LSU Problem Solving Seminar - Fall 2015
Nov. 4: Sequences and Recurrences

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Useful facts:

- **Geometric Series.** If $|x| < 1$, then
$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$
- **Ratio Test.** Let $L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. If $L < 1$, then $\sum_{n\geq 1} a_n$ converges, and if $L > 1$, then the sum diverges. If $L = 1$, the test is inconclusive.
- **Monotone Convergence.** If $a_1 \leq a_2 \leq \ldots$ and all $a_n \leq B$ for some constant, then $\lim_{n\to\infty} a_n$ exists.
- **Alternating Series.** If $a_1 \geq a_2 \geq \ldots$ and $\lim_{n\to\infty} a_n = 0$, then $a_1 - a_2 + a_3 - \ldots$ converges.
- **Linear Recurrences.** The *characteristic polynomial* associated to a (homogeneous) recurrence $a_{n+k} = c_{k-1}a_{n+k-1} + \cdots + c_1 a_{n+1} + c_0 a_n$ is $p(x) := x^k - c_{k-1}x^{k-1} - \cdots - c_1 x - c_0$. If $p(x)$ has (distinct) roots $\lambda_1, \ldots, \lambda_k$, then the general solution to the recurrence is
$$a_n = b_1\lambda_1^n + \cdots + b_k\lambda_k^n,$$
where the constants are determined by k initial values.

Warm Up:

1. Evaluate the following series:
   (a) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = ?$
   (b) $\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \cdots = ?$
   Hint: Apply partial fractions and write $\frac{1}{x(x+2)} = \frac{a}{x} + \frac{b}{x+2}$ *for constants a and b.*
2. Solve the following recurrences; this means finding a closed-form expression for a_n.
   (a) $a_1 = 4, a_2 = 8$, and $a_{n+2} = 2a_{n+1} + 3a_n$ for $n \geq 1$.
   (b) $a_1 = 2$, and $a_{n+1} = 2a_n - 1$ for $n \geq 1$.
   Hint: The general solution has the form $a_n = c2^n + d$ for some constants c and d.
   (c) $a_1 = 1$, and $a_{n+1} = 2a_n - n + 1$ for $n \geq 1$.

Main Problems:

3. The *Harmonic Series* is the sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.

(a) It is known that the sum diverges. Try to prove this for yourself!

(b) Prove that the "9-free" Harmonic sum converges. This is the sum of $\frac{1}{n}$ for all integers n that do **not** contain the digit 9.

(c) What about the sum of $\frac{1}{n}$ for all n that do not **begin** with 9 - does this converge?

4. (a) Find the value of $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$.

   Hint: Let $a_1 = 1$ and $a_{n+1} = \sqrt{1+a_n}$. Assume that $L = \lim_{n\to\infty} a_n$ exists, and determine its possible values. Then prove that the given sequence converges.

   (b) Find the value of $\sqrt{1-\sqrt{1-\sqrt{1-\sqrt{1-\cdots}}}}$.

5. Define a sequence by $a_1 = \frac{1}{2}$ and

$$3a_{n+1}a_n = 2a_n + 1$$

for $n \geq 1$. Determine $\lim_{n\to\infty} a_n$.

Hint: First, assume that the limit is some value L and plug in to the recurrence to determine the possible values. Then prove convergence to the correct value.

6. The *Fibonacci numbers* are defined by $F_1 = 1$, $F_2 = 1$, and

$$F_{n+2} = F_{n+1} + F_n, \qquad \text{for } n \geq 1.$$

   (a) Prove that for $n \geq 1$,

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

   (b) The quantity $\phi := \frac{1+\sqrt{5}}{2} = 1.618\ldots$ is known as the *Golden Ratio*. Prove that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi.$$

   (c) Evaluate the sum

$$\frac{1}{F_1F_3} + \frac{1}{F_2F_4} + \frac{1}{F_3F_5} + \cdots.$$

   Hint: Try to find a formula for the finite partial sums.

7. (a) Suppose that $a_n > 0$ for all n. Is it possible that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{a_n}$ both diverge? Is it possible that they both converge?

   (b) [VTRMC **1979 # 6**] Suppose $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ diverges. Determine whether $\sum_{n=1}^{\infty} \frac{a_n}{S_n^2}$ converges, where $S_n = a_1 + \cdots + a_n$.

8. [Putnam **1994 A1**] Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that $\sum_{n=1}^{\infty} a_n$ diverges.