Stability for Control Systems and Lyapunov Functions

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Control Systems

A *control system* is a system of the form

$$\dot{x} = f(x, u),$$

where the control u is undetermined at the outset. We can

- choose a 'pre-computed' control $u(t)$ and study the system $\dot{x} = f(x, u(t))$, or
- choose a *feedback* function $k(x)$ and study the *closed-loop* system $\dot{x} = f(x, k(x))$
- choose $u = 0$ and study the corresponding *unforced system*,

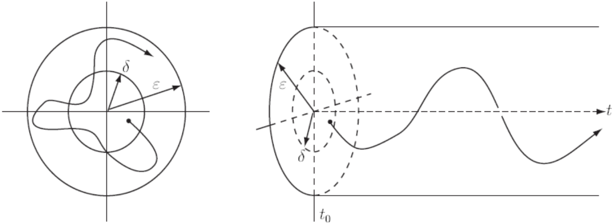
Internal notions of stability

Various notions of stability exist for forced and unforced systems. We will begin with notions of stability for unforced systems, also knowns as internal notions. They are applicable to dynamical systems $\dot{x} = f(x)$ in general.

Lyapunov Stability

Definition (Lyapunov Stability)

The system $\dot{x} = f(x)$ is *Lyapunov stable* if for every $\epsilon > 0$, there is $\delta > 0$ so that $||x(0)|| < \delta$ implies $||x(t)|| < \epsilon$ for every $t \geq 0$.

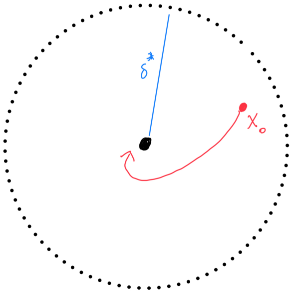


"Trajectories with sufficiently small initial values will remain small."

Asymptotic Stability

Definition (Asymptotic Stability)

The system $\dot{x} = f(x)$ is *locally asymptotically stable* if it is stable and in addition there is $\delta^* > 0$ so that $||x(0)|| < \delta^*$ implies $\lim_{t\to\infty} ||x(t)|| = 0$.

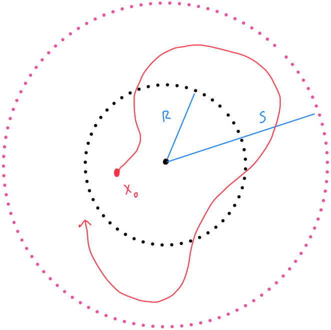


"Trajectories with sufficiently small initial values will converge to the origin," or more generally the equilibrium point.

Lagrange Stability

Definition (Lagrange Stability)

The system $\dot{x} = f(x)$ is *Lagrange stable* if for every $R > 0$, there is $S > 0$ so that $||x(0)|| < R$ implies $||x(t)|| < S$ for all $t \geq 0$.



"Trajectories cannot blow up."

Stability for Linear Systems

The following result characterizing the stability of linear systems is well-known.

Definition (Hurwitz) A real, square matrix A is *Hurwitz* if all of its eigenvalues have negative real part.

Proposition *The system* $\dot{x} = Ax$ *is asymptotically stable if and only if* A *is Hurwitz.*

Proposition *The system* $\dot{x} = Ax$ *is stable if and only if all its eigenvalues lie on the closed left half complex plane and eigenvalues lying on the imaginary axis are simple.*

In the above proposition, there is no distinction between Lyapunov and Lagrange stability; these notions are equalivalent in the case of linear systems.

Demonstrating stability: Lyapunov functions

Since we generally cannot find explicit soltuions to dynamics $\dot{x} = f(x)$, we need technical means to prove stability properties. We can use Lyapunov functions.

Definition A *smooth strict Lyapunov function* for the system $\dot{x} = f(x)$ is a map $V : B_r \to [0, \infty)$ so that

- $V(0) = 0$
- $V(x) > 0$ for $x \neq 0$
- $V(x)$ is of class C^1 on B_r
- $\nabla V(x) \cdot f(x) < 0$ for each $x \in B_r$, $x \neq 0$.

Theorem *If there is a smooth strict Lyapunov function for $\dot{x} = f(x)$, then the system is asymptotically stable.*

Lyapunov function example

Example

Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1 + 2x_1^3 x_2^2 \\ -x_2 \end{pmatrix}.$$

Let $V(x_1, x_2) = \frac{x_1^2}{1+x_1^2} + 2x_2^2$. Then,

$$\begin{aligned} \nabla V(x) \cdot f(x) &= \begin{pmatrix} \frac{2x_1}{(1+x_1^2)^2} \\ 4x_2 \end{pmatrix} \cdot \begin{pmatrix} -x_1 + 2x_1^3 x_2^2 \\ -x_2 \end{pmatrix} \\ &= \frac{-2x_1^2 + 4x_1^4 x_2^2}{(1+x_1^2)^2} - 4x_2^2 \\ &= \frac{-2x_1^2 - 4x_2^2 - 8x_2^2 x_1^2}{(1+x_1^2)^2} < 0, \end{aligned}$$

so the system is locally asymptotically stable.

External notions of stability

There are notions of stability for control systems $\dot{x} = f(x, u)$ as well. We call this *external* stability. These notions generally relate $||x(t)||$ to $||x(0)||$ and $||u(\cdot)||_\infty$ and reduce to internal notions when $u \equiv 0$.

Weak, Strong Finite Gain Property

We begin with a couple of notions specific to linear systems.

Definition (Weak Finite Gain Property) The system $\dot{x} = Ax + Bu$ has the *weak finite gain property* if there are positive constants γ_1, γ_2 so that

$$||x(t; u(\cdot))|| \leq \gamma_1 ||x(0)|| + \gamma_2 ||u(\cdot)||_\infty$$

for all $t \geq 0$.

Definition (Strong Finite Gain Property) The system $\dot{x} = Ax + Bu$ has the *strong finite gain property* if there are positive constants $\alpha, \gamma_1, \gamma_2$ such that

$$||x(t; u(\cdot))|| \leq \gamma_1 ||x_0|| e^{-\alpha t} + \gamma_2 ||u(\cdot)||_\infty$$

for all $t \geq 0$.

Comparison Functions

Before we can effectively generalize these notions, we need to introduce comparison functions.

Definition (Class $\overline{\mathcal{K}}_0$) A function $\alpha : [0, \infty) \to [0, \infty)$ is of class $\overline{\mathcal{K}}_0$ if it is continuous, strictly increasing, and $\alpha(0) = 0$.

Definition (Class $\overline{\mathcal{K}}^\infty$) A function $\alpha : [0, \infty) \to [0, \infty)$ is of class $\overline{\mathcal{K}}^\infty$ if it is continuous, strictly increasing and $\lim_{r\to\infty} \alpha(r) = \infty$.

Definition (Class $\mathcal{L}$) A function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{L}$ if it is continuous, decreasing and satisfies $\lim_{r\to\infty} \alpha(r) = 0$.

Definition (Class $\mathcal{KL}$) A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is in class $\mathcal{KL}$ if $\beta(\cdot, t) \in \overline{\mathcal{K}}_0$ for every t and $\beta(s, \cdot) \in \mathcal{L}$ for every s.

UBIBS-Stability

The following is a generalization of the weak finite gain property.

Definition (Uniformly bounded-input bounded-state stable)

The system $\dot{x} = f(x, u)$ is called *uniformly bounded-input bounded-state stable* (UBIBS-stable) if for each $R > 0$, there is $S > 0$ so that $||x(0)||, ||u(\cdot)||_\infty \leq R$ implies $||x(t)|| \leq S$ for every $t \geq 0$.

It means that there is a bound on the trajectory proportional to the bounds on the initial value and control. This can be made more clear using comparison functions:

Proposition

If $\dot{x} = f(x, u)$ is UBIBS-stable, then there are $\gamma_1, \gamma_2 \in \overline{\mathcal{K}}^\infty$ so that

$$||x(t)|| \leq \gamma_1(||x(0)||) + \gamma_2(||u(\cdot)||_\infty)$$

for all $t \geq 0$.

ISS-Stability

The following is a generalization of the strong finite gain property. The key point is that the part of the bound on $||x(t)||$ corresponding to the initial value $x(0)$ goes to 0 as $t \to \infty$.

Definition (Input-to-state stable)

The system $\dot{x} = f(x, u)$ is called *input-to-state stable* (ISS-stable) if there are maps $\beta \in \mathcal{KL}$, $\gamma \in \overline{\mathcal{K}}_0$ so that

$$||x(t)|| \leq \beta(||x(0)||, t) + \gamma(||u||_\infty).$$

Demonstrating ISS-stability: ISS-Lyapunov functions

In the same way that we can use Lyapunov functions to prove the asymptotic stability of dynamical systems, we can use ISS-Lyapunov functions to prove the ISS-stability for control systems. A similar result holds for UBIBS-stability.

Definition (ISS-Lyapunov function)

A *smooth ISS-Lyapunov function* for the system $\dot{x} = f(x, u)$ is a positive definite, radially unbounded C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ with $V(0) = 0$ and such that there are functions $\rho, \chi \in \mathcal{K}_0^\infty = \overline{\mathcal{K}}_0 \cap \overline{\mathcal{K}}^\infty$ such that for all $x \in \mathbb{R}^n$, $x \neq 0$, and all $u \in \mathbb{R}^m$, if $||x|| \geq \rho(||u||)$, then

$$\nabla V(x) \cdot f(x, u) < -\chi(||x||).$$

Theorem

If an ISS-Lyapunov function exists for the system $\dot{x} = f(x, u)$, then the system is ISS-stable.

Conclusion

Thanks for your attention!

Bibliography:

Lionel Rosier and Andrea Bacciotti. *Liapunov functions and stability in control theory. Second edition*. Springer Verlag, second edition edition, 2005. URL https://hal.archives-ouvertes.fr/hal-00139067.