M7210 Lecture 25 Monday October 19, 2012

Abelian Groups IV

Analysis of the torsion part (continued)

Let T be finitely-generated torsion (abelian) group. We ascertained in the last lecture that T may be decomposed as a finite sum of cyclic groups of prime power order, so in particular T is finite.[1]

We will show that for each prime p and each positive integer l, the number of times that p^l appears in any two decompositions of T into cyclic groups of prime power order is the same. To be more explicit, suppose:

$$T \;\cong\; \mathbb{Z}/p_1^{k_1}\mathbb{Z} \oplus \mathbb{Z}/p_2^{k_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_M^{k_M}\mathbb{Z} \;\cong\; \mathbb{Z}/q_1^{\ell_1}\mathbb{Z} \oplus \mathbb{Z}/q_2^{\ell_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_N^{k_N}\mathbb{Z}. \tag{1}$$

Here the p_i and the q_i are primes, and in each sum, a given p^k (or q^ℓ) may occur more than once. We want to show:

$$M = N \text{ and after renumbering we have } p_i^{k_i} = q_i^{\ell_i} \text{ for } i = 1, \ldots, M. \tag{2}$$

Let $T_p := \{\, x \in T \mid p^a x = 0 \text{ for some integer } a\}$. This set forms a subgroup of T generally called the p-primary component of T. If $T = T_p$, we say that T is p-primary. For fixed p, it is clear that T_p is the sum of the summands of the form $\mathbb{Z}/p^\ell\mathbb{Z}$ in (1). Thus, if there are two decompositions of T that violate (2), we will see that violation when we restrict attention to T_p.

Therefore, let H be a p-primary finite abelian group. Let $p^jH = \{\, p^jh \mid h \in H \,\}$ and let $H^{(j)} := p^jH/p^{j+1}H$. Note that $p\left(\overline{p_jh}\right) = 0$ for each $\overline{p_jh} \in H^{(j)}$, so $H^{(j)}$ is a vector space over $\mathbb{Z}/p\mathbb{Z}$. Let $\dim_p H^{(j)}$ be its dimension.

Suppose $H = H_1 \oplus \cdots \oplus H_M$, where each H_i is p-primary cyclic. Then $p^jH = p^jH_1 \oplus \cdots \oplus p^jH_M$ and also $H^{(j)} = H_1^{(j)} \oplus \cdots \oplus H_M^{(j)}$. (One can check this by verifying the universal mapping property with a diagram.) Since,

$$\left(\mathbb{Z}/p^k\mathbb{Z}\right)^{(j)} = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{if } k \geq j+1; \\ 0, & \text{if } k \leq j, \end{cases}$$

we see that the number of summands among the H_i of order $\geq j+1$ is simply $\dim_p H^{(j)}$. The numbers $\dim_p H^{(1)}, \dim_p H^{(1)}, \ldots$ do not depend the sum decomposition, yet they determine what summands must be present. Thus, we have proved (2).

Elementary Divisors and Invariant Factors

The data that determines the structure of a finitely-generated abelian group is

a) The rank of the free summand

[1] Of course, we can see this directly: if the generators are $g_1, \ldots, g_n$ with orders $k_1, \ldots, k_n$, then every element has an expression of the form $m_1g_1 + \cdots m_ng_n$ with $0 \leq m_i < k_i$, so there are at most $(k_1)(k_2)\cdots(k_n)$ elements. It is interesting to note that without the abelian hypothesis the situation is entirely different. There are finitely-generated groups in which each element has finite order which are nonetheless infinite. There are even infinite groups that are finitely-generated and in which every element has order less than a fixed bound. For more on this topic, the search term is "Burnside's problem".

b) For each prime p, a list (possibly with repetitions) of the orders of the p-primary cyclic components.

We may arrange this data is a display as follows. The number of rows is the maximum number of cyclic summands that are primary to some p. In the picture below, for 5 and 7, there are 5 summands. The shorter lists are allowed to "fall down to the bottom". In each column, the exponents are listed in increasing order, with repeats as needed. All the data in this chart is recorded in the numbers corresponding to the rows. In the picture (assuming all primes in the decomposition are shown), these numbers are: $5 \cdot 7$, $5 \cdot 7$, $5 \cdot 7^2 \cdot 13$, $3 \cdot 5 \cdot 7^2 \cdot 13^3$, $2 \cdot 3^2 \cdot 5 \cdot 7^3 \cdot 13^4$. These are called the "invariant factors." We can write G as a sum of cyclic groups whose orders are the invariant factors. We can—by exercising care—find matrices P and Q so that the invariant factors appear as the diagonal entries of PAQ. When this occurs, PAQ is in Smith Normal Form (see the discussion of Smith Normal Form at the end of Lecture 23

primes	2	3	5	7	13	$\cdots$
	–	–	–	–	–	–
exponents			1	1		$\cdots$
			1	1		$\cdots$
			1	2	1	$\cdots$
		1	1	2	3	$\cdots$
	2	2	1	3	4	$\cdots$