Lecture 19. Generating Functions and Moment Generating Functions

This lecture is related to the material in Chapter 11.

Recall the following from the last lecture:

Definition. Let X be a random variable that takes values in the set $\{0, 1, 2, \ldots\}$ and has pmf f. The *generating function* of X is

$$G_X(t) := f(0) + f(1)t + f(2)t^2 + \cdots = \sum_{i=0}^{\infty} f(i)t^i$$

In Lecture 18, I called this $p_X(t)$. But we use p for so many other things that it seems like a good idea to use a different letter. G has the advantage that it is the first letter of "generating." The generating function is useful because of the useful way in which it keeps track of all the information in the pmf, as the applications we see below will demonstrate.

Remark. Note that if X is defined on the probability space Ω, then

$$G_X(t) = \sum_{\omega \in \Omega} f(\omega)t^{X(\omega)}.$$

Example. The Binomial Theorem states:

$$(A+B)^m = \sum_{i=0}^{m} \binom{m}{i} A^i B^{m-i}.$$

If X is binomial (m, p), then $f(i) = \binom{m}{i}p^i(1-p)^{m-i}$, so

$$G_X(t) = \sum_{i=0}^{m} f(i)\, t^i = \sum_{i=0}^{m} \binom{m}{i} (pt)^i (1-p)^{m-i} = \big(p\, t + (1-p)\big)^m.$$

Remark. A generating function can be viewed in two ways: either as a formal, symbolic object $f(0) + f(1)t + f(2)t^2 + \cdots$ or as a real-valued function $\sum_{i=0}^{\infty} f(i)t^i$ of t. If there are only finitely many values of i such that $f(i) \neq 0$, the difference is inconsequential, for then we have a polynomial expression. Each polynomial expression gives us a polynomial function, and every polynomial function has an expression. However, if $f(i)$ is non-zero for infinitely many values of i, then we need to know that $\sum_{i=0}^{\infty} f(i)t^i$ converges before we can regard $G_X(t)$ as a real-valued function of t. But there is good news! Since $\sum_{i=0}^{\infty} f(i) = 1$, $\sum_{i=0}^{\infty} f(i)t^i$ converges at least on $[-1, 1]$.

Generating functions and addition of random variables

When we introduced the generating function in Lecture 18, we were motivated by the formula for adding independent random variables. As we showed:

Fact. If X and Y are both random variable that take values in the set $\{0, 1, 2, \ldots\}$, then

$$G_{X+Y}(t) = G_X(t) \cdot G_Y(t).$$

Problem. Show that if X is binomial (m, p) and Y is binomial (n, p), then $X + Y$ is binomial $(m + n, p)$.

Solution. We can accomplish this by showing that the generating function of $X + Y$ is $\big(p\,t + (1-p)\big)^{m+n}$, because *if two random variables have the same generating function, then they have the same pmf*. Now,

$$G_{X+Y}(t) = G_X(t) \cdot G_Y(t) = \big(p\,t + (1-p)\big)^m \cdot \big(p\,t + (1-p)\big)^n = \big(p\,t + (1-p)\big)^{m+n}.$$

Now the right hand side is the generating function of a binomial $(m+n, p)$ random variable.

Example. Suppose X is Poisson, with parameter λ. Then:

$$P(X = n) = e^{-\lambda} \frac{\lambda^n}{n!}.$$

Thus,

$$G_X(t) = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} t^n = e^{-\lambda} \sum_{i=0}^{\infty} \frac{(\lambda t)^n}{n!} = e^{-\lambda} e^{t\lambda} = e^{\lambda(t-1)}$$

1. **Homework.** Show that if X is Poisson, with parameter λ and Y is Poisson, with parameter μ, then $X + Y$ is Poisson, with parameter $\lambda + \mu$.

Generating functions, expectation and variance

Let $G_X{}'(t) = \frac{d}{dt} G_X(t)$. Observe that

$$G_X{}'(1) = \sum_{i=0}^{\infty} f(i) \cdot i\, t^{i-1} \,|_{t=1} = \sum_{i=0}^{\infty} i\, f(i) = \mathrm{E}(X).$$

$$G_X{}''(1) = \sum_{i=0}^{\infty} f(i) \cdot i(i-1)\, t^{i-2} \,|_{t=1} = \sum_{i=0}^{\infty} i^2\, f(i) - \sum_{i=0}^{\infty} i\, f(i) = \mathrm{E}(X^2) - \mathrm{E}(X).$$

2. **Homework.** Express $\mathrm{Var}(X)$ in terms of ${G_X}'(1)$ and ${G_X}''(1)$.

3. **Homework.** Suppose X is Poisson with parameter λ. Find $E(X)$ and $\mathrm{Var}(X)$.

   We can find a formula that includes all the moments of X by applying G_X to the exponential function, expanding the latter, reversing the order of summation and simplifying:

$$
\begin{aligned}
G_X(e^t) &= \sum_{k=0}^{\infty} f(k) e^{kt} \\
&= \sum_{k=0}^{\infty} f(k) \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(k) \frac{(kt)^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} k^n\, f(k) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \mathrm{E}(X^n)\, \frac{t^n}{n!}.
\end{aligned}
$$

   Definition. Let X be a random variable that takes values in the set $\{0, 1, 2, \ldots\}$ and has pmf f. The *moment generating function* of X, denoted $M_X(t)$ is

$$M_X(t) := G_X(e^t).$$

4. **Homework.** Let ${M_X}^{(j)}$ be the j^{th} derivative of $M_X(t)$. Show that

$${M_X}^{(j)}(0) = E(X^j).$$

   Recall that $E(X^j)$ is called *the* j^{th} *moment of* X. This accounts for the name "moment generating function."

5. **Homework.** Show that

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

Moment generating functions of continuous random variables

Definition. Suppose X is a continuous random variable with values in $\mathbb{R}$. The *moment generating function of* X, denoted $M_X(t)$ is defined by

$$M_X(t) = \mathrm{E}(e^{tX}) = \int_{\mathbb{R}} e^{tx} f(x)\, dx.$$

6. **Homework.** Find $M_X(t)$ in each of the following cases:

a) X is uniform on (a, b);

b) X is standard normal;

c) X is normal with parameters $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$;

d) X is exponential with parameter λ.