_0 = T$. Assume that we have defined a sequence $(G_s, T_0), (G_{s+1}, T_1), \dots, (G_{s+i}, T_i)$ where each G_{s+j} is 3-edge-connected having the tree T_j as a subgraph. If $T_i \neq T'$, then T_i has a leaf $l \notin V(T')$. Since G_{s+i} is 3-edge-connected, there are two edges joining c and l that form a cycle O_i in G_{s+i}. Define $G_{s+i+1} = G_{s+i}/O_i$ and $T_{i+1} = T - l$. Repeating this process, we eventually obtain a pair (G_{s+h}, T_h) with $T_h = T'$. By the choice of T', we see that if T' is a vertex of degree more than t in G_{s+h}, then G_{s+h} is a bond graph with more than t edges; if T' is a path on more than t vertices, then G_{s+h} is a fan-type graph with more than t sets of parallel spokes; and if T' is a path P such that $w(V(P)) > t$, then G_{s+h} is a fan-type graph with more than t spokes. Therefore, 7.3.1 holds.

To complete the proof of the theorem, we shall show that the required result holds for the function $f_{7.1}(t) = \binom{f_{7.2} \circ f_{7.3}(t)}{2}$. Since G is simple and has more than $\binom{f_{7.2} \circ f_{7.3}(t)}{2}$ edges, G has more than $f_{7.2} \circ f_{7.3}(t)$ vertices. By Lemma 7.2, G has a subgraph isomorphic to a subdivision of one of W_k, V_k, or $K_{3,k}$ where $k = f_{7.3}(t)$. In each of these three cases, let C be the bold cycle and I be the C-island containing the white vertices, as shown in Figure 16. It is straightforward to verify that the choices of C and I satisfy the conditions in 7.3.1. Hence, by 7.3.1, G has a fan-type graph $F_{t_1,t_2,\dots,t_n}$ as a cc-minor such that $\sum_{i=1}^{n} t_i \geq t$. □

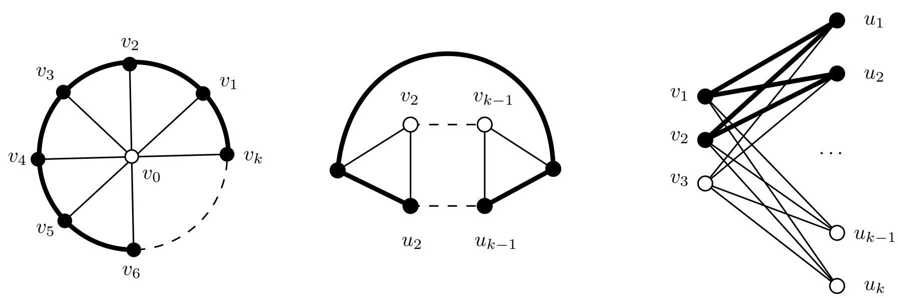

Figure 16: C and a C-island I in each of W_k, V_k, and $K_{3,k}$

8 Proof of Theorem 1.4

Before we present the proof of Theorem 1.4, we prove two lemmas.

Lemma 8.1. *If H is a cc-minor of a loopless graph G and G' is a parallel-path extension of G, then G' has a cc-minor H' that is a parallel-path extension of H.*

Proof. By Proposition 2.2, there is a collection $\{G_1, G_2, \dots, G_k\}$ of disjoint 2-edge-connected subgraphs of G such that $H = G/(\cup_{i=1}^{k} E(G_k))$. Now G' is obtained from G by replacing some edges with internally disjoint paths joining their ends. For each $i \in \{1, 2, \dots, k\}$, let G'_i be the graph that is obtained from G_i by replacing all such edges in G_i with the same set of internally disjoint paths joining their ends as in G'. Clearly, G'_i is 2-edge-connected for all $i \in \{1, 2, \dots, k\}$. Let $H' = G'/(\cup_{i=1}^{k} E(G'_i))$. It is straightforward to check that H' is a parallel-path extension of H. □

Lemma 8.2. *Let G be the 2-sum of two loopless graphs G_1 and G_2 on the basepoint b. Suppose that, for each $i \in \{1, 2\}$, H_i is a cc-minor of G_i that has b as a non-loop edge. If H is the 2-sum of H_1 and H_2 on the basepoint b, then H is a cc-minor of G.*

Proof. By Proposition 2.2, for each i in $\{1, 2\}$, there is a collection $\mathcal{J}_i$ of disjoint 2-edge-connected subgraphs of G_i such that $H_i = G_i/(\cup_{J \in \mathcal{J}_i} E(J))$. Because $E(H_1) \cap E(H_2) = \{b\}$ and $b \notin E(J)$ for each J in $\mathcal{J}_1 \cup \mathcal{J}_2$, we know $\mathcal{J}_1 \cup \mathcal{J}_2$ is a collection of edge-disjoint 2-edge-connected subgraphs of G. By Corollary 4.2, we deduce that H, which equals $G/(\cup_{J \in \mathcal{J}_1 \cup \mathcal{J}_2} E(J))$, is a cc-minor of G. □

We are now ready to prove the main theorem of the paper.

Proof of Theorem 1.4. Note that proving Theorem 1.4 is equivalent to proving the following.

8.3.1. *Let r be a positive integer. There is an integer $g(r)$ such that every loopless 2-connected graph G with $|E(G)| \geq g(r)$ has a cc-minor H that is a parallel-path extension of a template with at least r parts.*

We shall show that G has a parallel-path extension of a template with at least r parts as a cc-minor when $g(r) = \sum_{i=1}^{r} \left(f_{7.1}(r+2)\right)^i$ where $f_{7.1}$ is the function whose existence was established in Theorem 7.1. Let T be a tree decomposition of G such that each vertex of T is either a simple 3-connected graph, or K_3, or B_3. First, we show the following.

8.3.2. *If there is a vertex $G_v \in V(T)$ such that $|E(G_v)| > f_{7.1}(r+2)$, then 8.3.1 holds.*

Note that $|f_{7.1}(t)| \geq t$ for all $t \geq 3$, so, for any positive integer r, we have $|E(G_v)| > f_{7.1}(r+2) \geq 3$. Thus, G_v cannot be isomorphic to K_3 or B_3. Therefore, G_v must be a simple 3-connected graph. Note that each component of $T - G_v$ is a tree decomposition for a 2-connected graph. Let $\{J_1, J_2, \dots, J_n\}$ be the collection of such graphs. Then G can be obtained by repeatedly gluing each J_i to G_v via a 2-sum on the basepoint b_i where $\{b_i\} = E(J_i) \cap E(G_v)$ for all $i \in \{1, 2, \dots, n\}$. By Lemma 4.3, for each $i \in \{1, 2, \dots, n\}$, the graph J_i has a cc-minor J'_i that is a parallel connection, with basepoint b_i, of a collection of cycles containing b_i. Thus, G has a cc-minor G'_v that is obtained by, for each i in $\{1, 2 \dots, n\}$, gluing J'_i to G_v via a 2-sum on the basepoint b_i. It is straightforward to verify that G'_v is a parallel-path extension of G_v.

By Lemma 8.1, it suffices to show that G_v has a cc-minor that is a template with at least r parts. By Theorem 7.1, G_v has a cc-minor that is a fan-type graph with at least $r + 2$ spokes, which constitutes a template with at least r parts.

Hence 8.3.2 holds.

We may now assume that $|E(G_v)| \leq f_{7.1}(r+2)$ for every vertex G_v in $V(T)$. First we show the following.

8.3.3. *T has a path with at least r vertices.*

Since the basepoints are deleted after 2-sums, for each $G_v \in V(T)$, we have $d_T(G_v) \leq |E(G_v)|$. Therefore, we conclude that $d_T(G_v) \leq f_{7.1}(r+2)$ for every $G_v \in V(T)$. For any two vertices G_u and G_v, the *distance* $d(G_u, G_v)$ between them is the number of edges of the shortest G_uG_v-path in T. For an arbitrary vertex G_w in $V(T)$, we have the following.

8.3.4. *For each non-negative integer h,*

$$|\{G_v \in V(T) : d(G_w, G_v) = h\}| \leq \left(f_{7.1}(r+2)\right)^h.$$

Since $|E(G)| \geq g(r)$ and $|E(G_v)| \leq f_{7.1}(r+2)$ for each vertex G_v in $V(T)$, the tree T has at least $\frac{g(r)}{f_{7.1}(r+2)}$ vertices. As $g(r) = \sum_{i=1}^{r} \left(f_{7.1}(r+2)\right)^i$, we deduce that T has at least $\sum_{j=0}^{r-1} \left(f_{7.1}(r+2)\right)^j$ vertices. Therefore, by 8.3.4, there is a vertex $G_q \in V(T)$ such that $d(G_w, G_q) \geq r-1$. Hence 8.3.3 holds.

Let P be a path $G_1G_2 \dots G_r$ in T. Note that P is a graph-labelled tree representing a graph G_P obtained from the graphs $G_1, G_2, \dots, G_r$ by applying a sequence of 2-sums. Note that each component of $T - V(P)$ is a tree decomposition for a 2-connected graph. Let $\{F_1, F_2, \dots, F_m\}$ be the collection of these graphs. Then G can be obtained by, for each i in $\{1, 2, \dots, m\}$, gluing F_i to G_P via a 2-sum on the basepoint p_i where $\{p_i\} = E(F_i) \cap E(G_P)$. By Lemma 4.3, for each $i \in \{1, 2, \dots, m\}$, the graph F_i has a cc-minor F_i' that is a parallel connection, with basepoint p_i, of a collection of cycles containing p_i. Thus, G has a cc-minor G_P' that is obtained by, for each i in $\{1, 2, \dots, m\}$, gluing F_i' to G_P via a 2-sum on the basepoint p_i. It is straightforward to verify that G_P' is a parallel-path extension of G_P. By Lemma 8.1, it remains only to show that G_P has a cc-minor that is a template with at least r parts.

For each $i \in \{1, 2, \dots, r-1\}$, let e_i be the unique edge in $E(G_i) \cap E(G_{i+1})$ that is used as the basepoint between G_i and G_{i+1} in the construction of G_P. For each $i \in \{2, 3, \dots, r-1\}$, by Theorem 6.1, G_i has a cc-minor G_i' that is

(i) a bond graph containing e_{i-1} and e_{i+1}; or

(ii) a fan-type graph containing e_{i-1} and e_{i+1} as distinct outer spokes that are not parallel; or

(iii) a parallel extension of K_4 that has e_{i-1} and e_{i+1} as non-adjacent edges.

For $i \in \{1, r\}$, if G_i is isomorphic to K_3 or B_3, let $G_i' = G_i$. Now suppose that G_1 is not isomorphic to K_3 or B_3. Then, by Corollary 4.4, G_1 has a cc-minor G_1' that contains e_1 and is isomorphic to a bond graph with at least three edges. Define G'_r symmetrically when G_r is not isomorphic to K_3 or B_3. Let G'_P be the graph that is obtained by applying a sequence of 2-sums to the graphs $G'_1, G'_2, \dots, G'_r$, using the edges $e_1, e_2, \dots, e_{r-1}$ as basepoints. Applying Lemma 8.2 inductively, we see that G'_P is a cc-minor of G_P. It is straightforward to verify that G'_P is a template on at least r parts. □

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