- Virginia Tech Mathematics Contest. Sat., Oct. 21. **Sign-up deadline: Sep. 30**.
- Putnam Mathematical Competition. Sat., Dec. 2. **Sign-up deadline: Oct. 6**.

LSU Problem Solving Seminar - Fall 2017
Sep. 27: Enumeration

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Useful facts: (n and k are non-negative integers)

- **Inclusion-Exclusion.** Suppose that $A_1, A_2, \ldots, A_n$ are sets. Then

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|,$$
$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|,$$

and in general, with $A_I := \bigcap_{i \in I} A_i$,

$$|A_1 \cup \cdots \cup A_n| = \sum_{I \subseteq [1,n]} (-1)^{|I|} A_I.$$

- **Permutations.** The number of ordered lists of k distinct elements chosen from a set of n objects is $P(n,k) := \dfrac{n!}{(n-k)!}$.
- **Binomial Coefficients.** Given two non-negative integers n and k, the number of ways of choosing k (unordered) objects from a set of n is $\dbinom{n}{k} := \dfrac{n!}{k!(n-k)!}$ (this is read as "n choose k"). They satisfy the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
- **Binomial Theorem.** For an integer $n \geq 0$, $(1+x)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k$.
- **Number of subsets.** There are 2^n distinct subsets of a set with n elements.

Warm Up

1. (a) How many integers from 1 to 1000 are multiples of 2 or 3?
   (b) How many integers from 1 to 1000 have at least one 7 among their decimal digits? Answer this using Inclusion-Exclusion: write each integer with 3 decimal digits, possibly beginning with 0 (ignoring 1000 - why?). Now let A_i be the set of 3-digit integers with 7 in the i-th position....
   (c) Check your answer for (b) by observing that the complementary set is all of those 3-digit integers where each digit is 0, 1, 2, 3, 4, 5, 6, 8, or 9.
2. A Mexican restaurant offers a number of fillings/toppings: Adobo sauce, Black Beans, Chorizo, Diced Tomatoes, Epazote, Fresh Queso, Grilled Onions, and Hot Peppers. How many distinct choices are there for each of the following menu options?

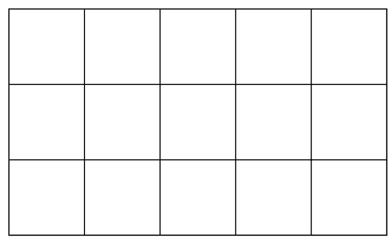
(a) A *Torta sandwich* consists of 3 distinct items layered on bread. Since the upper ingredients drip into the lower ones, the order matters!

(b) A *Burrito* consists of 4 distinct items rolled up in a tortilla; note that the order does not matter.

(c) A *Burrito Bowl* consists of 5 items layered over lettuce and rice, where same item may be chosen multiple times.

(d) A *Combination Plate* consists of a taco, a quesadilla, and covered tortilla chips. The taco is filled with 2 distinct items, the quesadilla with 1 item, and the tortilla chips are covered with 2 items, with repetition allowed.

3. (a) How many rectangles of any size are there in the following figure?



(b) What is the general formula; i.e., in an $m \times n$ grid, how many total rectangles are there?

Main Problems

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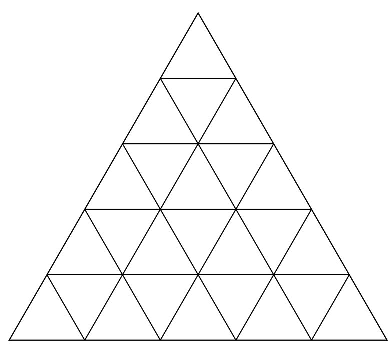
4. (a) How many triangles of any size are there in the following diagram?



(b) Find the general formula: suppose that the figure contains n smaller equilateral triangles along each outer edge. How many total triangles are there?

5. Evaluate

$$\sum_{j=0}^{n} \sum_{k=j}^{n} \binom{n}{k} \binom{k}{j}.$$

For example, when $n = 1$ this is

$$\binom{1}{0}\binom{0}{0} + \binom{1}{1}\binom{1}{0} + \binom{1}{1}\binom{1}{1} = 1 + 1 + 1 = 3.$$

Find the value for $n = 2$ before trying to prove the general case.

Hint: Write the second summation index as $k \geq j$, and then change the order of summation in order to use the Binomial Theorem.

6. The summands in Problem 5 are related to *multinomial* coefficients; if $i + j + k = n$, we write
$$\binom{n}{i, j, k} := \frac{n!}{i!j!k!}.$$

Remark: It would also be natural to call these "trinomial" coefficients, but that term is actually used for something slightly different.

(a) Prove that if $i + j + k = n$,
$$\binom{n}{i, j, k} = \binom{n}{i}\binom{n-i}{j}.$$

(b) Prove that if $0 \leq j \leq k \leq n$,
$$\binom{n}{k}\binom{k}{j} = \binom{n}{j, k-j, n-k}.$$

(c) Prove the 3-variable case of the **Multinomial Theorem**:
$$(x + y + z)^n = \sum_{i+j+k=n} \binom{n}{i, j, k} x^i y^j z^k.$$

Here the summation index means all nonnegative integers i, j, k that satisfy $i + j + k = n$. This can also be written as an explicit double sum (there is no sum over i, since it must be that $i = n - j - k$): $\displaystyle\sum_{k=0}^{n}\sum_{j=0}^{n-k}$.

7. [Gelca-Andreescu **823**] Prove that a list can be made of all the subsets of a finite set such that

(i) the empty set is the first set;

(ii) each subset occurs once;

(iii) each subset is obtained from the preceding by adding or deleting an element.

Remark: For example, the following is such a list for the set $\{a, b, c\}$ (written in shorthand where ab means the subset $\{a, b\}$):

$$\emptyset,\ a,\ ab,\ abc,\ ac,\ c,\ bc,\ b.$$

8. In a certain country postage stamps are produced in 6 and 11-cent denominations.

(a) The current postage rate is 46 cents; it was previously 41 cents, and before that 39 cents. Show that each of these values can be achieved with some combination of stamps.

(b) The legislature recently proposed raising the rate to 49 cents, until a mathematician pointed out that this value cannot be formed with any combination of stamps! However, the mathematician further explained that if the rate is instead raised to 50 cents, then any future rate increases will cause no problems. In other words, the largest **impossible** value is 49 cents!

Prove all of these claims.

9. [Putnam **1991 B3**] Does there exist a value L such that if m and n are integers greater than L, then an $m \times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two of which intersect at most along their boundaries?