Bellman function and a local Chang-Wilson-Wolff theorem

Abstract

Following [1], we study functions whose square function is from $L^\infty([0,1])$ and establish the corresponding exponential integrability result. Using a concise Bellman-function proof, we expand the original result to the case of an arbitrary measurable subset E of $[0,1]$. Let D the dyadic lattice rooted in $[0,1]$. Let

$$S\varphi(x) = \sum_{I\in D, x\in I} \left(\langle\varphi\rangle_{I_+} - \langle\varphi\rangle_{I_-}\right)^2.$$

We prove that if $S_{[0,1]} = \|S\varphi(x)\|^2_{L^\infty(E)} \le 1$, then there exist absolute constants $\alpha > 0$ and $C > 0$ such that

$$\int_E e^{\alpha\left(\varphi - \langle\varphi\rangle_{[0,1]}\right)^2} \le C. \tag{1}$$

This result follows immediately from the local version of the dyadic Chang-Wilson-Wolff theorem. Let $F(E) = \left\{\varphi \in L^1 : \ \|S\varphi(x)\|_{L^\infty(E)} < \infty\right\}$.

Theorem. *For every $\varphi \in F(E)$ and every $t \ge 0$,*

$$\int_E e^{t\left(\varphi(s) - \langle\varphi\rangle_{[0,1]}\right)} ds \le e^{At^2 S_{[0,1]}}. \tag{2}$$

The proof is a short Bellman-type argument, allowing the extension to the case of $E \ne [0,1]$ efortlessly. We also consider the continuous version of the theorem, as well as its implications for the g-function problem. This is a joint result with A. Volberg.

References

[1] S.-Y. A. Chang, J. M. Wilson, T. H. Wolff. Some weighted norm inequalities concerning the Schrödinger operators. *Comment. Math. Helvetici*, Vol. 60 (1985), no. 2, pp. 217-246.