Results and conjectures on the "size" of the space of smooth functions on a compact manifold.

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Let M be a smooth compact manifold and let $C^\infty(M)$ be the space of all real (or if it is convenient complex) valued functions on M. Fix a Riemannian metric on M and let Δ be the Laplacian on M with the sign chosen so that Δ is a positive semi-definite operator. For $k \geq 0$ define $q_k \colon C^\infty(M) \to [0,\infty)$ by

$$q_k(u) = \|(I+\Delta)^{k/2}u\|_{L^2(M)}.$$

Then the collection of semi-norms $\{q_k\}_{k=1}^\infty$ defines a Fréchet topology on $C^\infty(M)$. This topology is independent of the choice of the Riemannian metric on M and is the standard topology on $C^\infty(M)$.

Definition. If M and N are smooth compact Riemannian manifolds. Then a linear operator $L\colon C^\infty(M) \to C^\infty(N)$ is ***of finite order*** iff there is an integer ℓ, the ***order*** of L, so that for all k, there is $C_k > 0$ such that

$$q_k(Lu) \leq C_k q_{k+\ell}(u)$$

for all $u \in C^\infty(M)$.

Proposition. *Given any two compact Riemannian manifolds M and N (with no restrictions on the dimensions) there is an injective linear $L\colon C^\infty(M) \to C^\infty(N)$ of order 0. In fact L can be taken to be a an integral operator with smooth kernel.*

This gives counterexamples to a conjecture of mine that if $\dim M > \dim N$, then the kernel of L must be infinite dimensional. The goal had been to explain why the Radon transforms $R\colon C^\infty(\mathbf{Gr}_j(\mathbf{R}^n)) \to C^\infty(\mathbf{Gr}_k(\mathbf{R}^n))$ between function spaces on Grassmann manifolds have infinite dimensional kernels when $\dim \mathbf{Gr}_j(\mathbf{R}^n) > \dim \mathbf{Gr}_k(\mathbf{R}^n)$ by a method independent of representation theory that would generalize to geometric transforms not related to group actions.

The following would make the intuitive idea that $C^\infty(M)$ is somehow "larger" than $C^\infty(N)$ when $\dim M > \dim N$.

Conjecture. *Let $A\colon C^\infty(M) \to C^\infty(N)$ and $B\colon C^\infty(N) \to C^\infty(M)$ be linear operators each of finite order with*

$$AB = I_{C^\infty(M)} \qquad \text{and} \qquad BA = I_{C^\infty(N)}.$$

Then $\dim M = \dim N$.

This is known to be false if the condition that A and B have finite order is dropped. More details and a proof of the proposition above can be found at `http://www.math.sc.edu/~howard/Notes/counterexample.pdf`.