for some functions and for small intervals, the tangent line at the point halfway between the endpoints of the secant line has the same slope as the secant line

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Theorem 1 (Mean Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there is some $c \in (a, b)$ so that the slope of the line tangent to the graph of f at c is equal to the slope of the secant line going through the points $(a, f(a))$ and $(b, f(b))$, which means that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Theorem 2 (L'Hospital's rule)**.** *Let a be a real number and let $f(x)$ and $g(x)$ be functions that are differentiable on some open interval containing a. Assume also that $g'(x) \neq 0$ on this interval, except perhaps at the point a itself. If $f(a) = g(a) = 0$, then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists.

Theorem 1 and Theorem 2 can be found on pages 76 and 401 of [1] respectively.

Theorem 3. *Let $f(x) = x^2$. Suppose that $a \in \mathbb{R}$ and $h > 0$. Theorem 1 implies that there is some $\theta \in (0, 1)$ so that $\frac{f(a+h)-f(a)}{h} = f'(a + \theta h)$. The only solution to this equation is $\theta = 1/2$.*

Proof. Firstly,

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a+h)^2 - a^2}{h} \\ &= \frac{(a+h+a)(a+h-a)}{h} \\ &= 2a + h. \end{aligned}$$

Meanwhile, $f'(a + \theta h) = 2(a + \theta h) = 2a + 2\theta h$. We therefore have $2a + h = 2a + 2\theta h$, so $\theta = 1/2$. □

Theorem 4. *Let $f(x) = x^3$. Suppose that $a, h > 0$. Theorem 1 implies that there is some $\theta \in (0,1)$ so that $\frac{f(a+h)-f(a)}{h} = f'(a+\theta h)$. This equation defines a function $\theta(h) : (0,\infty) \to (0,1)$, and we have that $\lim_{h\to 0} \theta(h) = 1/2$.*

Proof. Firstly,

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{(a+h)^3 - a^3}{h}\\ &= \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h}\\ &= 3a^2 + 3ah + h^2.\end{aligned}$$

Meanwhile, $f'(a+\theta h) = 3(a+\theta h)^2 = 3a^2 + 6a\theta h + 3\theta^2 h^2$. We therefore have that $3h^2\theta^2 + 6ah\theta + 3a^2 = 3a^2 + 3ah + h^2$, which implies $3h^2\theta^2 + 6ah\theta - 3ah - h^2 = 0$. The quadratic formula implies

$$\begin{aligned}\theta &= \frac{-6ah \pm \sqrt{36a^2h^2 - 4(3h^2)(-3ah - h^2)}}{6h^2}\\ &= \frac{-6ah \pm \sqrt{36a^2h^2 + 36ah^3 + 12h^4}}{6h^2}\\ &= \frac{-6ah \pm \sqrt{4h^2(9a^2 + 9ah + 3h^2)}}{6h^2}\\ &= \frac{-6ah \pm 2h\sqrt{9a^2 + 9ah + 3h^2}}{6h^2}\\ &= \frac{-3a \pm \sqrt{9a^2 + 9ah + 3h^2}}{3h}.\end{aligned}$$

Subtracting the square root will yield a negative number, so the $\theta \in (0,1)$ that we are looking for is obtained by choosing to add the square root:

$$\theta = \frac{-3a + \sqrt{9a^2 + 9ah + 3h^2}}{3h}.$$

We see that the limit of the numerator is $\lim_{h\to 0}(-3a + \sqrt{9a^2 + 9ah + 3h^2}) = -3a + \sqrt{9a^2} = 0$, so we may apply Theorem 2 to compute the limit of θ as follows:

$$\begin{aligned}\lim_{h\to 0} \theta &= \lim_{h\to 0} \frac{-3a + \sqrt{9a^2 + 9ah + 3h^2}}{3h}\\ &= \lim_{h\to 0} \frac{9a + 6h}{6\sqrt{9a^2 + 9ah + 3h^2}}\\ &= \frac{9a}{6\sqrt{9a^2}}\\ &= \frac{9a}{18a}\\ &= \frac{1}{2}.\end{aligned}$$

□

Theorem 5. *Let $f(x) = e^x$. Suppose that $a \in \mathbb{R}$ and $h > 0$. Theorem 1 implies that there is some $\theta \in (0,1)$ so that $\frac{f(a+h)-f(a)}{h} = f'(a+\theta h)$. This equation defines a function $\theta(h) : (0,\infty) \to (0,1)$, and we have that $\lim_{h\to 0} \theta(h) = 1/2$.*

Proof. Firstly,

$$\frac{f(a+h)-f(a)}{h} = \frac{e^{a+h}-e^a}{h} = \frac{e^a(e^h-1)}{h}.$$

Meanwhile, $f'(a+\theta h) = e^{a+\theta h} = e^a e^{\theta h}$. We therefore have that $e^a(e^h-1)/h = e^a e^{\theta h}$, which implies $e^{\theta h} = (e^h-1)/h$. Since $h > 0$, the right side of the previous equation is positive, so we can take the natural logarithm of both sides of the equation and then divide both sides by h to obtain

$$\theta = \frac{\ln\left(\frac{e^h-1}{h}\right)}{h}.$$

Theorem 2 implies

$$\lim_{h\to 0} \frac{e^h-1}{h} = \lim_{h\to 0} e^h = 1.$$

Hence, the continuity of $\ln(x)$ implies

$$\lim_{h\to 0} \ln\left(\frac{e^h-1}{h}\right) = \ln\left(\lim_{h\to 0} \frac{e^h-1}{h}\right) = \ln(1) = 0.$$

We can therefore use Theorem 2 three more times to compute the limit of θ as follows:

$$\begin{aligned}
\lim_{h\to 0} \theta &= \lim_{h\to 0} \frac{\ln\left(\frac{e^h-1}{h}\right)}{h} \\
&= \lim_{h\to 0} \left(\frac{h}{e^h-1}\right) \cdot \left(\frac{he^h-(e^h-1)}{h^2}\right) \\
&= \lim_{h\to 0} \frac{he^h-e^h+1}{h(e^h-1)} \\
&= \lim_{h\to 0} \frac{he^h+e^h-e^h}{he^h+(e^h-1)} \\
&= \lim_{h\to 0} \frac{he^h}{he^h+e^h-1} \\
&= \lim_{h\to 0} \frac{he^h+e^h}{he^h+e^h+e^h} \\
&= \frac{1}{2}.
\end{aligned}$$

□

Theorem 6. *Let $f(x) = \ln(x)$. Suppose that $a, h > 0$. Theorem 1 implies that there is some $\theta \in (0,1)$ so that $\frac{f(a+h)-f(a)}{h} = f'(a+\theta h)$. This equation defines a function $\theta(h) : (0,\infty) \to (0,1)$, and we have that $\lim_{h\to 0} \theta(h) = 1/2$.*

Proof. Since $f'(x) = 1/x$, the equation in question is

$$\frac{\ln(a+h)-\ln(a)}{h} = \frac{1}{a+\theta h}.$$

Taking reciprocals, we have

$$a + \theta h = \frac{h}{\ln(a+h)-\ln(a)},$$

and so

$$\theta h = \frac{h - a\ln(a+h) + a\ln(a)}{\ln(a+h)-\ln(a)},$$

hence finally

$$\theta = \frac{h - a\ln(a+h) + a\ln(a)}{h(\ln(a+h)-\ln(a))}.$$

We see that the numerator and the denominator both converge to 0 as $h \to 0$, so we may apply Theorem 2 to compute the limit of θ.

$$\begin{aligned}
\lim_{h\to 0}\theta &= \lim_{h\to 0} \frac{h - x\ln(x+h) + x\ln(x)}{h(\ln(x+h)-\ln(x))} \\
&= \lim_{h\to 0} \frac{1 - x\frac{1}{x+h}}{h\frac{1}{x+h} + (\ln(x+h)-\ln(x))} \\
&= \lim_{h\to 0} \frac{\frac{x+h-x}{x+h}}{\frac{h+(x+h)(\ln(x+h)-\ln(x))}{x+h}} \\
&= \lim_{h\to 0} \frac{h}{h+(x+h)(\ln(x+h)-\ln(x))}.
\end{aligned}$$

We see that we can apply Theorem 2 once more.

$$\begin{aligned}
\lim_{h\to 0}\theta &= \lim_{h\to 0} \frac{h}{h+(x+h)(\ln(x+h)-\ln(x))} \\
&= \lim_{h\to 0} \frac{1}{1+(x+h)\frac{1}{x+h} + (\ln(x+h)-\ln(x))} \\
&= \lim_{h\to 0} \frac{1}{2+\ln(x+h)-\ln(x)} \\
&= \frac{1}{2}.
\end{aligned}$$

□

Theorem 7. *Let $f(x) = \sqrt{x}$. Suppose that $a, h > 0$. Theorem 1 that there is some $\theta \in (0,1)$ so that $\frac{f(a+h)-f(a)}{h} = f'(a+\theta h)$. This equation defines a function $\theta(h) : (0,\infty) \to (0,1)$, and we have that $\lim_{h\to 0} \theta(h) = 1/2$.*

Proof 1. Since $f'(x) = 1/(2\sqrt{x})$, the equation in question is

$$\frac{\sqrt{a+h}-\sqrt{a}}{h} = \frac{1}{2\sqrt{a+\theta h}}.$$

After squaring both sides and then taking reciprocals, we have

$$4(a+\theta h) = \frac{h^2}{2a+h-2\sqrt{a^2+ah}}.$$

We then solve for θ to obtain

$$\begin{aligned}\theta &= \frac{h}{8a+4h-8\sqrt{a^2+ah}} - \frac{a}{h}\\ &= \frac{h^2 - a(8a+4h-8\sqrt{a^2+ah})}{h(8a+4h-8\sqrt{a^2+ah})}\\ &= \frac{h^2-8a^2-4ah+8a\sqrt{a^2+ah}}{8ah+4h^2-8h\sqrt{a^2+ah}}.\end{aligned}$$

We see that the numerator and the denominator both converge to 0 as $h \to 0$, so we may apply Theorem 2 to compute the limit of θ.

$$\begin{aligned}\lim_{h\to 0}\theta &= \lim_{h\to 0}\frac{h^2-8a^2-4ah+8a\sqrt{a^2+ah}}{8ah+4h^2-8h\sqrt{a^2+ah}}\\ &= \lim_{h\to 0}\frac{2h-4a+8a\frac{a}{2\sqrt{a^2+ah}}}{8a+8h-8h\frac{a}{2\sqrt{a^2+ah}}-8\sqrt{a^2+ah}}\\ &= \lim_{h\to 0}\frac{2h-4a+4a^2(a^2+ah)^{-1/2}}{8a+8h-4ah(a^2+ah)^{-1/2}-8\sqrt{a^2+ah}}.\end{aligned}$$

We can again check that the conditions for Theorem 2 are satisfied.

$$\begin{aligned}\lim_{h\to 0}\theta &= \lim_{h\to 0}\frac{2h-4a+4a^2(a^2+ah)^{-1/2}}{8a+8h-4ah(a^2+ah)^{-1/2}-8\sqrt{a^2+ah}}\\ &= \lim_{h\to 0}\frac{2+4a^2\left(-\frac{1}{2}(a^2+ah)^{-3/2}a\right)}{8-\left[4ah\left(-\frac{1}{2}(a^2+ah)^{-3/2}a\right)+4a(a^2+ah)^{-1/2}\right]-8\frac{a}{2\sqrt{a^2+ah}}}\\ &= \lim_{h\to 0}\frac{2-2a^3(a^2+ah)^{-3/2}}{8+2a^2h(a^2+ah)^{-3/2}-8a(a^2+ah)^{-1/2}}.\end{aligned}$$

We see once more that we can apply Theorem 2.

$$\begin{aligned}
\lim_{h\to 0}\theta &= \lim_{h\to 0}\frac{2-2a^3(a^2+ah)^{-3/2}}{8+2a^2h(a^2+ah)^{-3/2}-8a(a^2+ah)^{-1/2}}\\
&= \lim_{h\to 0}\frac{3a^3(a^2+ah)^{-5/2}a}{2a^2h\left[-\frac{3}{2}(a^2+ah)^{-5/2}a\right]+2a^2(a^2+ah)^{-3/2}+4a^2(a^2+ah)^{-3/2}}\\
&= \lim_{h\to 0}\frac{3a^4(a^2+ah)^{-5/2}}{-3a^3h(a^2+ah)^{-5/2}+6a^2(a^2+ah)^{-3/2}}\\
&= \frac{1}{2}.
\end{aligned}$$

□

Proof 2. Since $f'(x) = 1/(2\sqrt{x})$, the equation in question is

$$\frac{\sqrt{a+h}-\sqrt{a}}{h} = \frac{1}{2\sqrt{a+\theta h}}.$$

We multiple both the numerator and the denominator on the left by $\sqrt{a+h}+\sqrt{a}$ to obtain

$$\frac{h}{h(\sqrt{a+h}+\sqrt{a})} = \frac{1}{\sqrt{a+h}+\sqrt{a}} = \frac{1}{2\sqrt{a+\theta h}}.$$

After squaring both sides, we have

$$\frac{1}{2a+h+2\sqrt{a^2+ah}} = \frac{1}{4(a+\theta h)},$$

which implies

$$4a+4\theta h = 2a+h+2\sqrt{a^2+ah}$$

and so

$$4\theta h = h+2\sqrt{a^2+ah}-2a$$

and finally we have

$$\theta = \frac{h+2\sqrt{a^2+ah}-2a}{4h}.$$

We see that the numerator and the denominator both converge to 0, so me may apply Theorem 2 to compute the limit of θ.

$$\begin{aligned}
\lim_{h\to 0}\theta &= \lim_{h\to 0}\frac{h+2\sqrt{a^2+ah}+2a}{4h}\\
&= \lim_{h\to 0}\frac{1+2\frac{a}{2\sqrt{a^2+ah}}}{4}\\
&= \frac{1}{2}.
\end{aligned}$$

□

Theorem 8. *Let $f(x) = \frac{1}{x}$. Suppose that $a, h > 0$. Theorem 1 implies that there is some $\theta \in (0,1)$ so that $\frac{f(a+h)-f(a)}{h} = f'(a+\theta h)$. This equation defines a function $\theta(h) : (0,\infty) \to (0,1)$, and we have that $\lim_{h\to 0} \theta(h) = 1/2$.*

Proof. Firstly, we have that

$$\frac{f(a+h)-f(a)}{h} = \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \frac{\frac{a-(a+h)}{a(a+h)}}{h} = \frac{\frac{-h}{a^2+ah}}{h} = \frac{-h}{h(a^2+ah)} = \frac{-1}{a^2+ah}.$$

Meanwhile, $f'(x) = -1/x^2$, so

$$f'(a+\theta h) = \frac{-1}{(a+\theta h)^2} = \frac{-1}{a^2 + 2a\theta h + \theta^2 h^2}.$$

The equation in question is therefore

$$\frac{-1}{a^2+ah} = \frac{-1}{a^2 + 2a\theta h + \theta^2 h^2},$$

which implies $h^2\theta^2 + 2ah\theta - ah = 0$. The quadratic formula implies

$$\begin{aligned}\theta &= \frac{-2ah \pm \sqrt{4a^2h^2 - 4(h^2)(-ah)}}{2h^2} \\ &= \frac{-2ah \pm \sqrt{4a^2h^2 + 4ah^3}}{2h^2} \\ &= \frac{-2ah \pm 2h\sqrt{a^2+ah}}{2h^2} \\ &= \frac{-a \pm \sqrt{a^2+ah}}{h}.\end{aligned}$$

Since choosing the negative square root will yield a negative number, we know that since θ is positive, we must choose the positive square root. We then see that the conditions of Theorem 2 are satisfied, so we can use it to compute the limit of θ.

$$\begin{aligned}\lim_{h\to 0} \theta &= \lim_{h\to 0} \frac{-a + \sqrt{a^2+ah}}{h} \\ &= \lim_{h\to 0} \frac{a}{2\sqrt{a^2+ah}} \\ &= \frac{1}{2}.\end{aligned}$$

□

References

[1] George Finlay Simmons. *Calculus with Analytic Geometry*. 2nd ed. McGraw-Hill, 1996.