SPDEs of fluctuating hydrodynamics type and large deviations

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IV. Weak solutions of the skeleton equation

Equivalence of weak and renormalized kinetic solutions [F., Gess; 2023]

Under assumptions including $\Phi(\xi) = \xi^m$ for every $m \in [1, \infty)$, a nonnegative function $\rho \in \mathrm{C}([0,T]; L^1(\mathbb{T}^d))$ that satisfies

$$\Phi^{\frac{1}{2}}(\rho) \in L^2([0,T]; H^1(\mathbb{T}^d))$$

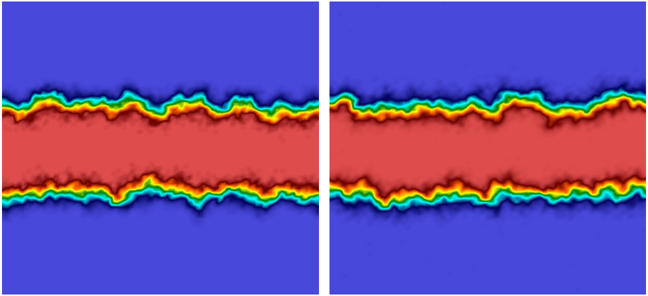
is a renormalized kinetic solution of the skeleton equation

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \ \text{ in } \ \mathbb{T}^d \times (0,T) \ \text{ with } \ \rho(\cdot, 0) = \rho_0,$$

for a nonnegative ρ_0 with finite entropy if and only if ρ is a weak solution. In particular, weak solutions exist and are unique.

- equivalence of renormalized and weak solutions [DiPerna, Lions; 1989], [Ambrosio; 2004].
- strong continuity with respect to weak convergence of the control
- for example, $\Phi^{\frac{1}{2}}$ convex or concave or Φ satisfies that $0 < \lambda \leq \Phi' \leq \Lambda$.

V. SPDEs of fluctuating hydrodynamics type



- a miscible mixture developing a rough diffusive interface due to the effect of thermal fluctuations [Donev; 2018]
- Fluctuating hydrodynamics, for example, [Spohn; 1991]
  — in the zero range case, the formal SPDE

$$\partial_t \rho_\varepsilon = \Delta \Phi(\rho_\varepsilon) - \sqrt{\varepsilon} \nabla \cdot (\Phi^{\frac{1}{2}}(\rho_\varepsilon)\xi).$$

  — fluctuation-dissipation relation, for the free energy $\Psi'_\Phi(\xi) = \log(\Phi(\xi))$,

$$\Phi'(\rho) = \Phi(\rho)\Psi''_\Phi(\rho).$$

  — coarse-graining and correlated noise

V. SPDEs of fluctuating hydrodynamics type

The mean behavior: the hydrodynamic limit

$$\partial_t \overline{\rho} = \Delta\sigma(\overline{\rho}) = \nabla \cdot J(\overline{\rho}),$$

for the flux $J(\overline{\rho}) = \nabla\sigma(\overline{\rho})$.

Fluctuating hydrodynamics: the isotropic non-equilibrium fluctuations ρ described by the continuity equation

$$\partial_t = \nabla \cdot j(\rho) \;\text{ with }\; j(\rho) = J(\rho) + \alpha,$$

for the mobility m and a Gaussian noise α satisfying [Spohn; 1991]

$$\langle \alpha_i(x,t)\alpha_j(y,s)\rangle = m(\rho)\delta_{ij}\delta_0(x-y)\delta_0(y-s).$$

The formal SPDE: the noise $\alpha = \sqrt{m(\rho)}\xi$ for ξ a space-time white noise,

$$\partial_t \rho = \Delta\sigma(\rho) - \nabla \cdot (\sqrt{m(\rho)}\xi).$$

The zero range process: $\sigma(\rho) = \Phi(\rho)$ and $m(\rho) = \Phi(\rho)$ and

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho)\xi).$$

The exclusion process: $\sigma(\rho) = \rho$ and $m(\rho) = \rho(1-\rho)$ and

$$\partial_t \rho = \Delta\rho - \nabla \cdot (\sqrt{\rho(1-\rho)}\xi).$$

V. SPDEs of fluctuating hydrodynamics type

Space-time white noise: a Gaussian noise ξ on $\mathbb{T}^d$ defined by

$$\mathrm{d}\xi = \sum_{k\in\mathbb{Z}^d} e_k\,\mathrm{d}B^k_t \quad \text{for a smooth orthonormal } L^2(\mathbb{T}^d) \text{ basis ``} e_k = \sqrt{2}\sin(k\cdot x).\text{''}$$

Schilder's theorem: for a Brownian motion B and $A \subseteq \mathrm{C}([0,T])$,

$$\mathbb{P}[\sqrt{\varepsilon}B \in A] \simeq \exp\big(-\varepsilon^{-1}\inf_{x\in A} I(x)\big) \quad \text{for} \quad I(x) = \frac{1}{2}\int_0^T |\dot{x}(s)|^2\,\mathrm{d}s.$$

The contraction principle: for the solutions

$$\partial_t\rho_\varepsilon = \Delta\Phi(\rho_\varepsilon) - \sqrt{\varepsilon}\nabla\cdot(\Phi^{\frac{1}{2}}(\rho_\varepsilon)\xi),$$

we have formally that, for $A \subseteq L^1_t L^1_x$,

$$\mathbb{P}[\rho_\varepsilon \in A] \simeq \exp\big(-\varepsilon^{-1}\inf_{\rho\in A} I(\rho)\big),$$

for the rate function

$$I(\rho) = \frac{1}{2}\inf\big\{\,\|g\|_2^2 : \partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot\big(\Phi^{\frac{1}{2}}(\rho)g\big)\big\}.$$

The Dean–Kawasaki equation: we have,

$$\partial_t \rho_\varepsilon = \frac{1}{2}\Delta \rho_\varepsilon - \sqrt{\varepsilon}\nabla \cdot (\sqrt{\rho_\varepsilon}\xi).$$

The Zero Range Process: the formal SPDE describing non-equilibrium behavior,

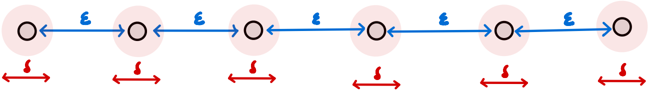
$$\partial_t \rho_\varepsilon = \Delta\Phi(\rho_\varepsilon) - \sqrt{\varepsilon}\nabla \cdot (\Phi^{\frac{1}{2}}(\rho_\varepsilon)\xi).$$

- Supercritical in the language of regularity structures [Hairer; 2014]
- Ill-posedness vs. triviality
  - — for example, [Konarovskyi, Lehmann, von Renesse; 2019]
- Degenerate diffusions
  - — porous media and fast diffusions, $\Phi(\xi) = \xi^m$ for every $m \in (0, \infty)$
- Irregular noise coefficients

The Dean–Kawasaki equation: for independent Brownian motions,

$$\partial_t \rho = \frac{1}{2}\Delta\rho - \sqrt{\varepsilon}\nabla\cdot(\sqrt{\rho}\xi).$$

- White noise is too singular (particles systems, course graining...):



- Spatially correlated noise:

$$\xi^\delta = \xi * \kappa^\delta \text{ for a convolution kernel } \kappa^\delta \text{ of scale } \delta\in(0,1).$$

Dean–Kawasaki equation with correlated noise: the Stratonovich equation,

$$\partial_t \rho = \frac{1}{2}\Delta\rho - \sqrt{\varepsilon}\nabla\cdot(\sqrt{\rho}\circ\xi^\delta).$$

The Stratonovich-to-Itô correction: we consider the Stratonovich SPDE

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\sigma(\rho) \circ f(x)\,\mathrm{d}\xi^\delta_t),$$

for the noise $\mathrm{d}\xi^\delta = \sum_{k=1}^\infty e^\delta_k\,\mathrm{d}B^k_t$.

$$\int_0^t \int_{\mathbb{T}^d} \sigma(\rho_s) \circ f\,\mathrm{d}\xi^\delta = \sum_{k=1}^\infty \int_{\mathbb{T}^d} e^\delta_k \sum_{|\mathcal{P}|\to 0} \frac{\sigma(\rho_{t_{i+1}}) + \sigma(\rho_{t_i})}{2}(B^k_{t_{i+1}} - B^k_{t_i})$$

$$= \sum_{k=1}^\infty \int_{\mathbb{T}^d} e^\delta_k \Big(\frac{1}{2}\sum_{|\mathcal{P}|\to 0} (\sigma(\rho_{t_{i+1}}) - \sigma(\rho_{t_i}))(B_{t_{i+1}} - B_{t_i}) + \sum_{|\mathcal{P}|\to 0} \sigma(\rho_{t_i})(B_{t_{i+1}} - B_{t_i})\Big)$$

$$= \frac{1}{2}\sum_{k=1}^\infty \int_0^t \int_{\mathbb{T}^d} e^\delta_k \sigma'(\rho)\,\mathrm{d}\langle \partial_t \rho, B^k\rangle_s + \int_0^t \int_{\mathbb{T}^d} \sigma(\rho)\,\mathrm{d}\xi^\delta$$

$$= \frac{1}{2}\sum_{k=1}^\infty \int_0^t \int_{\mathbb{T}^d} (e^\delta_k \sigma'(\rho)\nabla(\sigma(\rho)e^\delta_k)\,\mathrm{d}s + \int_0^t \int_{\mathbb{T}^d} f\sigma(\rho)\,\mathrm{d}\xi^\delta$$

$$= \frac{1}{2}\sum_{k=1}^\infty \int_{\mathbb{T}^d} \Big((e^\delta_k)^2 \sigma'(\rho)^2 \nabla\rho + \sigma'(\rho)\sigma(\rho)e^\delta_k \nabla e^\delta_k\Big)\,\mathrm{d}s + \int_0^t \int_{\mathbb{T}^d} f\sigma(\rho)\,\mathrm{d}B_s$$

The Itô-form of the SPDE: we have that

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\sigma(\rho)\,\mathrm{d}\xi^\delta) + \frac{\langle \xi^\delta \rangle}{2} \nabla \cdot (\sigma'(\rho)^2 \nabla\rho).$$

A general SPDE with conservative noise: for consider the Stratonovich SPDE

$$\partial_t \rho = \Delta\Phi(\rho) - \sqrt{\varepsilon}\nabla\cdot(\sigma(\rho)\circ \mathrm{d}\xi^\delta),$$

for probabilistically stationary noise $\xi^\delta = (\xi * \kappa^\delta)$ and scalar σ.

A choice of "renormalization": for the stochastic integral

$$\int_0^t f(B_s)\circ_\theta \, \mathrm{d}B_s = \lim_{|\mathcal{P}|\to 0}\sum \big(\theta f(B_{t_{i+1}}) + (1-\theta) f(B_{t_i})\big)(B_{t_{i+1}} - B_{t_i}),$$

yields the equation

$$\partial_t \rho = \Delta\Phi(\rho) - \sqrt{\varepsilon}\nabla\cdot(\sigma(\rho)\circ_\theta \, \mathrm{d}\xi^\delta),$$

and the Itô formulation

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)\,\mathrm{d}\xi^\delta) + \theta\langle\xi^\delta\rangle\nabla\cdot(\sigma'(\rho)^2\nabla\rho).$$

An Itô equation with a correction for $\theta\in[1/2,\infty)$,

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)\,\mathrm{d}\xi^\delta) + \theta\langle\xi^\delta\rangle\nabla\cdot(\sigma'(\rho)^2\nabla\rho).$$

— Stratonovich integration yields $\theta = 1/2$
— Klimontovich integration yields $\theta = 1$
— choice of correction motivated by gradient flow formulation

A general SPDE with conservative noise: for consider the Stratonovich SPDE

$$\partial_t \rho = \Delta\Phi(\rho) - \sqrt{\varepsilon}\nabla\cdot(\sigma(\rho)\circ \mathrm{d}\xi^\delta),$$

for probabilistically stationary noise $\xi^\delta = (\xi * \kappa^\delta)$ and scalar σ.

The Itô-formulation for the spatially constant quadratic variation $\langle \xi^\delta \rangle$,

$$\partial_t \rho = \Delta\Phi(\rho) - \sqrt{\varepsilon}\nabla\cdot(\sigma(\rho)\,\mathrm{d}\xi^\delta) + \frac{\varepsilon\langle \xi^\delta\rangle}{2}\nabla\cdot(\sigma'(\rho)^2\nabla\rho).$$

Logarithmic divergence of the correction: if $\sigma(\rho) = \sqrt{\rho}$ then

$$\frac{\varepsilon\langle \xi^\delta\rangle}{2}\nabla\cdot(\sigma'(\rho)^2\nabla\rho) = \frac{\varepsilon\langle \xi^\delta\rangle}{8}\nabla\cdot\left(\frac{1}{\rho}\nabla\rho\right) = \frac{\varepsilon\langle \xi^\delta\rangle}{8}\Delta\log(\rho),$$

and we have, in the Dean–Kawasaki case,

$$\partial_t \rho = \Delta\rho - \sqrt{\varepsilon}\nabla\cdot(\sqrt{\rho}\,\mathrm{d}\xi^\delta) + \frac{\varepsilon\langle \xi^\delta\rangle}{8}\Delta\log(\rho).$$

The SPDE: we specialize to the case

$$\partial_t \rho = \Delta\rho - \nabla\cdot(\rho^{\frac{1}{2}}\,\mathrm{d}\xi^\delta) + \frac{\langle\xi^\delta\rangle}{2}\nabla\cdot(\frac{1}{4\rho}\nabla\rho).$$

The PDE renormalization: for a smooth $S\colon\mathbb{R}\to\mathbb{R}$ and $\phi\in\mathrm{C}^\infty(\mathbb{T}^d)$,

$$\begin{aligned}
\partial_t\Big(\int_{\mathbb{T}^d} S(\rho)\phi(x)\Big) &= \int_{\mathbb{T}^d} S'(\rho)\phi(x)\big(\Delta\rho - \nabla\cdot(\rho^{\frac{1}{2}}\,\mathrm{d}\xi^\delta) + \frac{\langle\xi^\delta\rangle}{2}\nabla\cdot(\frac{1}{4\rho}\nabla\rho)\big)\\
&= -\int_{\mathbb{T}^d} S'(\rho)\nabla\phi(x)\cdot\Big(\nabla\rho - \rho^{\frac{1}{2}}\,\mathrm{d}\xi^\delta + \frac{\langle\xi^\delta\rangle}{8\rho}\nabla\rho\Big)\\
&\quad -\int_{\mathbb{T}^d} S''(\rho)\phi(x)\Big(|\nabla\rho|^2 - \rho^{\frac{1}{2}}\,\mathrm{d}\xi^\delta\cdot\nabla\rho + \frac{\langle\xi^\delta\rangle}{8\rho}|\nabla\rho|^2\Big)\\
&\quad + \frac{1}{2}\int_{\mathbb{T}^d} S''(\rho)\phi(x)\sum_{k=1}^\infty |\nabla(e_k^\delta\rho^{\frac{1}{2}})|^2.
\end{aligned}$$

The Itô correction: We have that

$$\begin{aligned}
\sum_{k=1}^\infty |\nabla(e_k^\delta\rho^{\frac{1}{2}})|^2 &= \sum_{k=1}^\infty \Big(\rho|\nabla e_k^\delta|^2 + 2\rho^{\frac{1}{2}}\nabla\rho^{\frac{1}{2}}\cdot e_k^\delta\nabla e_k^\delta + (e_k^\delta)^2|\nabla\rho^{\frac{1}{2}}|^2\\
&= \langle\nabla\cdot\xi^\delta\rangle\rho + \langle\xi^\delta\rangle\frac{1}{4\rho}|\nabla\rho|^2.
\end{aligned}$$

The SPDE: for the equation

$$\partial_t \rho = \Delta\rho - \nabla\cdot(\rho^{\frac{1}{2}}\,\mathrm{d}\xi^\delta) + \frac{\langle\xi^\delta\rangle}{2}\nabla\cdot\left(\frac{1}{4\rho}\nabla\rho\right).$$

The PDE renormalization: for a smooth $S\colon\mathbb{R}\to\mathbb{R}$ and $\phi\in \mathrm{C}^\infty(\mathbb{T}^d)$,

$$\partial_t\left(\int_{\mathbb{T}^d} S(\rho)\phi(x)\right) = -\int_{\mathbb{T}^d} S'(\rho)\nabla\phi(x)\cdot\left(\nabla\rho - \rho^{\frac{1}{2}}\,\mathrm{d}\xi^\delta + \frac{\langle\xi^\delta\rangle}{8\rho}\nabla\rho\right)$$
$$-\int_{\mathbb{T}^d} S''(\rho)\phi(x)\left(|\nabla\rho|^2 - \rho^{\frac{1}{2}}\,\mathrm{d}\xi^\delta\cdot\nabla\rho + \frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\rho\right).$$

The kinetic formulation: for $\chi(x,\xi,t) = \mathbf{1}_{\{0<\xi<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\xi<0\}}$,

$$\partial_t\left(\int_{\mathbb{T}^d}\int_{\mathbb{R}} S'(\rho)\phi(x)\chi\right) = -\int_{\mathbb{T}^d}\int_{\mathbb{R}} \nabla(S'(\xi)\phi(x))\cdot\left(\nabla\chi + \frac{\langle\xi^\delta\rangle}{8\xi}\nabla\chi\right)$$
$$-\int_{\mathbb{T}^d}\int_{\mathbb{R}} \partial_\xi(S'(\xi)\phi(x))\delta_\rho\left(|\nabla\rho|^2 + \frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\xi\right)$$
$$-\int_{\mathbb{T}^d}\int_{\mathbb{R}} \nabla(S'(\xi)\phi(x))\cdot(\partial_\xi\chi)\xi^{\frac{1}{2}}\,\mathrm{d}\xi^\delta + \partial_\xi(S'(\xi)\phi(x))\xi^{\frac{1}{2}}\nabla\chi\cdot\mathrm{d}\xi^\delta.$$

— we use $\nabla\chi = \delta_\rho\nabla\rho$ and $\partial_\xi\chi = \delta_0 - \delta_\rho$

— the test function $\psi(x,\xi) = S'(\xi)\phi(x)$

The kinetic formulation: for the solution

$$\partial_t \rho = \Delta\rho - \nabla\cdot(\rho^{\frac{1}{2}}\,\mathrm{d}\xi^\delta) + \frac{\langle\xi^\delta\rangle}{2}\nabla\cdot\big(\frac{1}{4\rho}\nabla\rho\big),$$

we for $\chi(x,\xi,t) = \mathbf{1}_{\{0<\xi<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\xi<0\}}$ that

$$\partial_t\chi = \Delta\chi + \frac{\langle\xi^\delta\rangle}{8\xi}\Delta\chi + \partial_\xi q - \partial_\xi\Big(\delta_\rho\big(\frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\xi\big)\Big)$$
$$+\nabla\cdot((\partial_\xi\chi)\xi^{\frac{1}{2}}\,\mathrm{d}\xi^\delta) - \partial_\xi(\xi^{\frac{1}{2}}\nabla\chi\cdot\,\mathrm{d}\xi^\delta),$$

for a locally finite nonnegative measure q satisfying the "entropy inequality"

$$q \geq \delta_\rho|\nabla\rho|^2.$$

Preservation of nonnegativity and mass: we have $\mathbb{P}$-a.e. that if $\rho_0 \geq 0$ then $\rho \geq 0$ with

$$\|\rho(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_0\|_{L^1(\mathbb{T}^d)}.$$

The kinetic formulation: we have that

$$\partial_t\chi = \Delta\chi + \frac{\langle \xi^\delta \rangle}{8\xi}\Delta\chi + \partial_\xi q - \partial_\xi\Big(\delta_\rho\big(\frac{\langle \nabla\cdot\xi^\delta\rangle}{2}\xi\big)\Big) + \nabla\cdot((\partial_\xi\chi)\xi^{\frac12}\,\mathrm{d}\xi^\delta) - \partial_\xi(\xi^{\frac12}\nabla\chi\cdot\,\mathrm{d}\xi^\delta).$$

The entropy estimate: testing with $\psi(\xi)=\log(\xi)$,

$$\begin{aligned}
&\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi\log(\xi)\Big|_{s=0}^{s=t}\\
&= -\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\frac{1}{\xi}q + \int_{\mathbb{T}^d}\int_{\mathbb{R}}\big(\delta_\rho\big(\frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\xi\big)\big)\frac{1}{\xi} + \int_{\mathbb{T}^d}\int_{\mathbb{R}}\big(\xi^{\frac12}\nabla\chi\cdot\,\mathrm{d}\xi^\delta\big)\frac{1}{\xi}\\
&= -\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\frac{1}{\xi}q + \frac12\int_{\mathbb{T}^d}\langle\nabla\cdot\xi^\delta\rangle + 2\int_{\mathbb{T}^d}\nabla\rho^{\frac12}\cdot\,\mathrm{d}\xi^\delta.
\end{aligned}$$

We therefore have using $\delta_\rho|\nabla\rho^{\frac12}|^2\lesssim\xi^{-1}q$ that

$$\max_{t\in[0,T]}\int_{\mathbb{T}^d}\rho\log(\rho) + \int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\frac{1}{\xi}q \lesssim \int_{\mathbb{T}^d}\rho_0\log(\rho_0) + \langle\nabla\cdot\xi^\delta\rangle + \max_{t\in[0,T]}\Big|\int_{\mathbb{T}^d}\nabla\rho^{\frac12}\cdot\,\mathrm{d}\xi^\delta\Big|.$$

Using the Burkholder–Davis–Gundy inequality

$$\mathbb{E}\Big[\max_{t\in[0,T]}\int_{\mathbb{T}^d}\rho\log(\rho) + \int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{R}}\frac{1}{\xi}q\Big] \lesssim \mathbb{E}\Big[\int_{\mathbb{T}^d}\rho_0\log(\rho_0) + \langle\nabla\cdot\xi^\delta\rangle\Big].$$

The kinetic formulation: we have that

$$\partial_t \chi = \Delta\chi + \frac{\langle \xi^\delta \rangle}{8\xi}\Delta\chi + \partial_\xi q - \partial_\xi\Big(\delta_\rho\big(\frac{\langle \nabla\cdot\xi^\delta\rangle}{2}\xi\big)\Big) + \nabla\cdot((\partial_\xi\chi)\xi^{\frac{1}{2}}\,\mathrm{d}\xi^\delta) - \partial_\xi(\xi^{\frac{1}{2}}\nabla\chi\cdot\,\mathrm{d}\xi^\delta).$$

The local H^1-estimate: testing with $\psi'_M(\xi) = \mathbf{1}_{\{M,M+1\}} = \mathbf{1}_M$,

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi\psi_M\Big|_{s=0}^{s=t}$$
$$= -\int_0^t\int_{\mathbb{T}^d}\int_M^{M+1} q + \frac{1}{2}\int_{\mathbb{T}^d}\langle\nabla\cdot\xi^\delta\rangle\rho\mathbf{1}_M(\rho) + 2\int_{\mathbb{T}^d}\mathbf{1}_M(\rho)\rho^{\frac{1}{2}}\nabla\rho\cdot\,\mathrm{d}\xi^\delta.$$

We have using $\delta_\rho|\nabla\rho|^2 \lesssim q$ and the Burkholder–Davis–Gundy inequality that

$$\mathbb{E}\Big[\int_0^t\int_{\mathbb{T}^d}\int_M^{M+1} q\Big] \lesssim \int_{\mathbb{T}^d}(\rho_0 - M)_+ + \mathbb{E}\Big[(M+1)\int_{\mathbb{T}^d}\langle\nabla\cdot\xi^\delta\rangle\mathbf{1}_M(\rho)\Big].$$

Vanishing of the measure at infinity: using the real analysis lemma, $\mathbb{P}$-a.e.,

$$\liminf_{M\to\infty} q(\mathbb{T}^d\times(M,M+1)\times[0,T]) = 0.$$

Stochastic kinetic solutions [F. Gess; 2021]

A *stochastic kinetic solution* is a continuous $L^1(\mathbb{T}^d)$-valued, $\mathcal{F}_t$-predictable process ρ and a random *kinetic measure* q that satisfy the following five properties.

(i) *Preservation of mass*: we have that $\mathbb{E}\,\|\rho(\cdot,t)\|_{L^1} = \mathbb{E}\,\|\rho_0\|_{L^1}$.

(iii) *Local regularity*: $\mathbb{P}$-a.e. for every $K \in \mathbb{N}$,

$$(\rho \wedge K) \vee (1/K) \in L^2(\Omega \times [0,T]; H^1(\mathbb{T}^d)).$$

(iv) *Regularity and vanishing of the measure at infinity*: $\mathbb{P}$-a.e. we have that

$$\liminf_{M\to\infty}(q(\mathbb{T}^d \times (M, M+1) \times [0,T])) = 0 \;\text{ and }\; q \geq \delta_\rho |\nabla\rho|^2.$$

(v) *The equation*: for every $\psi \in \mathrm{C}_c^\infty(\mathbb{T}^d \times (0,\infty))$ and $t \in (0,\infty)$,

$$\int_{\mathbb{R}}\int_{\mathbb{T}^d} \psi\chi|_{s=0}^{t} = -\int_0^t\int_{\mathbb{T}^d} \nabla\rho\cdot(\nabla\psi)(x,\rho) - \frac{\varepsilon\langle\xi^\delta\rangle}{8}\int_0^t\int_{\mathbb{T}^d}\frac{1}{\rho}\nabla\rho\cdot(\nabla\psi)(x,\rho)$$
$$-\int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\partial_\xi\phi\,\mathrm{d}q - \int_0^t\int_{\mathbb{T}^d}\psi(x,\rho)\nabla\cdot(\sigma(\rho)\,\mathrm{d}\xi^\delta) + \frac{\langle\nabla\xi^\delta\rangle}{2}\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\rho.$$

V. SPDEs of fluctuating hydrodynamics type

Extensions: we consider general equations of the type

$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot \Big(\sigma(\rho) \circ \xi^\delta + \nu(\rho) \Big) + \lambda(\rho) + \phi(\rho) \xi^\delta,$$

including non-equilibrium fluctuations of asymmetric systems, mean-field games, stochastic geometric PDEs, and branching interacting diffusions.

- The generalized Dean-Kawasaki equation with correlated noise

$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot \big(\Phi(\rho) + \Phi^{\frac{1}{2}}(\rho) \circ \xi^\delta \big).$$

- Nonlinear Dawson-Watanabe equation

$$\partial_t \rho = \Delta \Phi(\rho) + \sqrt{\rho} \xi^\delta.$$

- Fluctuating mean-curvature equation

$$\partial_t \rho = \nabla \cdot \big(\frac{\nabla \rho}{1 + \rho^2} \big) + \nabla \cdot \big((1 + \rho^2)^{\frac{1}{4}} \circ \xi^\delta \big).$$

- Fast diffusion and porous media: $\Phi(\xi) = \xi^m$ for any $m \in (0, \infty)$.
- ϕ is globally $1/2$-Hölder continuous, λ is globally Lipschitz continuous.

Vanishing of the defect measure: for the equation

$$\partial_t \chi = \Delta\chi + \frac{\langle \xi^\delta \rangle}{8\xi}\Delta\chi + \partial_\xi q - \partial_\xi\Big(\delta_\rho\big(\frac{\langle \nabla\cdot\xi^\delta\rangle}{2}\xi\big)\Big) + \nabla\cdot\big((\partial_\xi\chi)\xi^{\frac{1}{2}}\,\mathrm{d}\xi^\delta\big) - \partial_\xi\big(\xi^{\frac{1}{2}}\nabla\chi\cdot\,\mathrm{d}\xi^\delta\big),$$

for the test functions $\psi'_\beta = \frac{2}{\beta}\mathbf{1}_{\{\frac{\beta}{2}<\xi<\beta\}}$ and $\zeta'_M = -\mathbf{1}_{\{M<\xi<M+1\}}$,

$$\mathbb{E}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi\psi_\beta\zeta_M\Big|_{s=0}^{s=t} = \mathbb{E}\Big[-\frac{2}{\beta}\int_0^t\int_{\mathbb{T}^d}\int_{\frac{\beta}{2}}^{\beta} q + \int_0^t\int_{\mathbb{T}^d}\int_M^{M+1} q\Big]$$
$$+\frac{2}{\beta}\int_0^t\int_{\mathbb{T}^d}\frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\rho\mathbf{1}_{\{\frac{\beta}{2}<\rho<\beta\}} + \int_0^t\int_{\mathbb{T}^d}\frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\rho\mathbf{1}_{\{M<\rho<M+1\}}.$$

The righthand side vanishes as $M\to\infty$ and $\beta\to 0$. Therefore,

$$\text{“}\mathbb{E}\int_0^t\int_{\mathbb{T}^d} q(x,0,s)\text{”} = \lim_{\beta\to 0}\mathbb{E}\Big[\frac{2}{\beta}\int_0^t\int_{\mathbb{T}^d}\int_{\frac{\beta}{2}}^{\beta} q\Big] = \mathbb{E}\Big[\int_{\mathbb{T}^d}\big(\rho_0(x)-\rho(x,t)\big)\Big] = 0.$$

We have $\mathbb{P}$-a.e. that

$$\liminf_{\beta\to 0}\Big(\frac{2}{\beta}q(\mathbb{T}^d\times({}^{\beta}\!/_{2},\beta)\times[0,T])\Big) = 0.$$

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$$(\rho \wedge K) \vee (1/K) \in L^2(\Omega \times [0,T]; H^1(\mathbb{T}^d)).$$

(iv) *Regularity and vanishing of the measure at infinity*: $\mathbb{P}$-a.e. we have that

$$\liminf_{M\to\infty}(q(\mathbb{T}^d \times (M, M+1) \times [0,T])) = 0 \;\text{ and }\; q \geq \delta_\rho |\nabla \rho|^2.$$

(v) *The equation*: for every $\psi \in \mathrm{C}_c^\infty(\mathbb{T}^d \times (0,\infty))$ and $t \in (0,\infty)$,

$$\int_{\mathbb{R}}\int_{\mathbb{T}^d} \psi\chi|_{s=0}^t = -\int_0^t\int_{\mathbb{T}^d} \nabla\rho\cdot(\nabla\psi)(x,\rho) - \frac{\varepsilon\langle\xi^\delta\rangle}{8}\int_0^t\int_{\mathbb{T}^d}\frac{1}{\rho}\nabla\rho\cdot(\nabla\psi)(x,\rho)$$
$$-\int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^d}\partial_\xi\phi\,\mathrm{d}q - \int_0^t\int_{\mathbb{T}^d}\psi(x,\rho)\nabla\cdot(\sigma(\rho)\,\mathrm{d}\xi^\delta) + \frac{\langle\nabla\xi^\delta\rangle}{2}\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\rho.$$

— we have that $\liminf_{\beta\to 0}\left(\frac{2}{\beta}q(\mathbb{T}^d \times (\beta/2, \beta) \times [0,T])\right) = 0.$

V. SPDEs of fluctuating hydrodynamics type

A useful identity: if ρ_1 and ρ_2 are kinetic solutions, for

$$\chi_i(x,\xi,t) = \mathbf{1}_{\{0<\xi<\rho_i(x,t)\}} - \mathbf{1}_{\{\rho_i(x,t)<\xi<0\}},$$

we have

$$\begin{aligned}\int_{\mathbb{T}^d} |\rho_1 - \rho_2| = \int_{\mathbb{T}^d}\int_{\mathbb{R}} |\chi_1 - \chi_2|^2 &= \int_{\mathbb{T}^d}\int_{\mathbb{R}} (\chi_1)^2 + (\chi_2)^2 - 2\chi_1\chi_2 \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi_1 \mathrm{sgn}(\xi) + \chi_2 \mathrm{sgn}(\xi) - 2\chi_1\chi_2 \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi_1 + \chi_2 - 2\chi_1\chi_2.\end{aligned}$$

The cutoff functions: the cutoff at zero, for $\beta \in (0,1)$,

$$\psi_\beta(0) = 0 \ \text{ and } \ \psi_\beta' = \frac{2}{\beta}\mathbf{1}_{\{\frac{\beta}{2}<\xi<\beta\}},$$

and the cutoff at infinity, for $M \in (1,\infty)$,

$$\zeta_M(0) = 1 \ \text{ and } \ \zeta_M' = -\mathbf{1}_{\{M<\xi<M+1\}}.$$

The essential identity: we will use that

$$\int_{\mathbb{T}^d} |\rho_1 - \rho_2| = \lim_{\beta\to 0}\lim_{M\to\infty}\left(\int_{\mathbb{T}^d}\int_{\mathbb{R}} (\chi_1 + \chi_2 - 2\chi_1\chi_2)\psi_\beta\zeta_M\right).$$

The equation: we have that

$$\partial_t \chi = \Delta\chi + \frac{\langle \xi^\delta \rangle}{8\xi}\Delta\chi + \partial_\xi q - \partial_\xi\Big(\delta_\rho\big(\frac{\langle \nabla\cdot\xi^\delta\rangle}{2}\xi\big)\Big) + \nabla\cdot((\partial_\xi\chi)\xi^{\frac{1}{2}}\,\mathrm{d}\xi^\delta) - \partial_\xi(\xi^{\frac{1}{2}}\nabla\chi\cdot\mathrm{d}\xi^\delta),$$

and we use that

$$\int_{\mathbb{T}^d}|\rho_1-\rho_2| = \lim_{\beta\to 0}\lim_{M\to\infty}\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\big(\chi_1+\chi_2-2\chi_1\chi_2\big)\psi_\beta\zeta_M\Big).$$

The singletons: we have that

$$\begin{aligned}\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_i\psi_\beta\zeta_M\Big) &= -\frac{2}{\beta}q_i\big(\mathbb{T}^d\times(\frac{\beta}{2},\beta)\times(0,t)\big) + q_i\big(\mathbb{T}^d\times(M,M+1)\times(0,t)\big)\\ &+\frac{2}{\beta}\int_{\mathbb{T}^d}\mathbf{1}_{\{\frac{\beta}{2}<\rho_i<\beta\}}\zeta_M(\rho_i)\rho_i^{\frac{1}{2}}\nabla\rho_i\cdot\mathrm{d}\xi^\delta + \int_{\mathbb{T}^d}\mathbf{1}_{\{M<\rho_i<M+1\}}\psi_\beta(\rho_i)\rho_i^{\frac{1}{2}}\nabla\rho_i\cdot\mathrm{d}\xi^\delta\\ &+\frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\big(\frac{2}{\beta}\mathbf{1}_{\{\frac{\beta}{2}<\rho_i<\beta\}}\zeta_M(\rho_i)\rho_i + \mathbf{1}_{\{M<\rho_i<M+1\}}\psi_\beta(\rho_i)\rho_i\big).\end{aligned}$$

These terms vanish in the limit $M\to\infty$ and $\beta\to 0$.

The equation: we have that

$$\partial_t \chi = \Delta\chi + \frac{\langle \xi^\delta \rangle}{8\xi}\Delta\chi + \partial_\xi q - \partial_\xi\Big(\delta_\rho\big(\frac{\langle \nabla\cdot\xi^\delta\rangle}{2}\xi\big)\Big) + \nabla\cdot((\partial_\xi\chi)\xi^{\frac{1}{2}}\,\mathrm{d}\xi^\delta) - \partial_\xi(\xi^{\frac{1}{2}}\nabla\chi\cdot\,\mathrm{d}\xi^\delta),$$

The mixed term: we have that

$$\begin{aligned}\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_1\chi_2\psi_\beta\zeta_M\Big) &= \text{“terms identical or analogous to the skeleton equation''}\\ &-\frac{\langle\xi^\delta\rangle}{8}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\frac{1}{\xi}\nabla\chi_1\cdot\nabla\chi_2\psi_\beta\zeta_M - \frac{\langle\xi^\delta\rangle}{8}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\frac{1}{\xi}\nabla\chi_2\cdot\nabla\chi_1\psi_\beta\zeta_M\\ &+\frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\delta_{\rho_1}(\partial_\xi\chi_2)\xi + \frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\delta_{\rho_2}(\partial_\xi\chi_1)\xi\\ &+\int_{\mathbb{T}^d}\int_{\mathbb{R}}\mathrm{d}\langle\chi_1,\chi_2\rangle\psi_\beta\zeta_M\\ &+\frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\xi(\psi_\beta\zeta_M)\chi_2\delta_{\rho_1}\xi + \frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\xi(\psi_\beta\zeta_M)\chi_1\delta_{\rho_2}\xi.\end{aligned}$$

The covariation term: for the covariation term

$$\begin{aligned}
&\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_1\chi_2\psi_\beta\zeta_M\Big) \\
&= -\frac{\langle\xi^\delta\rangle}{8}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\frac{1}{\xi}\nabla\chi_1\cdot\nabla\chi_2\psi_\beta\zeta_M - \frac{\langle\xi^\delta\rangle}{8}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\frac{1}{\xi}\nabla\chi_2\cdot\nabla\chi_1\psi_\beta\zeta_M \\
&\quad + \frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\delta_{\rho_1}(\partial_\xi\chi_2)\xi\psi_\beta\zeta_M \\
&\quad + \frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\delta_{\rho_2}(\partial_\xi\chi_1)\xi + \int_{\mathbb{T}^d}\int_{\mathbb{R}}\mathrm{d}\langle\chi_1,\chi_2\rangle\psi_\beta\zeta_M + \ldots
\end{aligned}$$

Since we have the identity

$$\mathrm{d}\langle\chi_1,\chi_2\rangle = \sum_{k=1}^\infty \delta_{\rho_1}\delta_{\rho_2}\nabla(\rho_1^{\frac{1}{2}}e_k^\delta)\cdot\nabla(\rho_2^{\frac{1}{2}}e_k^\delta) = \delta_{\rho_1}\delta_{\rho_2}\Big(\frac{\langle\xi^\delta\rangle}{4\xi}\nabla\rho_1\cdot\nabla\rho_2 + \langle\nabla\cdot\xi^\delta\rangle\rho_1^{\frac{1}{2}}\rho_2^{\frac{1}{2}}\Big)$$

we have using Hölder's and Young's inequality that

$$\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_1\chi_2\psi_\beta\zeta_M\Big) \geq -\frac{\langle\nabla\cdot\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\delta_{\rho_1}\delta_{\rho_2}(\rho_1^{\frac{1}{2}}-\rho_2^{\frac{1}{2}})^2\phi_\beta\zeta_M + \ldots,$$

where after regularization $\delta_{\rho_1}\delta_{\rho_2} \simeq \delta^{-1}\mathbf{1}_{\{|\rho_1-\rho_2|<\delta\}}$.

The cutoff term: for the cutoff term,

$$\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_1\chi_2\psi_\beta\zeta_M\Big) \geq \ldots + \frac{\langle \nabla\cdot\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\xi(\psi_\beta\zeta_M)\chi_2\delta_{\rho_1}\xi$$
$$+ \frac{\langle \nabla\cdot\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\xi(\psi_\beta\zeta_M)\chi_1\delta_{\rho_2}\xi.$$

We have that

$$\Big|\frac{\langle \nabla\cdot\xi^\delta\rangle}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\xi(\psi_\beta\zeta_M)\chi_2\delta_{\rho_1}\xi\Big|$$
$$\lesssim \frac{\beta}{2}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\rho_1\mathbf{1}_{\{\frac{\beta}{2}<\rho_1<\beta\}} + \int_{\mathbb{T}^d}\int_{\mathbb{R}}\rho_1\mathbf{1}_{\{M<\rho_1<M+1\}}$$
$$\lesssim \int_{\mathbb{T}^d}\int_{\mathbb{R}}\mathbf{1}_{\{\frac{\beta}{2}<\rho_1<\beta\}} + (M+1)\int_{\mathbb{T}^d}\int_{\mathbb{R}}\mathbf{1}_{\{M<\rho_1<M+1\}}.$$

The conclusion: we have $\mathbb{P}$-a.e. that

$$\partial_t\Big(\int_{\mathbb{T}^d}|\rho_1-\rho_2|\Big) = \partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\big(\chi_1+\chi_2-2\chi_1\chi_2\big)\Big)\leq 0,$$

and $\mathbb{P}$-a.e. that

$$\max_{t\in[0,T]}\|\rho_1-\rho_2\|_{L^1(\mathbb{T}^d)} = \|\rho_{1,0}-\rho_{2,0}\|_{L^1(\mathbb{T}^d)}.$$

V. SPDEs of fluctuating hydrodynamics type

Well-posedness of stochastic kinetic solutions [F. Gess; 2024]

Let $\rho_0 \in L^1(\Omega; L^1(\mathbb{T}^d))$ be nonnegative and $\mathcal{F}_0$-measurable. Then, there exists a unique stochastic kinetic solution with initial data ρ_0. Furthermore, two solutions ρ^1 and ρ^2 almost surely satisfy, for every $t \in [0, T]$,

$$\left\|\rho^1(\cdot,) - \rho^2(\cdot,t)\right\|_{L^1(\mathbb{T}^d)} \leq \left\|\rho_0^1 - \rho_0^2\right\|_{L^1(\mathbb{T}^d)}.$$

— Stochastic dynamics, random dynamical systems, and invariant measures [F., Gess, Gvalani; 2022].

Extensions: general equations of the type

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot \Big(\sigma(\rho) \circ \xi^{\delta_1} + \nu(\rho)\Big) + \lambda(\rho) + \phi(\rho)\xi^{\delta_2}.$$

— The generalized Dean-Kawasaki equation with correlated noise

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot \left(\Phi(\rho) + \Phi^{\frac{1}{2}}(\rho) \circ \xi^{\delta}\right).$$

— Nonlinear Dawson-Watanabe equation

$$\partial_t \rho = \Delta\Phi(\rho) + \sqrt{\rho}\xi^{\delta}.$$

VI. The large deviations principle

The large deviations rate function: A function $I\colon L^1_{t,x} \to \mathbb{R}$ is a (good) rate function if for each $M \in (0,\infty)$ the level set $\{\rho \in L^1_{t,x} \colon I(\rho) < M\}$ is a compact subset of $L^1_{t,x}$. We have that

$$I(\rho) = \frac{1}{2}\inf\{\|g\|^2_{L^2_{t,x}} : \partial_t \rho = \Delta\rho^\alpha - \nabla\cdot(\rho^{\frac{\alpha}{2}} g)\}.$$

The large deviations principle: the family of random variables ρ_ε satisfy a large deviations principle on $L^1_{t,x}$ with rate function I if, for every closed subset $A \subseteq L^1_{t,x}$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log\left(\mathbb{P}[\rho_\varepsilon \in A]\right) \leq -\inf_{\rho\in A} I(\rho),$$

and if, for every open subset $U \subseteq L^1_{t,x}$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log\left(\mathbb{P}[\rho_\varepsilon \in A]\right) \geq -\inf_{\rho\in U} I(\rho).$$

The Laplace principle: the family of random variables ρ_ε satisfy a Laplace principle on $L^1_{t,x}$ with rate function I if for all bounded continuous functions $h\colon L^1_{t,x} \to \mathbb{R}$,

$$\lim_{\varepsilon\to 0}\left(\varepsilon \log\left(\mathbb{E}\Big[\exp\Big(-\frac{1}{\varepsilon}h(\rho_\varepsilon)\Big)\Big]\right)\right) = -\inf_{\rho\in L^1_{t,x}}\left(f(\rho) + I(\rho)\right).$$

— these are equivalent, for example, [Budhiraja, DuPuis, Maroulas; 2008]

The large deviations rate function: the rate function

$$I(\rho) = \frac{1}{2} \inf\{\|g\|^2_{L^2_{t,x}} : \partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g)\}.$$

The space of bounded entropy: let $\mathrm{Ent}(\mathbb{T}^d)$ denote the space

$$\mathrm{Ent}(\mathbb{T}^d) = \left\{\rho \in L^1(\mathbb{T}^d) \colon \rho \geq 0 \;\text{ with }\; \int_{\mathbb{T}^d} \rho \log(\rho) < \infty\right\}.$$

The Uniform Laplace principle: the family of random variables ρ_ε satisfy a uniform Laplace principle on $L^1_{t,x}$ with respect to bounded subsets $K \subseteq \mathrm{Ent}(\mathbb{T}^d)$ if for all bounded continuous functions $h \colon L^1_{t,x} \to \mathbb{R}$,

$$\lim_{\varepsilon \to 0} \sup_{\rho_0 \in K} \left| \varepsilon \Big(\log \Big(\mathbb{E}_{\rho_0} \Big[\exp \Big(-\frac{1}{\varepsilon} h(\rho_\varepsilon) \Big) \Big] \Big) \Big) + \inf_{\rho \in L^1_{t,x}} \big(h(\rho) + I_{\rho_0}(\rho) \big) \right| = 0.$$

The large deviations principle [F., Gess; 2023]

The scaling limit: let $\delta(\varepsilon)$ be any sequence satisfying, as $\varepsilon \to 0$,

$$\varepsilon\delta(\varepsilon)^{-(d+2)} \to 0 \text{ and } \delta(\varepsilon) \to 0,$$

and for every $\varepsilon \in (0,1)$ let ρ_ε be the solution

$$\partial_t \rho_\varepsilon = \Delta\Phi(\rho_\varepsilon) - \sqrt{\varepsilon}\nabla \cdot (\Phi^{\frac{1}{2}}(\rho_\varepsilon) \circ \xi^{\delta(\varepsilon)}).$$

The large deviations principle: the solutions ρ_ε satisfy a uniform Laplace principle with rate function

$$I(\rho) = \frac{1}{2} \inf \left\{ \|g\|_{L^2}^2 : \partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \right\}.$$

The linear fluctuating hydrodynamics: the linear fluctuating hydrodynamics

$$\partial_t \tilde{\rho}^\varepsilon = \Delta\Phi(\tilde{\rho}^\varepsilon) - \sqrt{\varepsilon}\nabla \cdot (\Phi^{\frac{1}{2}}(\overline{\rho})\xi^{\delta(\varepsilon)}),$$

for the hydrodynamics limit $\partial_t \overline{\rho} = \Delta\Phi(\overline{\rho})$ satisfy an LDP with rate function

$$\tilde{I}(\rho) = \frac{1}{2} \inf \left\{ \|g\|_{L^2}^2 : \partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\overline{\rho}) g) \right\}.$$

VI. The large deviations principle

The scaling regime: the equation

$$\partial_t \rho_\varepsilon = \Delta \rho_{\varepsilon,\delta} - \sqrt{\varepsilon}\nabla \cdot (\rho^{\frac{1}{2}}_{\varepsilon,\delta(\varepsilon)} \circ \mathrm{d}\xi^\delta) - \nabla \cdot (\rho^{\frac{1}{2}}_\varepsilon g^{\delta(\varepsilon)}_\varepsilon).$$

for $\varepsilon\langle \nabla \cdot \xi^{\delta(\varepsilon)}\rangle \simeq \varepsilon\delta(\varepsilon)^{-(d+2)} \to 0.$

Convergence of the controls and initial data: suppose that there exists $M \in (0,\infty)$ such that

$$\sup_{\varepsilon\in(0,1)} \int_{\mathbb{T}^d} \rho_{0,\varepsilon} \log(\rho_{0,\varepsilon}) < M \;\text{ and }\; \sup_{\varepsilon\in(0,1)} \|g^\varepsilon\|_{L^\infty(\Omega;L^2_{t,x})} \leq M.$$

and $\rho_{0,\varepsilon} \rightharpoonup \rho_0$ weakly in L^1_x and $g_\varepsilon \to g$ in weakly in law on $L^2_{t,x}$.

The controlled SPDE: for every $\psi \in \mathrm{C}^\infty_c(\mathbb{T}^d \times (0,\infty))$ and $t \in [0,T]$,

$$\int_{\mathbb{R}}\int_{\mathbb{T}^d} \psi\chi_\varepsilon|^t_{s=0} = -\int_0^t\int_{\mathbb{T}^d} \nabla\rho_\varepsilon \cdot (\nabla\psi)(x,\rho_\varepsilon) - \frac{\varepsilon\langle\xi^\delta\rangle}{8}\int_0^t\int_{\mathbb{T}^d} \frac{1}{\rho_\varepsilon}\nabla\rho_\varepsilon\cdot(\nabla\psi)(x,\rho)$$
$$-\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}} \partial_\xi\phi \,\mathrm{d}q_\varepsilon - \int_0^t\int_{\mathbb{T}^d} \psi(x,\rho)\nabla\cdot(\sigma(\rho_\varepsilon)\,\mathrm{d}\xi^\delta) + \frac{\varepsilon\langle\nabla\cdot\xi^\delta\rangle}{2}\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\rho_\varepsilon$$
$$+\int_0^t\int_{\mathbb{T}^d} \rho^{\frac{1}{2}}_\varepsilon g^{\delta(\varepsilon)}_\varepsilon \cdot (\nabla\psi)(x,\rho_\varepsilon) - \int_0^t\int_{\mathbb{T}^d} (\partial_\xi\psi)(x,\rho_\varepsilon)\rho^{\frac{1}{2}}_\varepsilon g^{\delta(\varepsilon)}_\varepsilon \cdot \nabla\rho_\varepsilon.$$

Tightness: the condition that

$$\sup_{\varepsilon\in(0,1)}\int_{\mathbb{T}^d}\rho_{0,\varepsilon}\log(\rho_{0,\varepsilon})<M \;\text{ and }\; \sup_{\varepsilon\in(0,1)}\|g^{\varepsilon}\|_{L^\infty(\Omega;L^2_{t,x})}\leq M.$$

and $\rho_{0,\varepsilon}\rightharpoonup\rho_0$ weakly in L^1_x and $g_\varepsilon\to g$ in weakly in law on $L^2_{t,x}$ implies that the

$$\rho^{\varepsilon} \text{ are tight on } L^1_{t,x}.$$

The collapse of the controlled SPDE: for every $\varepsilon\in(0,1)$,

$$\int_{\mathbb{R}}\int_{\mathbb{T}^d}\psi\chi_\varepsilon|_{s=0}^{t}=-\int_0^t\int_{\mathbb{T}^d}\nabla\rho_\varepsilon\cdot(\nabla\psi)(x,\rho_\varepsilon)-\frac{\varepsilon\langle\xi^\delta\rangle}{8}\int_0^t\int_{\mathbb{T}^d}\frac{1}{\rho_\varepsilon}\nabla\rho_\varepsilon\cdot(\nabla\psi)(x,\rho)$$
$$-\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\xi\phi\,\mathrm{d}q_\varepsilon-\int_0^t\int_{\mathbb{T}^d}\psi(x,\rho)\nabla\cdot(\sigma(\rho_\varepsilon)\,\mathrm{d}\xi^\delta)+\frac{\varepsilon\langle\nabla\cdot\xi^\delta\rangle}{2}\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\phi)(x,\rho)\rho_\varepsilon$$
$$+\int_0^t\int_{\mathbb{T}^d}\rho_\varepsilon^{\frac12}g_\varepsilon^{\delta(\varepsilon)}\cdot(\nabla\psi)(x,\rho_\varepsilon)-\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\psi)(x,\rho_\varepsilon)\rho_\varepsilon^{\frac12}g_\varepsilon^{\delta(\varepsilon)}\cdot\nabla\rho_\varepsilon,$$

and, as $\varepsilon\to0$, we have that $\rho^\varepsilon(g)\to\rho(g)$ in law on $L^1_{t,x}$ for $\rho(g)$ solving

$$\partial_t\rho=\Delta\rho-\nabla\cdot(\rho^{\frac12}g).$$

The Laplace principle: the family of random variables ρ_ε satisfy a Laplace principle on $L^1_{t,x}$ with rate function I if for all bounded continuous functions $h\colon L^1_{t,x} \to \mathbb{R}$,

$$\lim_{\varepsilon\to 0}\left(\varepsilon \log\left(\mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}h(\rho_\varepsilon)\right)\right]\right)\right) = -\inf_{\rho\in L^1_{t,x}}\left(f(\rho)+I(\rho)\right).$$

The controlled SPDE: for $g \in L^\infty(\Omega; L^2_{t,x})$ the function $\rho_{\varepsilon,\delta}(g)$ solves

$$\partial_t \rho_{\varepsilon,\delta} = \Delta\rho_{\varepsilon,\delta} - \sqrt{\varepsilon}\nabla\cdot(\rho^{\frac{1}{2}}_{\varepsilon,\delta}\circ \mathrm{d}\xi^\delta) - \nabla\cdot(\rho^{\frac{1}{2}}_{\varepsilon,\delta} g^\delta).$$

The variational formulation: we have that, for $\varepsilon \in (0,1)$,

$$-\varepsilon\log\left(\mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}h(\rho_{\varepsilon,\delta})\right)\right]\right) = \inf_{g\in L^\infty(\Omega;L^2_{t,x})}\mathbb{E}\left[\frac{1}{2}\|g\|_2^2 + h(\rho_{\varepsilon,\delta}(g))\right].$$

— think of the control $g = \varepsilon^{\frac{1}{2}}(\varepsilon^{-\frac{1}{2}}g)$

— for example, [Budhiraja, Dupuis; 2000]

VI. The large deviations principle

The controlled SPDE: for $g \in L^\infty(\Omega; L^2_{t,x})$ the function $\rho_{\varepsilon,\delta}(g)$ solves

$$\partial_t \rho_\varepsilon = \Delta \rho_\varepsilon - \sqrt{\varepsilon}\nabla \cdot (\rho_\varepsilon^{\frac{1}{2}} \circ \mathrm{d}\xi^{\delta(\varepsilon)}) - \nabla \cdot (\rho_\varepsilon^{\frac{1}{2}} g^\delta).$$

The Laplace principle upper bound: for a bounded continuous h,

$$\begin{aligned} -\varepsilon \log\Big(\mathbb{E}\Big[\exp\Big(-\frac{1}{\varepsilon}h(\rho_\varepsilon)\Big)\Big]\Big) &= \inf_{g \in L^\infty(\Omega; L^2_{t,x})} \mathbb{E}\Big[\frac{1}{2}\|g\|_2^2 + h(\rho_\varepsilon(g))\Big] \\ &\geq \mathbb{E}\Big[\frac{1}{2}\|g_\varepsilon\|_2^2 + h(\rho_\varepsilon(g_\varepsilon))\Big] - \varepsilon. \end{aligned}$$

Since3 $\sup_{\varepsilon \in (0,1)} \|g_\varepsilon\|_{L^\infty(\Omega; L^2_{t,x})} \leq M$ we have that

$$g_\varepsilon \rightharpoonup g \ \text{ weakly in law on } (L^2_{t,x})^d \ \text{ and } \ \rho^\varepsilon(g) \to \rho \ \text{ strongly in law on } \ L^1_{t,x}.$$

Therefore,

$$\begin{aligned} \limsup_{\varepsilon \to 0}\Big[\varepsilon \log\Big(\mathbb{E}\Big[\exp\Big(-\frac{1}{\varepsilon}h(\rho_\varepsilon)\Big)\Big]\Big)\Big] &\leq -\Big(\frac{1}{2}\|g\|_2^2 + h(\rho(g))\Big) \\ &\leq -\inf_{\rho \in L^1_{t,x}} \big(I(\rho) + h(\rho)\big). \end{aligned}$$

— see, for example, [Budhiraja, DuPuis, Maroulas; 2008]

The controlled SPDE: for $g \in L^\infty(\Omega; L^2_{t,x})$ the function $\rho_{\varepsilon,\delta}(g)$ solves

$$\partial_t \rho_\varepsilon = \Delta \rho_\varepsilon - \sqrt{\varepsilon}\nabla \cdot (\rho_\varepsilon^{\frac{1}{2}} \circ \mathrm{d}\xi^{\delta(\varepsilon)}) - \nabla \cdot (\rho_\varepsilon^{\frac{1}{2}} g^\delta).$$

The Laplace principle lower bound: choose an almost minimizer g_0 of

$$\mathbb{E}\Big[\frac{1}{2}\|g\|_2^2 + h(\rho(g))\Big].$$

We then have that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\Big(-\varepsilon\log\Big(\mathbb{E}\Big[\exp\Big(-\frac{1}{\varepsilon}h(\rho_\varepsilon)\Big)\Big]\Big)\Big) \\
&= \limsup_{\varepsilon\to 0}\Big(\inf_{g\in L^\infty(\Omega;L^2_{t,x})}\mathbb{E}\Big[\frac{1}{2}\|g\|_2^2 + h(\rho_\varepsilon(g))\Big]\Big) \\
&\leq \limsup_{\varepsilon\to 0}\mathbb{E}\Big[\frac{1}{2}\|g_0\|_2^2 + h(\rho_\varepsilon(g_0))\Big] \\
&= \mathbb{E}\Big[\frac{1}{2}\|g_0\|_2^2 + h(\rho(g_0))\Big] = \inf_{\rho\in L^1_{t,x}}\big(I(\rho)+h(\rho)\big).
\end{aligned}$$

Therefore,

$$\liminf_{\varepsilon\to 0}\Big(\varepsilon\log\Big(\mathbb{E}\Big[\exp\Big(-\frac{1}{\varepsilon}h(\rho_\varepsilon)\Big)\Big]\Big)\Big) \geq -\inf_{\rho\in L^1_{t,x}}\big(I(\rho)+h(\rho)\big).$$

The large deviations principle [F., Gess; 2023]

The scaling limit: let $\delta(\varepsilon)$ be any sequence satisfying, as $\varepsilon \to 0$,

$$\varepsilon\delta(\varepsilon)^{-(d+2)} \to 0 \text{ and } \delta(\varepsilon) \to 0,$$

and for every $\varepsilon \in (0,1)$ let ρ_ε be the solution

$$\partial_t \rho_\varepsilon = \Delta\Phi(\rho_\varepsilon) - \sqrt{\varepsilon}\nabla \cdot (\Phi^{\frac{1}{2}}(\rho_\varepsilon) \circ \xi^{\delta(\varepsilon)}).$$

The large deviations principle: the solutions ρ_ε satisfy a uniform Laplace principle with rate function

$$I(\rho) = \frac{1}{2} \inf \left\{ \|g\|^2_{L^2} : \partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \right\}.$$

— also in $\mathrm{C}([0,T];\mathcal{M}_+(\mathbb{T}^d))$

— applies to $\Phi(\xi) = \xi^\alpha$ for every $\alpha \in [1,\infty)$

— uniformity with respect to sets of initial data with bounded entropy

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