Arch. Rational Mech. Anal. 229 (2018) 1197–1221
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-018-1237-1

Effective Maxwell's Equations for Perfectly Conducting Split Ring Resonators

Robert Lipton & Ben Schweizer

Communicated by G. Friesecke

Abstract

We analyze the time harmonic Maxwell's equations in a geometry containing perfectly conducting split rings. We derive the homogenization limit in which the typical size η of the rings tends to zero. The split rings act as resonators and the assembly can act, effectively, as a magnetically active material. The frequency dependent effective permeability of the medium can be large and/or negative.

1. Introduction

Meta-materials consist of a large number of small elements. While each of the elements contains only ordinary materials, the properties of the meta-material can be very different from those of its constituents. In the theory of light, the index of refraction of every ordinary material is positive, but today we know how to construct meta-materials that have, effectively, a negative index of refraction. The astonishing properties of negative index materials has been investigated already in 1968 by Veselago [32], the first ideas how to construct a negative index meta-material appeared around 2000, an experimental realization of such a material was reported in [31]. In the subsequent years, many possible applications have been investigated, e.g., perfect lensing [25], directing light-beams around obstacles [26], or cloaking by anomalous localized resonance [17,23].

The first ideas for negative index meta-materials were based on constructions with split rings. In these constructions the ring serves as an inductive element and the slit serves as a conductive element. The combination of the two elements forms an oscillator. When the oscillator (which is smaller than the wave-length of light) is in resonance with the frequency of the light, resonance effects can lead to large local fields. When this is the case, the macroscopic features of the meta-material are given by effective coefficients (permittivity and permeability) that can be very different from the coefficients of the materials that are used in the construction: they can be very large or they can have the opposite sign. For a recent general overview regarding the effect of resonances in many small resonators see [27].

The mathematical analysis of negative index meta-materials is possible with the methods of homogenization theory, which has roots in the 1970s [3]. Until today, the most powerful tool for the homogenization of periodic problems was the method of two-scale convergence [1]. The homogenization of Maxwell's equations was already performed in [15], but the analysis of homogenization settings that lead to unexpected effective equations was pioneered by Bouchitté. Together with co-authors, he derived effective systems with negative effective permittivity [14], with nonlocal effects [6], and with an effective magnetic response [5]. The three-dimensional Maxwell's equations have been analyzed first in [7] using the split ring geometry, later in [4] with dielectric Mie-resonators, and in [19] with flat rings. Very recently, the combination of resonator elements with wire elements was also successfully analyzed. The setting is very close to the experimental set-up of the early negative index constructions; the mathematical analysis reveals that it is possible to obtain an effective system with two coefficients $\varepsilon_{\mathrm{eff}}$ and μ_{eff}, which have both a negative real part [21]. With that contribution, we have a mathematical confirmation and an effective description of a negative index meta-material.

The possibility of light transmission in periodic media has been investigated for general microstructures in [29]. The possibility of transmitting light can be characterized in terms of topological properties (such as connectedness) of the inclusions. Recently backward propagating waves generated by effective surface impedances from sub-wavelength corrugated perfectly conducting waveguides have been developed via two scale homogenization in [22].

The above contributions regard results in the language of two-scale homogenization results, i.e., the derivation and justification of effective equations. For other approaches using local resonances see [18,24]. A recent development focuses on negative index metamaterials generated by subwavelength plasmon resonances associated with noble metals in the optical regime see [9–12]. There, frequency intervals are identified over which negative group velocity and power flow occur for coated inclusions and dispersed phases.

Regarding new ideas for the experimental design of negative index materials see e.g., [16,30]. Apart from constructing negative index materials (with all the possible applications of these materials), there are other examples where homogenization leads to unexpected effective equations, describing astonishing effects; for example, resonance with surface plasmons can induce perfect transmission through sub-wavelength holes [8]. The resonance in small Helmholtz resonators can lead to dispersive effects in the propagation of sound waves [20].

The results of this contribution We consider a split ring geometry with perfectly conducting materials. The latter means that the metal inclusion is not described by a large permittivity; instead, the electric and the magnetic field vanish inside the metal and the metal interacts only via boundary conditions with the fields. We perform the homogenization limit, i.e., we consider solutions to the time-harmonic Maxwell's equations and study their limiting behavior as $\eta \to 0$, where $\eta > 0$ is the typical size of the single split ring. We obtain that the limiting fields are described by time-harmonic Maxwell's equations with effective parameters ε_{eff} and μ_{eff}. We provide formulas for these two parameters; the formulas yield, in particular, that μ_{eff} can be large and that it can have a negative real part for an appropriate choice of parameters (connected to resonance).

The above description shows that we are, to some extent, following the path that was chosen in [7] (only that now the metal is perfectly conducting). The homogenization of split ring microstructures is very complex: one has to deal, at the same time, with a large contrast and with singular geometries in the homogenization setting, furthermore with three-dimensional vector calculus to perform the analysis. For these reasons, the proofs in [7] are involved.

With the contribution at hand, we analyze the case of *perfectly conducting* split rings. Many aspects of the analysis simplify considerably in this case. Most importantly, the magnetic cell-problem is no longer a coupled problem: in the present situation, the cell-problem can be solved with four quite explicit shape functions. Obviously, we also profit from the developments in homogenization theory of the last 10 years, e.g. the fact that we understand the system now better using the notion of geometric averaging (introduced in [4]). Neglecting the more technical parts of previous proofs, such as the description of the scattering problem, we can describe here the heart of the negative μ_{eff}-result in a quite accessible way.

1.1. Mathematical Problem and Main Result

We study the time-harmonic Maxwell equations in a complex three dimensional geometry. Let $\Omega \subset \mathbb{R}^3$ be a domain and let $R \subset \Omega$ be a subdomain in which many split ring resonators are distributed periodically. The metallic split rings are modelled as perfect conductors. We denote the set that is occupied by metal by $\Sigma_\eta \subset R$. We therefore study the system

$$\operatorname{curl} E^\eta = \quad i\omega\mu_0 H^\eta \tag{1.1}$$

$$\operatorname{curl} H^\eta = -i\omega\varepsilon_0 E^\eta \tag{1.2}$$

in the domain $\Omega_\eta = \Omega \setminus \bar{\Sigma}_\eta$. The number $\omega > 0$ denotes the frequency, the complex numbers ε_0, μ_0 are the permittivity and the permeability in vacuum. To simplify notation, we consider only the sequence $\eta_j = \eta_0 2^{-j}$.

We have to specify boundary conditions on $\partial\Sigma_\eta$. To this end we consider trivial extensions of the fields. We denote the trivial extensions again as $E^\eta, H^\eta : \Omega \to \mathbb{C}^3$ (setting $E^\eta(x) := H^\eta(x) := 0$ for $x \in \Sigma_\eta$). We demand that the extensions satisfy

$$\operatorname{curl} E^\eta = i\omega\mu_0 H^\eta \tag{1.3}$$

in Ω. Equation (1.3) provides a boundary condition: it implies that the tangential components of E^η are vanishing on the boundary $\partial\Sigma_\eta$ (since the curl of E^η has no singular part). We remark that (1.3) also implies that

$$\operatorname{div} H^\eta = 0 \tag{1.4}$$

holds in Ω for the trivial extensions. Equation (1.4) implies that the normal component of H^η is vanishing on $\partial\Sigma_\eta$.

Theorem 1.1. *Let the geometry of the problem be given by domains $\Sigma_\eta \subset R \subset \Omega \subset \mathbb{R}^3$ as specified in Sect. 1.2. Let (E^η, H^η) be a sequence of solutions of (1.1)–(1.2) with the boundary conditions expressed by (1.3). We assume that the sequence of solutions satisfies*

$$\int_\Omega |E^\eta|^2 + |H^\eta|^2 \leq C \tag{1.5}$$

for some $C \geq 0$, independent of $\eta > 0$. Let $(\hat{E}, \hat{H})$ be the geometric limit fields as defined in (1.15) and (1.16). Then there holds

$$\operatorname{curl} \hat{E} = \quad i\omega\mu_0\hat{\mu}\hat{H} \tag{1.6}$$

$$\operatorname{curl} \hat{H} = -i\omega\varepsilon_0\hat{\varepsilon}\hat{E} \tag{1.7}$$

in the distributional sense on Ω, where the effective permittivity and permeability are given as

$$\hat{\varepsilon}(x) := \begin{cases} \varepsilon_{\text{eff}} & \text{for } x \in R \\ 1 & \text{for } x \in \Omega \setminus R \end{cases} \qquad \hat{\mu}(x) := \begin{cases} \mu_{\text{eff}} & \text{for } x \in R \\ 1 & \text{for } x \in \Omega \setminus R. \end{cases} \tag{1.8}$$

The tensor ε_{eff} is provided in (2.7) in terms of cell solutions $E^k(y)$, $k = 1, 2, 3$. The frequency dependent coefficient $\mu_{\text{eff}} = \mu_{\text{eff}}(\omega)$ is provided in (2.25) with the cell solutions $H^k(y)$, $k = 0, 1, 2, 3$.

On assumption (1.5) we note that the assumption on the energies can be removed for many choices of boundary conditions by a compactness argument. Assuming that, for a fixed boundary condition, the solution sequence does not satisfy (1.5), one rescales to obtain a normalized family of solutions. Theorem 1.1 is applied to the normalized sequence and provides the convergence to a solution of a limit system. The boundary conditions of the limit system are trivial due to the rescaling, hence, for non-degenerate ε_{eff} and μ_{eff}, the solution vanishes. A compactness argument provides a contradiction to the fact that the solution sequence was normalized. For the details of the argument we refer to [7].

On negative eigenvalues of μ_{eff}. For generic shapes of Σ_Y, the tensor μ_{eff} has negative eigenvalues for an appropriate choice of α. Let Σ_Y be such that the real numbers D_0, D_2, and $\int_{Y\setminus\Sigma_Y} H_2^0$ are non-vanishing (the three numbers are averages of cell-solutions). Formula (2.25) for μ_{eff} implies that, for α close to $\omega^2\mu_0\varepsilon_0 D_0/2$, we find $(\mu_{\text{eff}})_{22}$ arbitrarily large and with any sign. For a special geometry that generates negative eigenvalues see [7], the geometry can be used also in the case of a perfectly conducting material.

1.2. Geometry

The geometry is constructed in two steps. In the first step we describe the inclusion of the split ring in a single periodicity cell Y. In a second step we combine many microscopic structures to obtain a macroscopic geometry.

Microscopic geometry We start from the periodicity cube $Y = (0, 1)^3$; since we will always impose periodicity conditions on the cube Y, we may also regard it as

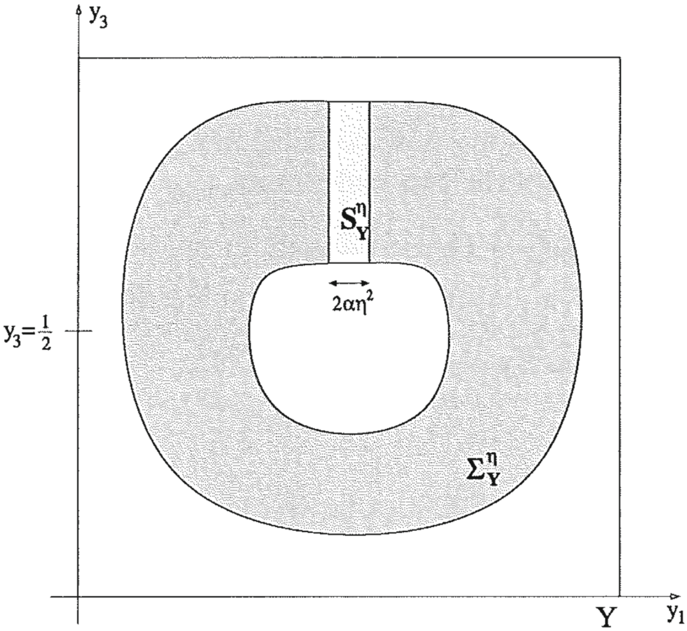

Fig. 1. The microscopic geometry, we show a cut with the (y_1, y_3)-plane. The cell $Y = (0, 1)^3$ contains the subdomain Σ_Y, which is topologically a full torus (the union of the two grey regions). The perfect conductor occupies $\Sigma_Y^\eta = \Sigma_Y \backslash S_Y^\eta$ (the dark region); it consists of Σ_Y with the slit region S_Y^η (light grey) removed. The slit has width $2\alpha\eta$, it hence vanishes in the limit $\eta \to 0$

the flat torus $\mathbb{T}^3$. Let a *closed* ring inside Y be given by a set $\Sigma_Y \subset Y$. We assume that Σ_Y is an open set with C^2-boundary, not touching the boundary of Y, i.e. with $\bar{\Sigma}_Y \cap \partial(0, 1)^3 = \emptyset$. We furthermore assume that Σ_Y is topologically a torus.

In order to define the split ring, we assume that $\Sigma_Y \backslash \{(y_1, y_2, y_3)|y_1 = 0, y_3 > \frac{1}{2}\}$ is an open, connected and simply connected set with Lipschitz boundary. The split ring is defined as

$$\Sigma_Y^\eta := \Sigma_Y \backslash \left\{(y_1, y_2, y_3)| -\alpha\eta^2 < y_1 < \alpha\eta^2, y_3 > \tfrac{1}{2}\right\}. \tag{1.9}$$

Our construction is such that the slit has the width $2\alpha\eta^2$ inside the single periodicity cell. For ease of notation, we furthermore assume that the slit is a cylinder:

$$\begin{aligned} S_Y^\eta &:= \Sigma_Y \cap \{(y_1, y_2, y_3)| -\alpha\eta^2 < y_1 < \alpha\eta^2, y_3 > \tfrac{1}{2}\} \\ &= \{(y_1, y_2, y_3)| -\alpha\eta^2 < y_1 < \alpha\eta^2,\ y_3 > \tfrac{1}{2},\ (0, y_2, y_3) \in \Sigma_Y\}. \end{aligned} \tag{1.10}$$

Macroscopic geometry We study electromagnetic waves in an open set $\Omega \subset \mathbb{R}^3$. Contained in Ω is a second domain $R \subset \Omega$ with $\bar{R} \subset \Omega$. The set R consists of meta-material, whereas on $\Omega \backslash R$ we have relative permeability and relative permittivity equal to unity (Fig. 1).

In order to define the microstructure in R, we use indices $k \in \mathbb{Z}^3$ and shifted small cubes $Y_k^\eta := \eta(k + Y)$. We denote by $\mathcal{K} := \{k \in \mathbb{Z}^3 | Y_k^\eta \subset R\}$ the set of indices k such that the small cube Y_k^η is contained in R. Here and in the following, in summations or unions over k, the index k takes all values in the index set $\mathcal{K}$. The number of relevant indices has the order $|\mathcal{K}| = O(\eta^{-3})$.

Using the unit-cell split rings $\Sigma_Y^\eta \subset Y$, we can now define the meta-material by setting

$$\Sigma_\eta := \bigcup_{k \in \mathcal{K}} \eta(k + \Sigma_Y^\eta) \subset R, \quad \Omega_\eta := \Omega \backslash \Sigma_\eta. \tag{1.11}$$

The set of slits is defined accordingly as $S_\eta := \bigcup_{k \in \mathcal{K}} \eta(k + S_Y^\eta)$. The above definition of Ω_η closes the description of the system (1.1)–(1.2). Note that we do not specify boundary conditions on $\partial\Omega$; if, for example, the solution sequence (E^η, H^η) satisfies a Dirichlet condition for fixed boundary data, the limit functions $(\hat{E}, \hat{H})$ will satisfy the same boundary condition.

1.3. Two-Scale Limits and Geometric Averaging

Our aim is to study properties of the sequence (E^η, H^η) of solutions to (1.1)–(1.2). We obtain these properties by characterizing limits. The most useful object is the two-scale limit of the sequence.

Two-scale limits Since E^η and H^η are, by assumption, bounded in $L^2(\Omega)$ we can, after extraction of a subsequence, consider the two-scale limits as $\eta \to 0$:

$$E^\eta(x) \rightharpoonup E_0(x, y) \text{ weakly in two scales,} \tag{1.12}$$

$$H^\eta(x) \rightharpoonup H_0(x, y) \text{ weakly in two scales,} \tag{1.13}$$

for some limit functions $E_0, H_0 \in L^2(\Omega \times Y, \mathbb{C}^3)$.

Geometric averaging There are (at least) two possibilities to average a function $u : Y \to \mathbb{C}^3$. The standard averaging procedure is the volumetric average, given by the integral over Y. We introduce here another averaging procedure, which associates to curl-free fields $u : Y \to \mathbb{C}^3$ their line integrals.

In the subsequent definition, Σ is assumed to be connected, but we do not demand Σ to be simply connected. Actually, inclusions Σ that are not simply connected are our motivation to modify the original definition of a geometric average which was used in [4]. In contrast to the slightly generalized geometric average in [29] which combines the advantages of both approaches, we consider here only the case that the inclusion does not touch the sides of the cube, $\bar{\Sigma} \subset (0, 1)^3$. We will apply the lemma with $\Sigma = \Sigma_Y$.

The definition uses the unit vectors $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$, and three of the edges, parametrized with $\gamma_k : [0, 1] \to Y, t \mapsto te_k$. We note that, since opposite sides are identified, the cube has only these edges; they are in the interior of $Y \backslash \bar{\Sigma}$. The geometric average is defined with line integrals. We emphasize that line integrals are not well-defined for general functions of class H^1, but they are well-defined for functions with vanishing curl. The argument exploits that line integrals of smooth curl-free functions are invariant under deformations; for more details on the argument we refer to Definition 2.7 of [29].

Definition 1.2. *(Geometric averaging)* Let $\Sigma \subset \bar{\Sigma} \subset (0,1)^3$ be a connected subset of the flat torus Y and let $H : Y\setminus\Sigma \to \mathbb{C}^3$ be a field of class $H^1_\sharp(Y\setminus\Sigma, \mathbb{C}^3)$ with $\operatorname{curl}_y H = 0$ in $Y\setminus\bar{\Sigma}$. The geometric average of H is a vector $\oint H \in \mathbb{C}^3$; for $k = 1, 2, 3$, the k-th component of $\oint H$ is defined as a line integral over the edge $\gamma_k : [0,1] \to Y, t \mapsto te_k$,

$$\left(\oint H\right)_k := \int_{\gamma_k} H \cdot \tau, \tag{1.14}$$

where $\tau = e_k$ is the tangential vector of the curve γ_k.

We remark that, due to the periodicity of H, the edge γ_k of the torus Y is inside the domain $Y\setminus\Sigma$. More important is the following observation: for every curve $\tilde{\gamma}_k$ that connects a point $y_0 \in \partial(0,1)^3$ with $y_0 + e_k \in \partial(0,1)^3$ and that can be obtained with an homotopy in $Y\setminus\Sigma$ from the curve γ_k along the edge, the integral has the same value: $\int_{\tilde{\gamma}_k} H \cdot \tau = \int_{\gamma_k} H \cdot \tau$. This is a consequence of $\operatorname{curl}_y H = 0$ in $Y\setminus\bar{\Sigma}$. This fact also shows that the line integral is well defined even though we only assumed $H \in H^1(Y\setminus\Sigma)$.

Geometric limits Starting from $L^2(\Omega)$-bounded sequences E^η and H^η we want to define, in the appropriate sense of averaging, limit functions $\hat{E}$ and $\hat{H}$ in $L^2(\Omega)$.

Regarding the sequence E^η, we choose to define the limit $\hat{E}$ as the weak limit of E^η in $L^2(\Omega)$,

$$\hat{E} := w - \lim_{\eta} E^\eta, \quad \hat{E}(x) = \int_Y E_0(x, y)\, dy. \tag{1.15}$$

For convenience we recalled in the second equality of (1.15) a property of the two-scale limit E_0.

Instead, regarding H^η, we start from the two-scale limit $H_0(x, y)$ as defined in (1.13) and define $\hat{H}$ as the geometric average of that limit according to Definition 1.2,

$$\hat{H}(x) := \oint H_0(x, .). \tag{1.16}$$

A property of the geometric average The above definition of a geometric average is justified by the fact that geometric averages appear in the macroscopic equations. The underlying reason for this fact is the following property:

Lemma 1.3. (A property of the geometric average) *Let $\Sigma \subset Y$ be as in Definition 1.2. We consider $H \in H^1_\sharp(Y\setminus\Sigma, \mathbb{C}^3)$ with* $\operatorname{curl}_y H = 0$ *in $Y\setminus\bar{\Sigma}$. On a second function $E \in H^1_\sharp(Y, \mathbb{C}^3)$ we assume* $\operatorname{curl}_y E = 0$ *in Y and $E = 0$ on Σ. Then there holds*

$$\int_{Y\setminus\Sigma} H \wedge E = \left(\oint H\right) \wedge \int_Y E. \tag{1.17}$$

Proof. Since E has vanishing curl in Y, we can write E as a gradient of some function $\phi \in H^1((0,1)^3, \mathbb{C})$. Since E vanishes on the connected set Σ, the function ϕ is constant on Σ. We may assume the constant to be zero, hence

$$E = \nabla\phi \text{ in } (0,1)^3, \quad \phi = 0 \text{ in } \Sigma.$$

We emphasize that ϕ is, in general, *not* a periodic function. We calculate the mismatch for two opposite points $z_0, z_1 := z_0 + e_k \in \partial(0,1)^3$. Connecting the two points with the curve $\gamma : t \mapsto z_0 + te_k$ we find

$$\phi(z_1) - \phi(z_0) = \int_\gamma \nabla\phi \cdot \tau = \int_\gamma E \cdot \tau = \int_Y E \cdot e_k.$$

The last equality requires some explanation, which we give for $k = 3$: The line integral over γ is independent of the point $z_0 \in (0,1)^2 \times \{0\}$ since E has vanishing curl (we recall that the sides are identified). We can integrate over all z_0 and obtain the last equality.

The result of the above calculation is that the difference of the ϕ-values on opposite faces is given by the volume average of E.

We can now calculate the left hand side of (1.17) with an integration by parts. Using that $\operatorname{curl}_y H = 0$ holds in the set where $\phi \neq 0$, we find

$$\int_{Y\setminus\Sigma} H \wedge E = \int_{(0,1)^3} H \wedge \nabla\phi = \int_{\partial(0,1)^3} H \wedge \nu\, \phi.$$

We evaluate this boundary integral. For every $k \le 3$ we consider the faces $F_k^- := \{y_k = 0\}$ with normal $\nu = -e_k$ and $F_k^+ := \{y_k = 1\}$, with normal $\nu = e_k$. This yields, by periodicity of H,

$$\begin{aligned}\int_{Y\setminus\Sigma} H \wedge E = \int_{\partial Y} H \wedge \nu\, \phi &= \sum_{k=1}^{3} \left[\int_{F_k^+} H \wedge e_k\, \phi - \int_{F_k^-} H \wedge e_k\, \phi \right] \\ &= \sum_{k=1}^{3} \left(\int_{F_k^+} H \wedge e_k \right) \left(\int_Y E \cdot e_k \right).\end{aligned}$$

On the face F_k^+, the integrand $H \wedge e_k$ consist of the two tangential components of H. The face F_k^+ can be written as a union of straight lines. Recalling that all line integrals over tangential components of H coincide because of $\operatorname{curl}_y H = 0$, we obtain that the F_k^+-integral coincides with the line integrals of the geometric average (1.14). We can therefore write

$$\int_{Y\setminus\Sigma} H \wedge E = \sum_{k=1}^{3} \left(\oint H \right) \wedge e_k \left(\int_Y E \cdot e_k \right) = \left(\oint H \right) \wedge \int_Y E.$$

This provides (1.17) and hence the lemma. □

We note that the original definition of a geometric average in [4] was expressed in terms of formula (1.17). This definition was also used, e.g., in [21]. Our new definition is in terms of line integrals and has the advantage that more complex obstacles Σ can be treated, e.g., obstacles where $Y\setminus\Sigma$ is not simply connected.

2. Cell Problems and Effective Coefficients

It is a standard procedure to collect the equations for the two-scale limit of a sequence of solutions. In this section, we collect the cell problems for E_0 and for H_0 from (1.12)–(1.13). The important step is to find linearly independent solutions to the cell-problems such that the two-scale limits E_0 and H_0 can be written as linear combinations of these basis functions.

2.1. Cell Problem for E_0

We investigate the two-scale limit function E_0 of (1.12). For $x \in R$ the map $Y \ni y \mapsto E_0(x, .) \in \mathbb{C}^3$ solves the following equations:

$$\operatorname{curl}_y E_0 = 0 \text{ in } Y, \tag{2.1}$$

$$\operatorname{div}_y E_0 = 0 \text{ in } Y \setminus \bar{\Sigma}_Y, \tag{2.2}$$

$$E_0 = 0 \text{ in } \Sigma_Y, \tag{2.3}$$

$$E_0 \text{ is periodic in } Y. \tag{2.4}$$

This set of equations follows easily from the properties of two-scale convergence: (1.3) implies (2.1) and (1.2) implies (2.2). Equation (2.3) follows from the fact that we considered trivially extended fields E^η and that the slit domains $S_Y^\eta \subset Y$ vanish in the limit $\eta \to 0$. The periodicity (2.4) is always satisfied by construction of two-scale limits.

The equations can be solved with the help of three shape functions. We observe that the first equation implies that $E_0(x, .) = E(x) + \nabla_y \phi(x, .)$, where ϕ is a scalar periodic potential in $H^1_\sharp(Y, \mathbb{C})$ and $E(x)$ denotes the average of $E_0(x, \cdot)$ on the unit cell. The second equation implies that ϕ is harmonic in $Y \setminus \bar{\Sigma}$, the third equation yields that $\phi(y) + E(x) \cdot y$ is constant on the connected subset Σ. Therefore, for a given average electric field $E(x)$, the periodic function ϕ is determined uniquely by the affine boundary values on $\partial\Sigma$, we demand $\phi(y) = -E(x) \cdot y$ for $y \in \Sigma$. In the above construction, there were only 3 degrees of freedom for E_0, the components of the vector $E(x)$ of the volume average. The solution space to (2.1)–(2.4) is therefore three-dimensional. We denoted the volume average by $E(x)$ since the Y-average of the two-scale limit E_0 coincides with the weak limit E.

We collect our results. The two-scale limiting electric field $E_0(x, .)$ can be written as a linear combination

$$E_0(x, y) = \sum_{k=1}^{3} E_k(x)\, E^k(y), \tag{2.5}$$

where the real valued shape functions $E^k(y) := e_k + \nabla \phi^k(y)$ are given in terms of ϕ^k, which is the unique solution in $H^1_\sharp(Y)$ of

$$\Delta \phi^k = 0 \text{ in } Y \setminus \bar{\Sigma}_Y\,, \qquad \phi^k = -y_k \text{ on } \bar{\Sigma}_Y\,. \tag{2.6}$$

The tensor $\boldsymbol{\varepsilon}_{\mathbf{eff}}$ By construction, the fields E^1, E^2, and E^3 satisfy $\int_Y E^k \cdot e_l = \delta_{kl}$ and form a basis of the space of solutions for the E_0-cell problem. However, they are not orthonormal with respect to the usual scalar product in $L^2(Y)$. We define the tensor $\varepsilon_{\text{eff}} := (\varepsilon_{\text{eff}})_{kl} \in \mathbb{R}^{3\times 3}$ by setting

$$(\varepsilon_{\text{eff}})_{kl} := \int_Y E^k(y) \cdot E^l(y)\, \mathrm{d}y. \tag{2.7}$$

2.2. Cell-Problem for H_0

The magnetic field $H_0(x, .)$ satisfies

$$\operatorname{curl}_y H_0 = 0 \text{ in } Y \setminus \bar{\Sigma}_Y, \tag{2.8}$$
$$\operatorname{div}_y H_0 = 0 \text{ in } Y, \tag{2.9}$$
$$H_0 = 0 \text{ in } \Sigma_Y, \tag{2.10}$$
$$H_0 \text{ is periodic in } Y. \tag{2.11}$$

Once more, this cell problem is an immediate consequence of two-scale convergence properties and the definition of H_0 in (1.13). Equation (2.8) follows from (1.2) and Eq. (2.9) follows from (1.4).

With respect to the cell-problem (2.1)–(2.4) for E_0, the role of div_y and curl_y are interchanged (in terms of the domain). This difference leads to a much richer structure of the solution space to (2.8)–(2.11); due to the fact that $Y \setminus \Sigma_Y$ is not simply connected but contains *one* nontrivial curve, the solution space of (2.8)–(2.11) has *one* extra dimension—it is four-dimensional.

We derive this result in the following proposition, which is central to the analysis of the ring geometry: for closed rings, the solution space for the H-problem is four-dimensional. The extra dimension in the H-problem makes resonance effects possible and leads to the negative effective permeability for appropriate parameters.

The result of the subsequent proposition is classical vector calculus. We construct non-trivial curl-free fields in domains that are not simply connected. Due to its importance in this study, we include the quite elementary proof for completeness.

Proposition 2.1. (The magnetic cell-problem) *Let $\Sigma_Y \subset Y$ be as described in Sect. 1.2, topologically a full torus. Then the solution space to problem* (2.8)–(2.11) *is four-dimensional and spanned by four shape functions $H^k(y)$, $k = 0, 1, 2, 3$. The shape functions are uniquely determined as solutions of* (2.8)–(2.11) *with the following normalization: for a closed curve $\gamma_0 : [0, 1] \to (0, 1)^3 \setminus \Sigma_Y$ with tangential vector τ which winds once through the closed ring Σ_Y, there holds*

$$\oint H^k = e_k, \quad \int_{\gamma_0} H^k \cdot \tau = 0 \quad \textit{for } k \in \{1, 2, 3\}, \textit{ and} \tag{2.12}$$

$$\oint H^0 = 0, \quad \int_{\gamma_0} H^0 \cdot \tau = 1. \tag{2.13}$$

Proof. In this proof, which concerns only functions on the unit cube Y, we write Σ instead of Σ_Y for brevity.

Step 1: Construction of H^k, $k = 1, 2, 3$ For fixed $k \in \{1, 2, 3\}$, our aim is to construct H^k as we constructed E^k in the lines before (2.6). We use the ansatz $H^k(y) = e_k + \nabla\phi^k(y)$, now ϕ^k is a periodic solution of the *Neumann*-problem

$$\Delta\phi^k = 0 \text{ in } Y\backslash\bar{\Sigma}_Y, \quad \nu \cdot \nabla\phi^k = -\nu \cdot e_k \text{ on } \partial\Sigma_Y, \tag{2.14}$$

where ν is the exterior normal vector field for Σ. This set of equations can be solved with $\phi^k \in H^1_\sharp(Y\backslash\Sigma)$ with the Lax-Milgram theorem (prescribing e.g. that the average of ϕ^k vanishes). We set $H^k(y) = e_k + \nabla\phi^k(y)$ for $y \in Y\backslash\bar{\Sigma}$ and $H^k(y) = 0$ for $y \in \bar{\Sigma}$.

Let us check that H^k is indeed a solution of the cell-problem. As a gradient field, H^k satisfies (2.8). Since ϕ^k is harmonic, H^k satisfies (2.9) in $Y\backslash\bar{\Sigma}$. The boundary condition for ϕ^k implies that the normal component of H^k vanishes on $\partial\Sigma$; this provides (2.9) in Y. Properties (2.10) and (2.11) hold by construction.

We next determine the geometric average of H^k by calculating line integrals. Along the curve γ_j, $j = 0, 1, 2, 3$, we obtain

$$\int_{\gamma_j} H^k \cdot \tau = \int_{\gamma_j} (e_k + \nabla\phi^k) \cdot \tau = \delta_{kj}\,. \tag{2.15}$$

This provides the normalization property (2.12).

Step 2: Construction of H^0 The construction of H^0 requires a refined construction, since we have to exploit the fact that the complement of the full torus is not simply connected. In a first step, we choose a smooth surface (with boundary) $D \subset Y\backslash\bar{\Sigma}$ that "closes the hole of the torus". We demand that the boundary curve $\bar{D} \cap \partial\Sigma$ is a closed curve and that $Y\backslash(\bar{\Sigma} \cup D)$ is simply connected.

For a function $\phi : Y\backslash\Sigma \to \mathbb{R}$ that possesses a trace on both sides of D, we denote by $[\phi]_D : D \to \mathbb{R}$ the jump of the function ϕ (we fix an arbitrary convention for the sign). We use the affine function space

$$Z := \left\{ \phi \in H^1_\sharp(Y\backslash(\bar{\Sigma} \cup D)) \,\middle|\, [\phi]_D = 1 \right\}. \tag{2.16}$$

On Z, we study the minimization problem for the Dirichelet energy. Find $\phi \in Z$ such that

$$I(\phi) := \frac{1}{2}\int_{Y\backslash(\bar{\Sigma}\cup D)} |\nabla\phi|^2 = \min_{\varphi\in Z} I(\varphi)\,. \tag{2.17}$$

We emphasize that we evaluate $\nabla\phi$ on the set $Y\backslash(\bar{\Sigma} \cup D)$, where the gradient exists by definition of Z because of $\phi \in Z$. In the definition of I, we integrate the squared norm of this function (in other words, we integrate the squared norm of the regular part of $\nabla\phi$). Minimization of the convex functional in (2.17) is possible with the direct method.

Let $\phi^0 \in Z$ be a solution of the minimization problem (2.17). We set $H^0 := \nabla\phi^0$ in $Y\backslash(\bar{\Sigma} \cup D)$. We extend trivially, setting $H^0(y) = 0$ for $y \in \bar{\Sigma}$.

It remains to check the properties of H^0. For every function $\rho \in C^1_\sharp(Y\setminus\Sigma)$ and every $\varepsilon \in \mathbb{R}$, the function $\phi^0 + \varepsilon\rho$ is contained in Z (the function $\phi^0 + \varepsilon\rho$ still has the jump 1 across D). This implies the Euler–Lagrange equation

$$\int_{Y\setminus\bar{\Sigma}} \nabla\phi^0 \cdot \nabla\rho = 0. \tag{2.18}$$

The Euler–Lagrange equation implies, in a first step, $\Delta\phi^0 = 0$ in $Y\setminus(\bar{\Sigma}\cup D)$. In a second step (with an integration by parts), (2.18) implies the boundary condition $\nu\cdot\nabla\phi^0 = 0$ along $\partial\Sigma$ (ν the normal on $\partial\Sigma$) and $[\nu\cdot\nabla\phi^0] = 0$ along D (ν the normal on D). These properties imply that H^0 satisfies (2.9) in Y. As a gradient, H^0 satisfies (2.8) in $Y\setminus(\bar{\Sigma}\cup D)$. Equations (2.10) and (2.11) are satisfied by construction.

It remains to check (2.8) across the interface D. In a neighborhood of a point $y \in D$, we can consider the function $\tilde{\phi}^0$, defined as $\tilde{\phi}^0 = \phi^0$ on one side of D and $\tilde{\phi}^0 = \phi^0 + 1$ on the other side of D. The construction is made such that $\tilde{\phi}^0$ has a vanishing jump across D. This fact implies that the function $\tilde{\phi}^0$ is of class H^1 in a neighborhood of y. By (2.18), $\tilde{\phi}^0$ is harmonic and hence locally of class H^2. This implies that the traces of $\nabla\phi^0 = \nabla\tilde{\phi}^0$ have no jumps across D; we conclude that H^0 is locally of class H^1. This yields (2.8) in $Y\setminus\bar{\Sigma}$.

The calculation of line integrals is as in (2.15) of Step 1. Line integrals of H^0 over the curves γ_j, $j = 1, 2, 3$, vanish by periodicity of ϕ^0. Instead, the line integral over γ_0 is (we assume that $\gamma_0 : [0, 1] \to Y$ starts and ends in a point $y_0 \in D$ and is entirely contained in $Y\setminus(\bar{\Sigma}\cup D)$)

$$\int_{\gamma_0} H^0 \cdot \tau = \int_{\gamma_0} \nabla\phi^0 \cdot \tau = \lim_{t\nearrow 1} \phi^0(\gamma_0(t)) - \lim_{t\searrow 0} \phi^0(\gamma_0(t)) = \pm 1. \tag{2.19}$$

Upon reversing the sign, we obtain the normalization property (2.13).

Step 3: Conclusion of the proof We have constructed four solutions H^k, $k = 0, 1, 2, 3$, to the cell problem. The normalizations (2.12)–(2.13) show that the solutions are linearly independent, hence the solution space is at least four-dimensional.

In order to show that the solution space has at most four dimensions, it suffices to show that if solves the cell-problem with vanishing normalization, i.e., with $\oint u = 0$ and $\int_{\gamma_0} u \cdot \tau = 0$, then u necessarily vanishes.

To show this result, we construct a scalar potential Φ for the vector field u in the classical way (as in the proof of the statement "closed one-forms are exact", known as "Poincaré lemma"). For any point $y \in Y\setminus\bar{\Sigma}$ we choose a smooth curve $\tilde{\gamma}$ with tangential vector τ that connects 0 and y, and set $\Phi(y) := \int_{\tilde{\gamma}} u\cdot\tau$. Due to $\operatorname{curl}_y u = 0$ and the normalizations, the potential Φ is well-defined (independent of the choice of $\tilde{\gamma}$) and periodic, it satisfies $u = \nabla\Phi$. Since u has a vanishing divergence, Φ is harmonic. Since u vanishes on Σ, the potential Φ satisfies a homogeneous Neumann condition on $\partial\Sigma$. Integration by parts gives $\int_{Y\setminus\Sigma} \nabla\Phi\cdot\nabla\Phi = 0$ and it follows that Φ is constant. In particular, the gradient $u = \nabla\Phi$ vanishes. This shows that the four normalization conditions determine u uniquely. □

The fast-scale microscopic magnetic field $H_0(x, .)$ is a solution of the magnetic cell problem. Proposition 2.1 implies that it can hence be written as a linear combination of the four cell solutions:

$$H_0(x,y) = \sum_{k=0}^{3} h_k(x)\, H^k(y). \tag{2.20}$$

The amplitude factor $h_k(x)$ tells us how much of the cell solution H^k is present in a macroscopic point $x \in R$. The result of the slit analysis of the next section will be an expression for $h_0(x)$ in terms of $h_1(x)$, $h_2(x)$, $h_3(x)$.

The Tensor $\boldsymbol{\mu}_{\mathbf{eff}}$ For points $x \in R$, the effective permeability is defined such that Y-averages can be expressed by geometric averages. More precisely, we want to define μ_{eff} such that, for $x \in R$,

$$H(x) = \mu_{\text{eff}}\, \hat{H}(x). \tag{2.21}$$

Let us investigate this relation with the help of (2.20). The weak limit is the arithmetic average of H_0, which takes the form

$$H(x) = \int_{Y\setminus\Sigma_Y} H_0(x,y)\mathrm{d}y = \sum_{k=0}^{3} h_k(x) \int_{Y\setminus\Sigma_Y} H^k. \tag{2.22}$$

By (1.16), the geometric average is

$$\hat{H}(x) = \oint H_0(x,.) = \sum_{k=0}^{3} h_k(x) \oint H^k = (h_1(x), h_2(x), h_3(x)). \tag{2.23}$$

Because of (2.21), our aim is to define the effective permeability tensor μ_{eff} such that

$$\left(\int_{Y\setminus\Sigma_Y} H^0\right) h_0(x) + \sum_{k=1}^{3} \left(\int_{Y\setminus\Sigma_Y} H^k\right) h_k(x) = \mu_{\text{eff}} \cdot (h_1, h_2, h_3)(x). \tag{2.24}$$

We anticipate the result of the next section. For constants $D_k \in \mathbb{R}$ we derive in (3.19) the formula

$$h_0(x) = \frac{\omega^2 \mu_0 \varepsilon_0}{2\alpha - \omega^2 \mu_0 \varepsilon_0 D_0} \sum_{k=1}^{3} D_k h_k(x).$$

We demand that (2.24) holds for every vector $(h_1, h_2, h_3)(x)$ if $h_0(x)$ is given as above. This leads to the choice, for $1 \le j, k \le 3$,

$$(\mu_{\text{eff}})_{jk} = \left(\int_{Y\setminus\Sigma_Y} H^k\right)_j + \frac{\omega^2 \mu_0 \varepsilon_0}{2\alpha - \omega^2 \mu_0 \varepsilon_0 D_0} D_k \left(\int_{Y\setminus\Sigma_Y} H^0\right)_j. \tag{2.25}$$

The numbers D_k are defined in (3.9).

3. Slit Analysis

The slit analysis is the core of the analysis of the split ring geometry. At this point, we make the connection between the capacitor (the slits S_η) and the inductance (the rings Σ_η). The result of the analysis is a relation between the strength h_0 of the special field H^0 (which points through the ring) and the other fields H^k, $k = 1, 2, 3$.

On a more technical level, our analysis is based on the following idea: we calculate, in two different ways, the average of the field E^η in the slit S_η. More precisely, we investigate, for a smooth cut-off function θ, limits of the expression

$$\int_{S_\eta} \frac{1}{\eta} E^\eta(x) \cdot e_1 \, \theta(x) \, \mathrm{d}x. \tag{3.1}$$

Once we have determined the limit of the expression in (3.1), we can conclude also a slightly stronger statement. To formulate this statement, we define a Radon measure $m_\eta \in \mathcal{M}(\Omega)$ with the help of the density of (3.1), and we set

$$m_\eta := \mathbf{1}_{S_\eta} \frac{1}{\eta} E^\eta \cdot e_1 \, \mathcal{L}^3 \,, \tag{3.2}$$

where $\mathcal{L}^3$ is the three dimensional Lebesgue measure and the function $\mathbf{1}_{S_\eta}$ is the characteristic function of the set S_η. The Cauchy–Schwarz inequality implies the boundedness of

$$\|m_\eta\|_{\mathcal{M}(\Omega)} = \left\| \mathbf{1}_{S_\eta} \frac{1}{\eta} E^\eta(x) \cdot e_1 \right\|_{L^1(\Omega)} \leq \left(\int_{S_\eta} |E^\eta|^2 \right)^{1/2} \left(\int_{S_\eta} \eta^{-2} \right)^{1/2} \leq C \,,$$

where we used in the last step that the volume of S_η is of order η^2. We can extract a subsequence $\eta \to 0$ and find a limit measure $m \in \mathcal{M}(\bar{\Omega})$ such that $m_\eta \rightharpoonup m$ weak-$*$ in the sense of measures.

Calculating the limit of the expression in (3.1), which is nothing else than $\int_\Omega \theta \, dm_\eta$, we determine the distributional limit of the sequence m_η. Since the distributional limit and the measure limit coincide, we have thus determined the limit measure m. In this way, we will compute a first expression for m in Sect. 3.1. We will then compute another formula for m by a different approach in Sect. 3.2. Comparison of the two expressions for m provides the desired formula (3.19).

3.1. Large Circles and E-Equation

In order to state the formulas for the limit measure m, we need the four effective quantities D_0, D_1, D_2, D_3. They are defined with the help of a special field $\chi : Y \to \mathbb{R}^3$.

Construction of Test-Functions We want to construct a field $\chi : Y \to \mathbb{R}^3$ that has similar character as our special H-field H^0; our specific requirement here is that $\mathrm{curl}_y \chi = e_1$ inside the slit of the ring.

Lemma 3.1. (The field χ) *There exists a number $\eta_0 > 0$ and a field $\chi \in L^2(Y, \mathbb{R}^3)$ with*

$$\tau := \mathrm{curl}_y\, \chi \in L^2(Y, \mathbb{R}^3), \tag{3.3}$$

$$\tau \equiv 0 \textit{ in } Y \backslash \Sigma \textit{ and } \tau \equiv e_1 \textit{ in } S_{\eta_0}, \tag{3.4}$$

with the normalization

$$\int_Y E^k(y) \wedge \chi(y)\, \mathrm{d}y = 0 \tag{3.5}$$

for $k = 1, 2, 3$.

We note that, by our assumption on the slits S_η, there holds $\tau \equiv e_1$ in S_η for every $\eta \le \eta_0$.

Proof. We start by constructing a field $\tau : Y \to \mathbb{R}^3$ which lives on the closed ring. For some η_0, we demand $\tau \equiv e_1$ in the slit S_{η_0}. We next choose an extension of τ to all of Y with the properties

$$\tau = 0 \quad \text{in } Y \backslash \bar{\Sigma}, \tag{3.6}$$

$$\mathrm{div}_y\, \tau = 0 \quad \text{in } Y, \tag{3.7}$$

Since the field τ has a vanishing divergence, it also has a vanishing average due to $\int_Y \tau \cdot e_l = \int_Y \tau \cdot \nabla y_l = 0$. This implies that τ possesses a periodic vector potential χ with $\mathrm{curl}_y\, \chi = \tau$. The potential χ is already the desired field, it only remains to choose an appropriate average value of χ in order to satisfy the normalization (3.5).

We start by calculating the wedge-product of χ with E^k, using $E^k(y) = e_k + \nabla\phi^k(y)$ and $\phi^k(y) = -y_k$ from (2.6):

$$\begin{aligned}\int_Y E^k(y) \wedge \chi(y)\, \mathrm{d}y &= \int_Y (e_k + \nabla\phi^k(y)) \wedge \chi(y)\, \mathrm{d}y \\ &= e_k \wedge \int_Y \chi(y)\, \mathrm{d}y - \int_Y \phi^k(y)\, \mathrm{curl}_y\, \chi(y)\, \mathrm{d}y \\ &= e_k \wedge \int_Y \chi(y)\, \mathrm{d}y + \int_\Sigma y_k\, \tau(y)\, \mathrm{d}y.\end{aligned}$$

The result contains the volume average P and the weighted τ-integrals N_{kl} for $k, l = 1, 2, 3$:

$$P := \int_Y \chi(y)\, \mathrm{d}y, \quad N_{kl} := \int_\Sigma y_k\, \tau_l(y)\, \mathrm{d}y.$$

We observe that, for $1 \le k, l \le 3$, there holds

$$0 = -\int_Y y_k y_l \,\mathrm{div}_y \tau(y)\,\mathrm{d}y = \int_\Sigma \nabla(y_k y_l)\cdot\tau(y)\,\mathrm{d}y$$
$$= \int_\Sigma y_k \tau_l(y) + y_l \tau_k(y)\,\mathrm{d}y = N_{kl} + N_{lk}.$$

Hence the matrix $N \in \mathbb{R}^{3\times 3}$ is skew-symmetric. It can therefore be expressed with a vector $P \in \mathbb{R}^3$ such that

$$e_l \cdot \int_Y E^k(y) \wedge \chi(y)\,\mathrm{d}y = e_l \cdot (e_k \wedge P) - N_{kl} = 0 \tag{3.8}$$

for $1 \le k, l \le 3$. This is the normalization condition (3.5).

Given the field τ (which provides the matrix N), we can choose a vector potential χ with the average P as in (3.8). This provides the field χ with all the desired properties. □

With the function χ of Lemma 3.1 and the cell solutions H^k we define coefficients $D_k \in \mathbb{R}$ as

$$D_k := \int_Y H^k(y)\cdot\chi(y)\,\mathrm{d}y. \tag{3.9}$$

Calculation of Limits This subsection is devoted to the first way of calculating the limit measure m, starting from the expression for the E^η-integral of (3.1). With χ of Lemma 3.1 and a cut-off function $\theta \in \mathcal{D}(R)$ we construct the function $x \mapsto \theta(x)\chi(x/\eta)$ and use it as a test-function in (1.1). Using two-scale convergence and (2.20) we obtain

$$\int_\Omega \mathrm{curl}\, E^\eta(x)\cdot\theta(x)\chi(x/\eta)\,\mathrm{d}x = i\omega\mu_0 \int_\Omega H^\eta(x)\cdot\theta(x)\chi(x/\eta)\,\mathrm{d}x$$
$$\to i\omega\mu_0 \int_\Omega \sum_{k=0}^{3} h_k(x)\theta(x) \int_Y H^k(y)\cdot\chi(y)\,\mathrm{d}y\,\mathrm{d}x$$
$$= i\omega\mu_0 \int_\Omega \sum_{k=0}^{3} D_k\, h_k(x)\theta(x)\,\mathrm{d}x.$$

On the other hand, we can evaluate the left hand side with an integration by parts,

$$\int_\Omega \mathrm{curl}\, E^\eta(x)\cdot\theta(x)\chi(x/\eta)\,\mathrm{d}x = \int_\Omega E^\eta(x)\cdot\mathrm{curl}\,(\theta(x)\chi(x/\eta))\,\mathrm{d}x$$
$$= \int_{S_\eta} E^\eta \cdot e_1 \theta \frac{1}{\eta} + \int_\Omega E^\eta \cdot (\nabla\theta \wedge \chi(./\eta)).$$

The first term is the slit integral of the E-field which we want to evaluate. The second term on the right hand side can be calculated using two-scale convergence. As $\eta \to 0$, there holds

$$\int_\Omega E^\eta \cdot (\nabla\theta \wedge \chi(./\eta)) \to \sum_{k,l=1}^{3} \int_\Omega E_k(x)\partial_l\theta(x) \int_Y E^k(y)\cdot(e_l \wedge \chi(y))\,dy\,dx\,.$$

The Y-integral vanishes by the normalization (3.5) in the construction of the field χ. Our result is the following limit expression for the slit integral of (3.1): as $\eta \to 0$, there holds

$$\int_{S_\eta} E^\eta(x) \cdot e_1\, \theta(x) \frac{1}{\eta}\, \mathrm{d}x \to i\omega\mu_0 \int_\Omega \sum_{k=0}^{3} D_k\, h_k(x)\, \theta(x)\, \mathrm{d}x\,. \tag{3.10}$$

With (3.10), we have calculated the distributional limit of the measures m_η of (3.2). The distributional limit coincides with the measure valued weak-$*$ limit, we have therefore determined the limit measure m:

$$m = i\omega\mu_0 \sum_{k=0}^{3} D_k\, h_k(x)\, \mathrm{d}\mathcal{L}^3. \tag{3.11}$$

3.2. Small Circles and H-Equation

The aim of this section is to calculate slit integrals of (3.1) in another way. This will provide a new formula for the limit measure m. Comparison with (3.11) yields the desired relation on the macroscopic factors $h_j(x)$.

Let us sketch the idea behind this second calculation: We think of a two-dimensional disk $D \subset Y$ that does not touch the split ring, $D \subset Y \setminus \Sigma_Y^\eta$. To obtain a nontrivial result, we assume that D lies (partially) in the slit S_η; more precisely, we assume that the boundary line of D is a closed curve $\Gamma \subset Y \setminus \Sigma_Y$, which is, topologically, a nontrivial curve in $Y \setminus \Sigma_Y$.

The Stokes theorem implies that the (two-dimensional) D-integral over $(\operatorname{curl} H) \cdot n$ (where n is the normal to D) coincides with the (one-dimensional) Γ-integral over $H \cdot \tau$ (where τ is a tangential vector on Γ). Using equation (1.2) we obtain, loosely speaking, that the D-integral over $-i\omega\varepsilon_0 E \cdot e_1$ coincides with the Γ-line-integral over H. The first is the slit integral (up to the factor $-i\omega\varepsilon_0$ and a factor 2α for the width of the slit). The latter is the amplitude of the part of H that winds through the ring—essentially h_0. We recover this result by a rigorous calculation in (3.17).

Construction of Test-Functions We now perform the argument rigorously with the help of test-functions. We recall that the slit S_Y^η in the single cell Y is contained in a slab F_Y^η of width $2\alpha\eta^2$, in formulas $S_Y^\eta \subset F_Y^\eta := \{(y_1, y_2, y_3) \in Y | -\alpha\eta^2 < y_1 < \alpha\eta^2\}$. We define the test function $\sigma^\eta : Y \to \mathbb{R}^3$ as follows:

$$\sigma^\eta(y) := \begin{cases} 0 & \text{for } y \in Y \setminus F_Y^\eta \\ e_1 & \text{for } y \in S_Y^\eta \\ \sigma_\delta(y_2, y_3) e_1 & \text{for } y \in F_Y^\eta \setminus S_Y^\eta, \end{cases} \tag{3.12}$$

where σ_δ is a smooth function $\mathbb{R}^2 \to \mathbb{R}$. It has values 1 for arguments (y_2, y_3) such that $(0, y_2, y_3) \in S_Y^\eta$, and it vanishes in a distance $\delta > 0$ from this set.

By construction, the curl of $\sigma^\eta(y)$ vanishes outside F_Y^η and in the slit S_Y^η. We consider the curl in an appropriate scaling,

$$\rho^\eta(y) := \frac{1}{\eta^2} \operatorname{curl}_y \sigma^\eta(y) = \frac{1}{\eta^2} \begin{pmatrix} 0 \\ \partial_{y_3}\sigma_\delta \\ -\partial_{y_2}\sigma_\delta \end{pmatrix} (y_2, y_3)\, \mathbf{1}_{F_Y^\eta}(y). \tag{3.13}$$

We furthermore have to localize in the macroscopic variable. In this calculation, we cannot use a smooth cut-off function, but have to use a different localization argument. In what follows, $P \subset R \subset \Omega$ shall be an arbitrary set that is given, for some $\eta_0 > 0$, by a union over cubes $P = \bigcup_{k \in \mathcal{K}_P} \eta_0(k+Y)$ for some $\mathcal{K}_P \subset \mathbb{Z}^n$. In this section, the cut-off function θ is *not* a smooth function, but the indicator function,

$$\theta(x) := \mathbf{1}_P(x) = \begin{cases} 1 & x \in P, \\ 0 & x \notin P. \end{cases} \tag{3.14}$$

Calculation of Limits With θ and σ^η as above, we use $x \mapsto \frac{1}{\eta}\theta(x)\sigma^\eta(x/\eta)$ as a test-function in (1.2). We obtain, on the one hand,

$$\begin{aligned} \int_\Omega \operatorname{curl} H^\eta(x) \cdot \frac{1}{\eta}\theta(x)\sigma^\eta(x/\eta)\,\mathrm{d}x &= -i\omega\varepsilon_0 \int_\Omega E^\eta(x) \cdot \frac{1}{\eta}\theta(x)\sigma^\eta(x/\eta)\,\mathrm{d}x \\ = -i\omega\varepsilon_0 \int_{S_\eta} E^\eta(x) \cdot e_1 \frac{1}{\eta}\theta(x)\,\mathrm{d}x &- i\omega\varepsilon_0 \int_{\Omega\setminus S_\eta} E^\eta(x) \cdot \frac{1}{\eta}\theta(x)\sigma^\eta(x/\eta)\,\mathrm{d}x. \end{aligned} \tag{3.15}$$

The first integral on the right hand side is the slit integral that we want to calculate. The other integral can be estimated by

$$\begin{aligned} &\left| \int_{\Omega\setminus S_\eta} E^\eta(x) \cdot \frac{1}{\eta}\theta(x)\sigma^\eta(x/\eta)\,dx \right|^2 \\ &\quad \le \|E^\eta\|^2_{L^2(\Omega_\eta)} \int_{\Omega\setminus S_\eta} \frac{1}{\eta^2} |\theta(x)\sigma^\eta(x/\eta)|^2\,dx \le C\delta, \end{aligned}$$

since the support of σ^η has a volume of order η^2 and a support of order $\delta\eta^2$ when the set S_η is removed. We therefore obtain, up to errors of order δ, the desired slit integral over E^η on the right hand side of (3.15).

On the other hand, we can calculate the left hand side of (3.15) with an integration by parts. We exploit the fact that σ^η vanishes on the boundary of Y, whence $\sigma^\eta(./\eta)$ vanishes on the boundary of the set P (which is aligned with the cells for all $\eta \le \eta_0$), so we have

$$\begin{aligned} &\int_\Omega \operatorname{curl} H^\eta(x) \cdot \frac{1}{\eta}\theta(x)\sigma^\eta(x/\eta)\,\mathrm{d}x \\ &\quad = \int_P H^\eta(x) \cdot \eta^{-2}(\operatorname{curl}_y \sigma^\eta)(x/\eta)\,\mathrm{d}x = \int_P H^\eta(x) \cdot \rho^\eta(x/\eta)\,\mathrm{d}x \\ &\quad = \int_P H^\eta(x) \cdot \rho^{\eta_0}(x/\eta)\,\mathrm{d}x + \int_P H^\eta(x) \cdot \big(\rho^\eta(x/\eta) - \rho^{\eta_0}(x/\eta)\big)\,\mathrm{d}x. \end{aligned} \tag{3.16}$$

In the last equality, we inserted a zero. The first integral has a limit as $\eta \to 0$ by the defining property of two-scale convergence,

$$\int_P H^\eta(x) \cdot \rho^{\eta_0}(x/\eta)\,\mathrm{d}x \to \int_P \int_Y H_0(x,y) \cdot \rho^{\eta_0}(y)\,\mathrm{d}y\,\mathrm{d}x.$$

The Y-integral can be evaluated by our choice of a test-function. For almost every $x \in \Omega$ that holds, using integration by parts with normal vector ν on $\partial\Sigma_Y$ and $\operatorname{curl}_y H_0 = 0$,

$$\begin{aligned}\int_Y H_0(x, y) \cdot \rho^{\eta_0}(y)\,\mathrm{d}y &= \int_{Y\setminus\Sigma_Y} H_0(x, y) \cdot \frac{1}{\eta_0^2}\operatorname{curl}_y \sigma^{\eta_0}(y)\,\mathrm{d}y \\ &= \frac{1}{\eta_0^2}\int_{F_Y^{\eta_0}\cap\partial\Sigma_Y} H_0(x, y) \cdot (\nu(y) \wedge e_1) \\ &= \frac{1}{\eta_0^2}\int_{F_Y^{\eta_0}\cap\partial\Sigma_Y} \sum_{k=0}^{3} h_k(x) H^k(y) \cdot \tau(y) \\ &= 2\alpha h_0(x).\end{aligned}$$

In the last step we used that $\tau = \nu \wedge e_1$ is a tangential field along curves and the Fubini theorem to write the surface integral as an average of line integrals (the average is over $y_1 \in (-\alpha\eta_0^2, \alpha\eta_0^2)$); all line integrals are, by our normalization of the functions $H^k(y)$, equal to $\delta_{0,k}$.

The other term on the right hand side of (3.16) is small. To show this, we write the difference $\hat{\rho}^\eta(y) := \rho^\eta(y) - \rho^{\eta_0}(y)$ as the curl of some vector potential ψ_Y^η, $\operatorname{curl}_y \psi_Y^\eta = \hat{\rho}^\eta$. Here, we use the fact that $\hat{\rho}^\eta$ has vanishing divergence and a vanishing flux around the torus Σ (since $\rho^\eta(y)$ and $\rho^{\eta_0}(y)$ have identical fluxes). This implies that we can find a potential ψ_Y^η that vanishes on the boundary $\partial(0, 1)^3$ of the unit cell and on the boundary $\partial\Sigma_Y$, see Lemma A.1 in the appendix for the existence of ψ_Y^η. We claim that the potentials ψ_Y^η are bounded in $L^2(Y)$. Using the estimate of Lemma A.1, it suffices to show an $H^{-1}(Y)$-bound for $\hat{\rho}^\eta$. The latter follows from the following calculation for a smooth test-function $\varphi : Y \to \mathbb{R}^3$, using $g(y_2, y_2) = (0, \partial_{y_3}\sigma_\delta, -\partial_{y_2}\sigma_\delta)(y_2, y_3)$,

$$\begin{aligned}\langle \hat{\rho}^\eta, \varphi\rangle &= \int_0^1\int_0^1 g(y_2, y_3) \int_0^1 \left[\frac{1}{\eta^2}\mathbf{1}_{F_Y^\eta}(y) - \frac{1}{\eta_0^2}\mathbf{1}_{F_Y^{\eta_0}}(y)\right] \varphi(y)\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3 \\ &= 2\alpha\int_0^1\int_0^1 g(y_2, y_3)\left[{\int\!\!\!\!\!\!-}_{-\alpha\eta^2}^{\alpha\eta^2} \varphi(y)\,\mathrm{d}y_1 - {\int\!\!\!\!\!\!-}_{-\alpha\eta_0^2}^{\alpha\eta_0^2} \varphi(y)\,\mathrm{d}y_1\right]\mathrm{d}y_2\,\mathrm{d}y_3,\end{aligned}$$

hence

$$\left|\langle \hat{\rho}^\eta, \varphi\rangle\right| \le 2\alpha \int_0^1\int_0^1 g(y_2, y_3)\int_{-\alpha\eta_0^2}^{\alpha\eta_0^2} |\partial_{y_1}\varphi(y)|\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3 \le C\|\varphi\|_{H^1(Y)}.$$

This calculation shows that the potentials ψ_Y^η are bounded in $L^2(Y)$.

Using the potential ψ_Y^η and the support $P_\eta \subset P$ of the function $P \ni x \mapsto \psi_Y^\eta(x/\eta)$ we can calculate

$$\begin{aligned}\int_P H^\eta(x) \cdot \left(\rho^\eta(x/\eta) - \rho^{\eta_0}(x/\eta)\right)\,\mathrm{d}x &= \int_{P_\eta} H^\eta(x) \cdot \operatorname{curl}_y \psi_Y^\eta(x/\eta)\,\mathrm{d}x \\ &= \eta\int_P \operatorname{curl}_x H^\eta(x) \cdot \psi_Y^\eta(x/\eta)\,\mathrm{d}x \to 0\end{aligned}$$

by $L^2(\Omega_\eta)$-boundedness of $\operatorname{curl}_x H^\eta(x)$ and $\psi_Y^\eta(./\eta)$.

Collecting the results from (3.15) and (3.16) we find for the limit $\eta \to 0$

$$m_\eta(P) = \int_{S_\eta} E^\eta(x) \cdot e_1 \frac{1}{\eta} \theta(x)\, dx \to \frac{2\alpha}{-i\omega\varepsilon_0} \int_P h_0(x)\, dx + O(\delta). \quad (3.17)$$

The limit of the left hand side exists by choice of the subsequence η, the limit is given by $m(P)$. Since δ was arbitrary, (3.17) gives the desired second relation for m.

Result of the Slit Analysis

It remains to compare the two expression for m obtained in (3.17) and in (3.11). We obtain

$$\frac{2\alpha}{-i\omega\varepsilon_0} h_0(x) = i\omega\mu_0 \sum_{k=0}^{3} D_k h_k(x). \quad (3.18)$$

This provides a frequency-dependent formula for h_0 in terms of h_1, h_2, h_3:

$$h_0(x) = \frac{\omega^2 \mu_0 \varepsilon_0}{2\alpha - \omega^2 \mu_0 \varepsilon_0 D_0} \sum_{k=1}^{3} D_k h_k(x). \quad (3.19)$$

4. Macroscopic Constitutive Laws

In the subsequent calculations we use, additionally to the geometric limits $\hat{E}$ and $\hat{H}$, also the weak limits $E^\eta \rightharpoonup E = \hat{E}$ and $H^\eta \rightharpoonup H$ in $L^2(\Omega)$.

Limit process in (1.1) We can take the distributional limit of (1.1) (or, more precisely, since we consider the trivial extensions, the distributional limit of Eq. (1.3)). We obtain, in the limit $\eta \to 0$,

$$\operatorname{curl} \hat{E} \leftarrow \operatorname{curl} E^\eta = i\omega\mu_0 H^\eta \to i\omega\mu_0 H = i\omega\mu_0 \hat{\mu} \hat{H}. \quad (4.1)$$

We recall that the last equation is a consequence of the definition of $\hat{\mu}$ and $\hat{H}$, see (2.21) together with $\hat{\mu} = \mu_{\text{eff}}$ in R and $\hat{\mu} = 1$ in $\Omega \backslash R$. The above distributional limit equation already provides (1.6), the first of the two effective equations.

Limit process in (1.2)**.** We use an oscillating test-function. We choose a smooth function $\theta : \Omega \to \mathbb{R}$ with compact support and fix $j \in \{1, 2, 3\}$. We consider $\psi_\eta(x) = E_\eta^j(x)\, \theta(x)$ with $E_\eta^j(x) = E^j(x/\eta)$. The second Maxwell equation (1.2) yields

$$\int_\Omega \operatorname{curl} H^\eta \cdot \psi_\eta = -i\omega\varepsilon_0 \int_\Omega E^\eta \cdot \psi_\eta. \quad (4.2)$$

It remains to evaluate the limits of both sides of (4.2). We start with the left hand side. In the subsequent calculation we use first integration by parts and $\operatorname{curl}_y E^j = 0$, then the property of two-scale convergence. We get

$$\begin{aligned}\int_\Omega \operatorname{curl} H^\eta \cdot \psi_\eta &= \int_\Omega H^\eta \cdot \operatorname{curl} \psi_\eta = -\int_\Omega H^\eta(x) \cdot (E^j_\eta(x) \wedge \nabla\theta(x))\, dx \\ &\to -\int_\Omega \int_Y \Big(H_0(x,y) \wedge E^j(y)\Big) \cdot \nabla\theta(x))\, dy\, dx \\ &\overset{(1.17)}{=} -\int_\Omega \Big(\oint H_0(x,.)\Big) \wedge \Big(\int_Y E^j\Big) \cdot \nabla\theta(x)\, dx \\ &\overset{(1.16)}{=} -\int_\Omega \hat{H}(x) \wedge e_j \cdot \nabla\theta(x)\, dx = \int_\Omega (\operatorname{curl} \hat{H}) \cdot e_j\, \theta\, .\end{aligned}$$

We now calculate the limit of the right hand side of (4.2). In the first equality, we use that $E^j_\eta(x)$ vanishes on $\Sigma_\eta \cup S_\eta$, in the second we use two-scale convergence. In the last equation we use the definition of ε_{eff} and the fact that $\hat{\varepsilon}$ from (1.8) coincides with ε_{eff} in R to get

$$\begin{aligned}\int_\Omega E^\eta \cdot \psi_\eta &= \int_{\Omega\setminus(\Sigma_\eta\cup S_\eta)} E^\eta(x) \cdot E^j_\eta(x)\, \theta(x)\, \mathrm{d}x \\ &\to \int_\Omega \int_Y E_0(x,y) \cdot E^j(y)\, \theta(x)\, \mathrm{d}y\, \mathrm{d}x \overset{(2.7)}{=} \int_\Omega \hat{\varepsilon} E(x) \cdot e_j\, \theta(x)\, \mathrm{d}x.\end{aligned}$$

Since $j \in \{1,2,3\}$ and $\theta = \theta(x)$ are arbitrary, (4.2) implies

$$\operatorname{curl} \hat{H}(x) = -i\omega\varepsilon_0 \hat{\varepsilon}(x)\, E(x)$$

for almost every $x \in \Omega$. Because of $\hat{E} = E$, we have obtained the effective equation (1.7). This concludes the proof of the main theorem.

Appendix A. Construction of Vector Potentials

In the slit analysis, we made use of Lemma A.1. The lemma is applied with the domain T that is *not* the torus Σ; the torus T lies in the complement $Y\setminus\Sigma$ and winds around the torus Σ. The essential part of this lemma can be found as Theorem 3.17 of [2]. Since very general domains are studied in [2], we have chosen to provide a sketch of proof here.

Lemma A.1. (Construction of vector potentials in tori) *Let $T \subset \bar{T} \subset (0,1)^3 = Y$ be an open subset which is, topologically, a torus. Let $D \subset T$ be a two-dimensional surface such that $T\setminus D$ is simply connected. We assume that ∂T and D are of class C^2 and denote a normal vector field on D by n and a normal vector field on ∂T by ν. Let $\rho : T \to \mathbb{R}^3$ be a vector field of class $L^2(T)$ with the properties*

$$\operatorname{div} \rho = 0 \quad \textit{in } T, \tag{A.1}$$

$$\rho \cdot \nu = 0 \quad \textit{on } \partial T, \tag{A.2}$$

$$\int_D \rho \cdot n = 0. \tag{A.3}$$

Then there exists a vector potential $\psi : T \to \mathbb{R}^3$ satisfying

$$\operatorname{curl} \psi = \rho \quad \textit{in } T, \tag{A.4}$$

$$\operatorname{div} \psi = 0 \quad \textit{in } T, \tag{A.5}$$

$$\psi \wedge \nu = 0 \quad \textit{on } \partial T. \tag{A.6}$$

For a constant $C \geq 0$ that depends on T but not on ρ, we can achieve

$$\|\psi\|_{L^2(T)} \leq C\|\rho\|_{H^{-1}(T)} . \tag{A.7}$$

Proof. *Construction of the potential* We consider the space

$$X := \left\{\psi \in H^1(T, \mathbb{R}^3) \,|\, \psi \wedge \nu = 0 \text{ on } \partial T\right\}, \tag{A.8}$$

and the bilinear form $a : X \times X \to \mathbb{R}$,

$$a(\psi, \varphi) := \int_T \operatorname{curl} \psi \cdot \operatorname{curl} \varphi + \int_T \operatorname{div} \psi \operatorname{div} \varphi. \tag{A.9}$$

We start with an investigation of the kernel of the corresponding operator. Let $\psi \in X$ be a function with $a(\psi, \psi) = 0$. Then $\operatorname{div} \psi = 0$ and $\operatorname{curl} \psi = 0$ hold by definition of a. As a curl-free function, ψ is locally the gradient of a harmonic function Φ. The boundary condition in the space X implies that all integrals over ψ over closed curves vanish, hence Φ is a global potential. The definition of X furthermore yields that Φ is constant on ∂T. We find that Φ is constant in T and $\psi \equiv 0$. We conclude that a has a trivial kernel.

The bilinear form a is coercive on X, see e.g., [13] for this classical result. We must be very careful in this statement. In [13], coerciveness is understood in the sense that $a(\psi, \psi) + \|\psi\|_{L^2}^2$ is equivalent to the squared H^1-norm, but in our case (namely when the bilinear form has a trivial kernel), the coercivity of a can be concluded also in the classical sense, i.e., $\|\psi\|_{H^1}^2 \leq C\, a(\psi, \psi)$ for all $\psi \in X$, compare [28].

The coercivity of a allows us to invoke the Lax-Milgram theorem. For ρ as in the lemma, we find a solution $\psi \in X$ of the problem

$$a(\psi, \varphi) = \int_T \rho \cdot \operatorname{curl} \varphi \quad \forall \varphi \in X. \tag{A.10}$$

We observe that an arbitrary scalar $L^2(T)$-function $h : T \to \mathbb{R}$ can be written as $h = \operatorname{div} \varphi$ for some $\varphi \in X$ with $\operatorname{curl} \varphi = 0$; indeed, we can choose $\varphi = \nabla \phi$ and solve the Poisson problem for $\Delta \phi = h$ with $\phi = 0$ on ∂T. Inserting φ in (A.10) shows that the solution ψ has a vanishing divergence.

It remains to collect the equations that are satisfied by the difference $u := \operatorname{curl} \psi - \rho$. The divergence of u vanishes by assumption (A.1) on ρ. The curl of u vanishes in T by (A.10). The normal component of u vanishes on ∂T, since $\psi \in X$ has no tangential components on ∂T and since the normal component of ρ vanishes on ∂T. Finally, u also satisfies the normalization (A.3): ρ satisfies it by assumption and $\operatorname{curl} \psi$ satisfies it by Stokes theorem. We conclude that u can be written as the gradient of a harmonic function on T (the normalization (A.3) allows us to extend a local potential to a global potential). The potential is harmonic and satisfies a homogeneous Neumann condition, hence it is constant and u vanishes. This provides (A.4). Property (A.6) holds by construction of $\psi \in X$ and (A.5) has been shown before.

Estimate for the potential We use that ψ posesses again a vector potential. This statement (with the correct boundary conditions for ψ) appears, e.g., as Theorem 3.12 in [2], but the proof can also be developed along the above lines. We write $\psi = \operatorname{curl} \Phi$ with $\operatorname{div} \Phi = 0$ in T and $\Phi \cdot \nu = 0$ on ∂T and the normalization (A.3). The potential Φ solves $\Delta\Phi = \rho$ and is therefore controlled in $H^1(T)$ by the $H^{-1}(T)$-norm of ρ. This implies the estimate (A.7). □

Acknowledgements. Support of the second author by DFG Grant Schw 639/6-1 is gratefully acknowledged. This material based on work supported by the U.S. Air Force Office of Scientific Research under Award FA9550-12-1-0489.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Allaire, G.: Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, 1992
2. Amrouche, C., Bernardi, C., Dauge, M., Girault, V.: Vector potentials in three-dimensional non-smooth domains. *Math. Methods Appl. Sci.*, 21(9):823–864, 1998
3. Bensoussan, A., Lions, J.-L., Papanicolaou, G.C.: Homogenization in deterministic and stochastic problems. In *Stochastic problems in dynamics (Sympos., Univ. Southampton, Southampton, 1976)*, pages 106–115. Pitman, London, 1977
4. Bouchitté, G., Bourel, C., Felbacq, D.: Homogenization of the 3D Maxwell system near resonances and artificial magnetism. *C. R. Math. Acad. Sci. Paris*, 347(9–10):571–576, 2009
5. Bouchitté, G., Felbacq, D.: Homogenization near resonances and artificial magnetism from dielectrics. *C. R. Math. Acad. Sci. Paris*, 339(5):377–382, 2004
6. Bouchitté, G., Felbacq, D.: Homogenization of a wire photonic crystal: the case of small volume fraction. *SIAM J. Appl. Math.*, 66(6):2061–2084, 2006
7. Bouchitté, G., Schweizer, B.: Homogenization of Maxwell's equations in a split ring geometry. *Multiscale Model. Simul.*, 8(3):717–750, 2010
8. Bouchitté, G., Schweizer, B.: Plasmonic waves allow perfect transmission through sub-wavelength metallic gratings. *Netw. Heterog. Media*, 8(4):857–878, 2013
9. Chen, Y., Lipton, R.: Tunable double negative band structure from non-magnetic coated rods. *New Journal of Physics*, 12(8):083010, 2010
10. Chen, Y., Lipton, R.: Double negative dispersion relations from coated plasmonic rods. *Multiscale Modeling and Simulation*, 209(3):835–868, 2013
11. Chen, Y., Lipton, R.: Multiscale methods for engineering double negative metamaterials. *Photonics and Nanostructures Fundamentals and Applications*, 11(4):442–454, 2013
12. Chen, Y., Lipton, R.: Resonance and double negative behavior in metamaterials. *Arch. Ration. Mech. Anal.*, 11(1):192–212, 2013
13. Costabel, M.: A coercive bilinear form for Maxwell's equations. *J. Math. Anal. Appl.*, 157(2):527–541, 1991

14. FELBACQ, D., BOUCHITTÉ, G.: Homogenization of a set of parallel fibres. *Waves Random Media*, 7(2):245–256, 1997
15. JIKOV, V., KOZLOV, S., OLEINIK, O.: *Homogenization of Differential Operators and Integral Functionals*. Springer, New York, 1994
16. KAFESAKI, M., TSIAPA, I., KATSARAKIS, N., KOSCHNY, Th., SOUKOULIS, C.M., ECONOMOU, E.N.: Left-handed metamaterials: The fishnet structure and its variations. *Phys. Rev. B* 75:235114–1–235114–9, 2007
17. KOHN, R.V., LU, J., SCHWEIZER, B., WEINSTEIN, M.I.: A variational perspective on cloaking by anomalous localized resonance. *Comm. Math. Phys.*, 328(1):1–27, 2014
18. KOHN, R.V., SHIPMAN, S.: Magnetism and homogenization of micro-resonators. *Multiscale Model. Simul.*, 7(1):62–92, 2007
19. LAMACZ, A., SCHWEIZER, B.: Effective Maxwell equations in a geometry with flat rings of arbitrary shape. *SIAM J. Math. Anal.*, 45(3):1460–1494, 2013
20. LAMACZ, A., SCHWEIZER, B.: Effective acoustic properties of a meta-material consisting of small Helmholtz resonators. *DCDS-S*, 2016
21. LAMACZ, A., SCHWEIZER, B.: A negative index meta-material for Maxwell's equations. *SIAM J. Math. Anal.*, 48(6):4155–4174, 2016
22. LIPTON, R., POLIZZI, A., THAKUR, L.: Novel metamaterial surfaces from perfectly conducting subwavelength corrugations. *SIAM J. Appl. Math.*, 77(4):1269–1291, 2017
23. MILTON, G.W., NICOROVICI, N.-A.P.: On the cloaking effects associated with anomalous localized resonance. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 462:3027–3059, 2006
24. MOVCHAN, A.B., GUENNEAU, S.: Split-ring resonators and localized modes. *Phys. Rev. B*, 70(12):125116, 2004
25. PENDRY, J.B.: Negative refraction makes a perfect lens. *Phys. Rev. Lett.*, 85:3966, 2000
26. PENDRY, J.B., SCHURIG, D., SMITH, D.R.: Controlling electromagnetic fields. *Science*, 312:5781, 2006
27. SCHWEIZER, B.: Resonance meets homogenization: construction of meta-materials with astonishing properties. *Jahresber. Dtsch. Math. Ver.*, 119(1):31–51, 2017
28. SCHWEIZER, B.: Friedrichs inequality, Helmholtz decomposition, vector potentials, and the div-curl lemma. *INdAM-Springer series*, Trends on Applications of Mathematics to Mechanics, 2018
29. SCHWEIZER, B., URBAN, M.: Effective maxwells equations in general periodic microstructures. *Appl. Anal.*, 0(0):1–21, 2017
30. SELLIER, A., BUROKUR, S. N., KANTÉ, B., DE LUSTRAC, A.: Novel cut wires metamaterial exhibiting negative refractive index. *IEEE*, 2009
31. SHELBY, R.A., SMITH, D.R., SCHULTZ, S.: Experimental verification of a negative index of refraction. *Science*, 292(5514):77–79, 2001
32. VESELAGO, V.G.: The electrodynamics of substances with simultaneously negative values of ε and μ. *Sov. Phys. Uspekhi*, 10:509–514, 1968

ROBERT LIPTON
Department of Mathematics,
Louisiana State University,
384 Lockett Hall,
Baton Rouge,
USA.

Ben Schweizer
Fakultät für Mathematik,
TU Dortmund,
Vogelpothsweg 87,
44227,
Dortmund,
Germany.
e-mail: ben.schweizer@tu-dortmund.de

(*Received December 20, 2016 / Accepted March 6, 2018*)
Published online March 22, 2018