Integrating along a curve: Distance traveled and length

- Let t denote time. Suppose the position of a particle moving in the plane is given by a function $\gamma(t)$. The position is described by its x- and y-coordinates, so for some functions x and y, we have:

$$\gamma(t) = \big(x(t), y(t)\big) = \text{position of the particle at time } t.$$

- The velocity of the particle at time t is $\gamma'(t) = \big(x'(t), y'(t)\big)$, and the speed of the particle is $|\gamma'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$.
- Distance is the integral of speed, so:

$$[\text{total distance traveled between time } a \text{ and time } b] = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

If the particle traverses the curve $\{\,\gamma(t) \mid a \le t \le b\,\}$ only once without ever doubling back or re-tracing any sections of the curve, then this integral also gives the length of the curve.

Example 1. Find the length of the graph of $y = f(x)$ from $x = a$ to $x = b$.
Solution: This portion of the graph is given parametrically by $\gamma(t) = (t, f(t))$, $t \in [a, b]$. Thus, the length is:

$$\int_a^b \sqrt{1 + (f'(t))^2}\, dt.$$

When specific functions are considered, this may be a very difficult integral. For example, the length of the graph of $f(x) = x^2$ from $(0, 0)$ to $(1, 1)$ is

$$\int_0^1 \sqrt{1 + 4t^2}\, dt = \frac{2\sqrt{5} + \text{ArcSinh}(2)}{4}.$$

Example 2. A wheel is 1 foot in radius. Find out how far a point on the rim travels when the wheel rolls one full revolution. (This example was not given in class, but it's very nice.)
Solution: Suppose the wheel rolls leftwards along the x-axis in such a way that the center is at point $(-t, 1)$ at time t. Then the wheel rotates counterclockwise making one full revolution in the t-interval $[0, 2\pi]$. Suppose there is a mark on the rim which is at position $(0, 0)$ at time $t = 0$. At time t, that mark will be at position

$$\gamma(t) = (-t, 1) + (\sin t, -\cos t) = (-t + \sin t, 1 - \cos t).$$

The velocity of the mark is $\gamma'(t) = (-1 + \cos t, \sin t)$ and the speed is

$$|\gamma'(t)| = \sqrt{(1 - 2\cos t + \cos^2 t) + \sin^2 t} = \sqrt{2 - 2\cos t}.$$

Thus, the distance traveled is:

$$\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt.$$

To solve this integral, note that the half-angle formula, $1 - \cos t = 2\sin^2(t/2)$, gives $\sqrt{1 - \cos t} = \sqrt{2}\sin(t/2)$, so

$$\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = \int_0^{2\pi} 2\sin(t/2)\, dt = \int_0^{\pi} 4\sin u\, du = 8.$$

Exercise 1. Find the length of the graph of $y = e^x$ from from $(0, 1)$ to $(\ln 2, 2)$. (Helpful hint: $\int \sqrt{1 + e^{2t}}\, dt = \sqrt{1 + e^{2\,t}} - \text{ArcTanh}(\sqrt{1 + e^{2\,t}})$.)

Weight of a piece of wire

- Suppose the density per unit length of a streight piece of wire lying on the x-axis is given by the function $\delta(x)$. Then the weight of the portion of the wire between $x = a$ and $x = b$ is given by
$$\int_a^b \delta(x)\, dx.$$
- The same idea can be applied to a curved piece of wire in the plane. Suppose that the parametric function $\gamma(t)$, $t \in [a, b]$ describes a uni-directional motion that goes once along a portion of the wire. Also, suppose that the density per unit length of the wire at any point (x, y) is $\delta(x, y)$. Then the weight of the portion of the wire covered between $t = a$ and $t = b$ is
$$\int_a^b \delta(\gamma(t))\, |\gamma'(t)|\, dt.$$

Example 3. Find the weight of a piece of wire lying on the graph of $y = x^2$ from from $(0, 0)$ to $(1, 1)$ if the density (in units of weight per unit length) is $\delta(x, y) = x$.
Solution: In this case, $\gamma(t) = (t, t^2)$ and $\delta(\gamma(t)) = t$. As in Example 1, $|\gamma'(t)| = \sqrt{1 + 4t^2}$. Thus, the weight is

$$\int_0^1 t\,\sqrt{1 + 4t^2}\, dt = \frac{5\sqrt{5} - 1}{12}.$$

Exercise 1. Find the weight of a piece of wire lying on the graph of $y = x^3$ from from $(0, 0)$ to $(1, 1)$ if the density (in units of weight per unit length) is $\delta(x, y) = y$.

Exercise 2. Find the weight of a ring of wire lying over the portion of the unit circle in the first quadrant, if the density is $\delta(x, y) = xy$. Note: the unit circle is parametrized by $\gamma(t) = (\cos t, \sin t)$, and for the portion in the first quadrant we want $t \in [0, \pi/2]$.