MATH 7230 Homework 1 - Spring 2014

Due Thursday, Feb. 6 at 1:30

The notation "I-K" is shorthand for the textbook.

1. (a) Fill in the details of the following proof that there are infinitely many primes:

   Pick an arbitrary $1 \neq N \in \mathbb{N}$ and set $a(1) := N$. Define

   $$\begin{aligned} a(2) &:= N \cdot (N+1), \\ a(3) &:= N \cdot (N+1) \cdot \big[N \cdot (N+1) + 1\big], \end{aligned}$$

   and so on, with $a(m) := a(m-1) \cdot (a(m-1)+1)$. Prove that for all m, $a(m)$ has a prime factor p_m that does not divide $a(1), \ldots, a(m-1)$. Conclude that there is an infinite list of distinct primes $p_1, p_2, \ldots$.

   (b) Generalize the above proof by letting $f : \mathbb{N} \to \mathbb{N}$ be any arithmetic function such that n is always coprime to $f(n)$, and constructing the iterative sequence $a(m) := a(m-1) \cdot f(a(m-1))$. For example, in part (a), $f(n) = n+1$.

2. (a) In this problem you will modify Euclid's proof to show that there are infinitely many primes of the form $4n+3$. Given a list $q_1, q_2, \ldots, q_k$ of primes of this form, let $N := 4q_1 \cdots q_k - 1$. Prove that N has a prime divisor $q \equiv 3 \pmod 4$ that is not one of the q_js.

   (b) Generalize your result from part (a) as much as possible.

   Hint: If p is prime and $m \in \mathbb{N}$, what are the possible residues $p \bmod m$?

3. In this problem you will prove that e is irrational.

   (a) As a warm-up, prove that $2 < e < 3$ by comparing the Taylor expansion $e - 2 = \sum_{n \geq 2} \frac{1}{n!}$ to the geometric series $\sum_{n \geq 1} \frac{1}{2^n}$.

   (b) Suppose to the contrary that $e = \frac{a}{b} \in \mathbb{Q}$. Find a contradiction by showing that

   $$b! \left(e - \sum_{n=0}^{b} \frac{1}{n!} \right)$$

   is an integer between 0 and 1.

   Remark: Actually, it is easier to consider e^{-1}, which was suggested by Pennisi (1953).

(Optional) To learn why e is transcendental, I recommend starting with Hurwitz's proof, which is Theorem 204 in Hardy and Wright. You can also find the argument outlined online at: `http://planetmath.org/eistranscendental`.

An alternative proof follows directly from Euler's continued fraction representation $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \ldots]$, though this requires the theory of continued fraction convergents and Bessel functions.

4. Read I-K Sections 1.1 – 1.4. Be sure that you understand all of the notation, definitions, and meanings of the following numbered formulas:

   (1.5), (1.6), (1.11), (1.13), (1.15), (1.18), (1.24), (1.26), (1.32), (1.35).

   Most of these require at least a few lines of calculation, and you are encouraged to hand in short proofs for all of them.

5. (a) Suppose that $a_n \geq 0$ for $n \geq 1$. Prove that $\prod_{n\geq 1}(1+a_n)$ converges if and only if $\sum_{n\geq 1} a_n$ converges.

   (b) Prove that $\sum_{n\geq 0} \frac{1}{p_n}$ diverges, where p_n denotes the n-th prime.

   Hint: Recall that $\zeta(s)$ diverges as $s \to 1^+$.

(Optional) Given an example of real $\{b_n\}$ such that $\prod(1+b_n)$ converges, but $\sum b_n$ diverges. Give an example of $\{c_n\}$ such that $\prod(1+c_n)$ diverges, but $\sum c_n$ converges.

6. Euler's ϕ-function is defined such that $\phi(n)$ is the number of integers $1 \leq a \leq n$ that are coprime to n. Provide an alternative proof of the formula ((1.36) from I-K)

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

   One option is to use an Inclusion-Exclusion argument, by first removing all multiples of $p_1, p_2, \ldots,$ and then correcting for over and under-counts.

   Another option is to prove that for any m,

$$\phi(pm) = \begin{cases} p \cdot \phi(m) & \text{if } p \mid m \\ (p-1) \cdot \phi(m) & \text{if } p \nmid m. \end{cases}$$

7. Prove and/or verify I-K equations (1.37) – (1.42). These definitions will be essential in the proof of the Prime Number Theorem.