M4056 Midterm Exam, Oct. 13, 2010   Name_____

Section I. Provide a precise mathematical definition for each of the following basic notions of mathematical statistics, and provide an example or illustration. (Five points each.)

1. Bernoulli random variable

2. Moment generating function

3. Chi squared random variable

4. Statistic

5. Sufficient statistic

6. Method of moments estimator

7. Maximum likelihood estimator

8. Mean squared error (of an estimator)

Section II.

1. [20 points] Let $S^2 = S^2(X_1, \ldots, X_n)$ be the sample variance of a sample of size n from a population described by a random variable X with distribution $f_X(x)$. (I.e., the X_i are independent, identically distributed, each with the same distribution as X.) Explain the difference in meaning between:

   a) the variance of X,
   b) the sample variance (i.e., S^2 itself),
   c) the expected value of S^2,
   d) the variance of the sample mean $(\operatorname{Var} M)$,
   e) the variance of S^2 $(\operatorname{Var} S^2)$.

2. [15 points] Let $(X_1, \ldots, X_n)$ be a sample of size n from an exponential distribution with parameter λ. Note that the distribution of X has *pdf* $f_X(x|\lambda) = \lambda e^{-\lambda x}$, $0 \leq x$.
   a) Write the sample distribution $f_{\vec{X}}(\vec{x}|\lambda)$.

   b) Find a sufficient statistic for λ.

   c) What is the MLE of λ?

   d) What is the mean squared error of the MLE?

Section III.

1. [25 points] Suppose the random variables $Y_1, \ldots, Y_n$ satisfy:

$$Y_i = \alpha + \beta x_i + \epsilon_i, \quad i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are fixed constants, and $\epsilon_1, \ldots, \epsilon_n$ are iid normal$(0, \sigma^2)$, σ^2 unknown.
   a) Find a three-dimensional sufficient statistic for $(\alpha, \beta, \sigma^2)$.

b) Express the MLE of β as a linear function of the statistics from part a) and α.

c) Express the MLE of α as a linear function of the statistics from part a) and β.

d) Suppose σ^2 and β are fixed. What is the MLE of α? Is this estimator biased or unbiased?

e) Suppose σ^2 and α are fixed. What is the MLE of β? Is this estimator biased or unbiased?