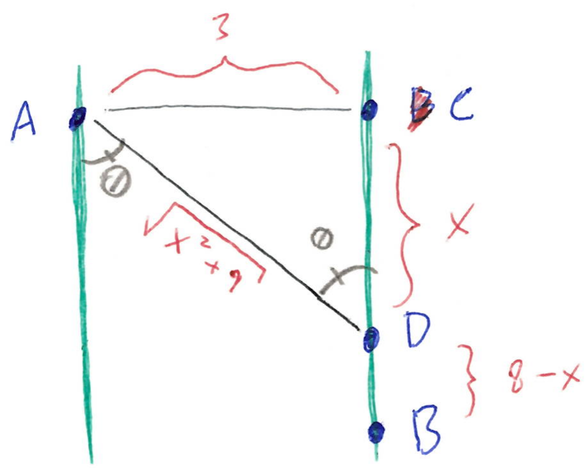
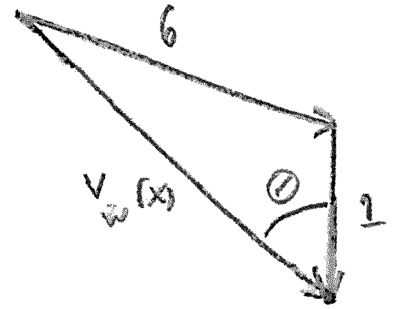
Boat Problem A woman launches her boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. She could row her boat directly across the river to point C and then run to B, or she could row directly to B, or she could row to some point D between C and B and then run to B. If she can row 6 km/h and run 8 km/h, where should she land to reach B as quickly as possible, if there is a downstream current flowing at a rate of 1 km/h?

solution:



$$T(x) = \sqrt{x^2+9} \cdot \frac{1}{V_w(x)} + (8-x) \cdot \frac{1}{8},$$

where $V_w(x)$ is the travel speed by boat, which will depend on the angle at which she travels down the river due to the current. To solve the problem, we need to find this function $V_w(x)$. The velocity vectors from the current and the woman's rowing will add up, so if she wants to travel directly towards point D, she will have to aim for a point upstream as seen in the triangle of velocity vectors:



To solve for $V_w(x)$, which is the length of the true velocity vector and therefore the true travel speed over the water, we can use the law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos(C) \quad \Rightarrow \quad 6^2 = 1^2 + v^2 - 2 \cdot 1 \cdot v \cos(\theta).$$

Let $z = \cos(\theta)$. Then,

$$36 = 1 + v^2 - 2vz \quad \Rightarrow \quad v^2 - 2zv - 35 = 0$$

$$\Rightarrow \quad v = \frac{2z \pm \sqrt{(2z)^2 - 4 \cdot 1 \cdot (-35)}}{2 \cdot 1}$$

$$= \frac{2z \pm \sqrt{4z^2 + 4 \cdot 35}}{2} = z \pm \sqrt{z^2 + 35}.$$

Since v must be positive, we must have

$$v = z + \sqrt{z^2 + 35}.$$

Now, from the original picture of the river, we see that $z = \cos(\theta) = \frac{x}{\sqrt{x^2+9}}$. Hence,

$$V_w(x) = z + \sqrt{z^2 + 35} = \frac{x}{\sqrt{x^2+9}} + \sqrt{\frac{x^2}{x^2+9} + 35} = \frac{x + \sqrt{36x^2 + 315}}{\sqrt{x^2+9}}.$$

We can plug this back into $T(x)$ to obtain

$$T(x) = \sqrt{x^2+9} \cdot \frac{1}{\left(\frac{x + \sqrt{36x^2+315}}{\sqrt{x^2+9}} \right)} + (8-x) \cdot \frac{1}{8}$$

$$= \frac{x^2+9}{x + \sqrt{36x^2+315}} + (8-x) \cdot \frac{1}{8}.$$

To optimize $T(x)$ using the closed interval method, we need to compute $T'(x)$:

$$T'(x) = \frac{\left(x + \sqrt{36x^2+315}\right) \cdot 2x - (x^2+9) \cdot \left(1 + \frac{36x}{\sqrt{36x^2+315}}\right)}{\left(x + \sqrt{36x^2+315}\right)^2} - \frac{1}{8}$$

$$= \frac{2x}{x + 3\sqrt{4x^2+35}} - \frac{(x^2+9)\left(3\sqrt{4x^2+35} + 36x\right)}{3\sqrt{4x^2+35}\left(x + 3\sqrt{4x^2+35}\right)^2} - \frac{1}{8}$$

$$= \frac{2x}{x + 3\sqrt{4x^2+35}} - \frac{(x^2+9)\left(\sqrt{4x^2+35} + 12x\right)}{\sqrt{4x^2+35}\left(x + 3\sqrt{4x^2+35}\right)^2} - \frac{1}{8}$$

$$= \frac{\left(\begin{array}{l} 16x\sqrt{4x^2+35}\left(x + 3\sqrt{4x^2+35}\right) - 8(x^2+9)\left(\sqrt{4x^2+35} + 12x\right) \\ \qquad - \sqrt{4x^2+35}\left(x + 3\sqrt{4x^2+35}\right)^2 \end{array} \right)}{8\left(x + 3\sqrt{4x^2+35}\right)^2 \sqrt{4x^2+35}}$$

$$= \frac{\left(\begin{array}{l} 16x\left[x\sqrt{4x^2+35} + 3(4x^2+35)\right] - 8\left[x^2\sqrt{4x^2+35} + 12x^3 \right. \\ \left. \quad + 9\sqrt{4x^2+35} + 108x\right] - \sqrt{4x^2+35}\left[x^2 + 6x\sqrt{4x^2+35} + 9(4x^2+35)\right] \end{array} \right)}{8\left(x + 3\sqrt{4x^2+35}\right)^2 \sqrt{4x^2+35}}.$$

Let $u = \sqrt{4x^2+35}$. We have

$$T'(x) = \frac{16x[xu + 12x^2 + 105] - 8[x^2u + 12x^3 + 9u + 108x] - u[x^2 + 6xu + 36x^2 + 315]}{8(x+3u)^2 \cdot u}.$$

We set it equal to 0 to begin searching for critical points, and then we can cancel the denominator to obtain the following equation:

$$\begin{aligned} &16x^2u + 192x^3 + 1680x \\ &\quad - 8x^2u - 96x^3 - 72u - 864x \\ &\quad - x^2u - 6x(4x^2+35) - 36x^2u - 315u \end{aligned}$$

$$\parallel$$

$$606x + 72x^3 - 387u - 29x^2u = 0$$

$$\Rightarrow \quad 606x + 72x^3 = 387u + 29x^2u$$

Now we square both sides...

$$\Rightarrow 367{,}236x^2 + 87{,}264x^4 + 5{,}184x^6 = 149{,}769u^2 + 22{,}446x^2u^2 + 841x^4u^2$$

$$= 149{,}769(4x^2+35) + 22{,}446x^2(4x^2+35) + 841x^4(4x^2+35)$$

$$= 599{,}076x^2 + 5{,}136{,}915 + 89{,}784x^4 + 785{,}610x^2 + 3{,}364x^6 + 29{,}435x^4$$

$$= 5{,}241{,}915 + 1{,}384{,}686x^2 + 119{,}219x^4 + 3{,}364x^6.$$

After combining like terms, we get

$$(*) \quad 1{,}820x^6 - 31{,}955x^4 - 1{,}017{,}450x^2 - 5{,}241{,}915 = 0$$

$$\| $$

$$35(x^2+9)^2(52x^2-1849).$$

We see that the only positive solution is $x = \frac{43}{2\sqrt{13}} \doteq 5.963.$

Hence, she should aim to land about 5.963 km downstream. Unsurprisingly, this is further downstream than the case with no current, where the solution was to land ~3.4 km downstream.

Question: why does polynomial (*) factor over the integers?