MATH 18.02A - Problem Set 7 Solutions
Spring 2007

Part II - Problem 1.

a) Green's Theorem states that the line integral through the field $\mathbf{F} = \left(xy + \frac{y^2}{2}\right)\hat{\mathbf{i}} + (2x - ye^y)\,\hat{\mathbf{j}}$ may be replaced by the double integral of

$$\operatorname{curl}\mathbf{F} = 2 - x - y.$$

In particular, if R is a simple region,

$$\oint_{\partial R}\left(xy + \frac{y^2}{2}\right)dx + (2x - ye^y)\,dy = \iint_R 2 - x - y\,dA.$$

In the first quadrant, $2 - x - y$ is positive precisely for the points that lie in the triangle with vertices $(0,0), (2,0)$, and $(0,2)$. Therefore, the integral is maximized when $\mathbf{c}$ is the clockwise boundary of this triangle.

b) The maximum value of the integral occurs for the region described above,

$$\int_0^2\int_0^{2-x}(2-x-y)\,dydx = \int_0^2 (2-x)^2 - \frac{(2-x)^2}{2}\,dx = \left.\frac{-(2-x)^3}{6}\right|_0^2 = \boxed{\mathbf{\frac{4}{3}}}.$$

Part II - Problem 2.

a) The divergence is

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(-x + 2y\sin x - \sqrt{1+y^3}) + \frac{\partial}{\partial y}(e^x + y^2\cos x + 3y) = \boxed{\mathbf{2 + 4y\cos x}}.$$

b) The path $\mathbf{c}$ is the positive boundary of the triangle R described in the problem, so by Green's Theorem

$$\begin{aligned}\int_{\partial R}\mathbf{F}\cdot\mathbf{n}\,ds &= \iint_R \operatorname{div}\mathbf{F}\,dA\\ &= \int_0^b\int_0^{\frac{ay}{b}} 2 + 4y\cos y\,dxdy = \int_0^b \frac{2ay}{b} + 4y\sin\frac{ay}{b}\,dy\\ &= \left.\left(\frac{ay^2}{b} - \frac{4by}{a}\cos\frac{ay}{b} + \frac{4b^2}{a^2}\sin\frac{ay}{b}\right|_0^b = \boxed{\boldsymbol{ab - \frac{4b^2}{a}\cos a + \frac{4b^2}{a^2}\sin a}}.\end{aligned}$$

Part II - Problem 3.

Call the given circular curve $\mathbf{c}_1$, and define $\mathbf{c}_2$ to be the line from $(0,1)$ to $(0,-1)$. Then $\mathbf{c} = \mathbf{c}_2 - \mathbf{c}_1$ is a closed, simple curve that is the positive boundary of a semicircular region R. The divergence of $\mathbf{F}$ is

$$\operatorname{div}\mathbf{F} = 4y^3 + 2y + -4y^3 = 2y.$$

Thus, Green's Theorem implies that the flux across $\mathbf{c}$ is

$$\oint_{\mathbf{c}}\mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \operatorname{div}\mathbf{F}\,dA = \iint_R 2y\,dA = 0,$$

because R is symmetric about the x-axis.

Finally, the properties of line integrals show that

$$0 = \oint_{\mathbf{c}} \mathbf{F}\cdot\mathbf{n}\,ds = \oint_{\mathbf{c}_2} \mathbf{F}\cdot\mathbf{n}\,ds - \oint_{\mathbf{c}_1} \mathbf{F}\cdot\mathbf{n}\,ds,$$

so

$$\begin{aligned}\int_{\mathbf{c}_1} \mathbf{F}\cdot\mathbf{n}\,ds = \int_{\mathbf{c}_2} \mathbf{F}\cdot\mathbf{n}\,ds &= \int_{\mathbf{c}_2} (4xy^3 + 2xy + 3y^2)\,dy - (2 - y^4)\,dx \\ &= \int_1^{-1} 3y^2\,dy = \boxed{\mathbf{-2}}.\end{aligned}$$

Remark. These calculations have a nice physical interpretation – since the net flow in and out of the boundary is zero (by the flux calculation), the inflow through the left boundary equals the outflow through the right boundary.

Part II - Problem 4.

a) In general, the surface of revolution around the z-axis of a parameterized curve $y = f(u), z = g(u), a \le u \le b$ can be written using a second parameter v :

$$x = f(u)\cos v, \qquad y = f(u)\sin v, \qquad z = g(u);$$
$$a \le u \le b, \qquad 0 \le v \le 2\pi.$$

This means that the torus may be parameterized by

$$\begin{aligned}\boldsymbol{x} &= \mathbf{(}\boldsymbol{R} + \mathbf{cos}\,\boldsymbol{u}\mathbf{)}\,\mathbf{cos}\,\boldsymbol{v}, & \mathbf{0} \le \boldsymbol{u} \le \mathbf{2}\boldsymbol{\pi},\\ \boldsymbol{y} &= \mathbf{(}\boldsymbol{R} + \mathbf{cos}\,\boldsymbol{u}\mathbf{)}\,\mathbf{sin}\,\boldsymbol{v}, & \mathbf{0} \le \boldsymbol{v} \le \mathbf{2}\boldsymbol{\pi}.\\ \boldsymbol{z} &= \mathbf{sin}\,\boldsymbol{u}; &\end{aligned}$$

b) The surface area of a general parameterized surface S is

$$\iint_S |\mathbf{T}_u \times \mathbf{T}_v|\,dudv = \iint_S \sqrt{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2 + \left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2}\,dudv.$$

In this case,

$$\begin{aligned}\mathbf{T}_u &= -\sin u\cos v\,\hat{\mathbf{i}} - \sin u\sin v\,\hat{\mathbf{j}} + \cos u\,\hat{\mathbf{k}},\\ \mathbf{T}_v &= -(R + \cos u)\sin v\,\hat{\mathbf{i}} + (R + \cos u)\cos v\,\hat{\mathbf{j}}.\end{aligned}$$

Therefore,

$$\mathbf{T}_u \times \mathbf{T}_v = -(R + \cos u)\left(\cos v\cos u\,\hat{\mathbf{i}} + \sin v\cos u\,\hat{\mathbf{j}} + \sin u(\sin^2 v + \cos^2 v)\,\hat{\mathbf{k}}\right),$$

and

$$\begin{aligned}A &= \iint_S |\mathbf{T}_u \times \mathbf{T}_v|\,dudv\\ &= \int_0^{2\pi}\int_0^{2\pi} (R + \cos u)\sqrt{\cos^2 v\cos^2 u + \sin^2 v\cos^2 u + \sin^2 u}\,dudv\\ &= \int_0^{2\pi}\int_0^{2\pi} (R + \cos u)\,dudv = 4\pi^2 R.\end{aligned}$$

Alternatively, the area can be calculated using Jacobians. The integrand is then

$$\sqrt{\begin{vmatrix} -\sin u\cos v & -(R+\cos u)\sin v \\ -\sin u\sin v & -(R+\cos u)\cos v \end{vmatrix}^2 + \begin{vmatrix} -\sin u\cos v & -(R+\cos u)\sin v \\ \cos u & 0 \end{vmatrix}^2 + \begin{vmatrix} -\sin u\sin v & -(R+\cos u)\cos v \\ \cos u & 0 \end{vmatrix}^2}$$

Note. The original problem statement asked only for an integral expression that could be used to calculate the answer, but not for its numerical evaluation. An answer is acceptable so long as either the tangent vectors or Jacobians have been computed symbolically (i.e., partial derivatives must have been written down).