The weak formulation of the skeleton equation

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II. The kinetic formulation of the skeleton equation

The skeleton equation: in the case of the zero range process,

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \ \text{ in } \ \mathbb{T}^d \times (0,T),$$

for an L^2-control $g \in (L^2_{t,x})^d$. We specialize to the case, for some $\alpha \in (0,\infty)$,

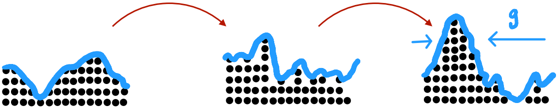
$$\partial_t \rho = \Delta\rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

The kinetic formulation: for $\chi = \mathbf{1}_{\{0<\xi<\rho\}} - \mathbf{1}_{\{\rho<\xi<0\}}$,

$$\partial_t \chi = \alpha\xi^{\alpha-1}\Delta\chi + \partial_\xi q - \partial_\xi(\xi^{\frac{\alpha}{2}} g \cdot \nabla\chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}}(\partial_\xi \chi) g),$$

for a locally finite, nonnegative measure q on $\mathbb{T}^d \times \mathbb{R} \times [0,T]$ with

$$q \geq \delta_\rho(\alpha\xi^{\alpha-1}|\nabla\rho|^2).$$



A renormalized kinetic solution of the skeleton equation

A nonnegative function $\rho \in \mathrm{C}([0,T];L^1(\mathbb{T}^d))$ is a renormalized kinetic solution of the skeleton equation if there exists a nonnegative, locally finite measure q on $\mathbb{T}^d \times \mathbb{R} \times [0,T]$ such that ρ and q satisfy the following four properties.

— *Preservation of mass*: $\|\rho(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_0\|_{L^1(\mathbb{T}^d)}$ for every $t \in [0,T]$.

— *Local H^1-regularity*: $\left((\rho \wedge K) \vee \frac{1}{K}\right) \in L^2([0,T];H^1(\mathbb{T}^d))$ for every $K \in \mathbb{N}$.

— *Regularity and vanishing of the measure at infinity*: we have that

$$\delta_\rho\left(\alpha \xi^{\alpha-1}|\nabla\rho|^2\right) \leq q \;\text{ and }\; \liminf_{M\to\infty} q(\mathbb{T}^d \times [M,M+1] \times [0,T]) = 0.$$

— *The equation*: for every $\psi \in \mathrm{C}_c^\infty(\mathbb{T}^d \times (0,\infty))$ and $t \in [0,T]$,

$$\int_{\mathbb{T}^d} \chi\psi\Big|_{s=0}^{s=t} = -\int_0^t\int_{\mathbb{T}^d} \alpha\rho^{\alpha-1}\nabla\rho\cdot(\nabla\psi)(x,\rho) - \int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\partial_\xi\psi)(x,\xi)q$$
$$+\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\psi)(x,\rho)\rho^{\frac{\alpha}{2}}g\cdot\nabla\rho + \int_0^T\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}}g\cdot(\nabla\psi)(x,\rho).$$

— we have that $\lim_{\beta\to 0}\left(\beta^{-1}q\left(\mathbb{T}^d \times (\frac{\beta}{2},\beta) \times [0,T]\right)\right) = 0$.

II. The kinetic formulation of the skeleton equation

Well-posedness of renormalized kinetic solutions [F., Gess; 2023]

Let $T \in (0,\infty)$, $d \in \mathbb{N}$, and let $\Phi \in \mathrm{C}^1_{\mathrm{loc}}((0,\infty)) \cap \mathrm{C}([0,\infty))$ satisfy that

— $\Phi(0) = 0$ with $\Phi' > 0$ on $(0,\infty)$,

— Φ' is locally $1/2$-Hölder continuous on $(0,\infty)$,

— and $\max_{\{0<\xi\leq M\}} \frac{\Phi(\xi)}{\Phi'(\xi)} \leq cM$.

Then for every nonnegative $\rho_0 \in L^1(\mathbb{T}^d)$ there exists a unique renormalized kinetic solution of the equation

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{\frac{1}{2}}(\rho)g) \ \text{ in } \ \mathbb{T}^d \times (0,T) \ \text{ with } \ \rho(\cdot,0) = \rho_0.$$

Furthermore, if ρ_1 and ρ_2 are two solutions with initial data $\rho_{1,0}$ and $\rho_{2,0}$, then

$$\max_{t\in[0,T]} \|\rho_1(x,t) - \rho_2(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_{1,0} - \rho_{2,0}\|_{L^1(\mathbb{T}^d)}.$$

— including $\Phi(\xi) = \xi^\alpha$ for every $\alpha \in (0,\infty)$, for which

$$\partial_t \rho = \Delta\rho^\alpha - \nabla\cdot(\rho^{\frac{\alpha}{2}} g).$$

III. Weak solutions of the skeleton equation

The skeleton equation: $\partial_t\rho = \Delta\rho^\alpha - \nabla\cdot(\rho^{\frac{\alpha}{2}}g) = 2\nabla\cdot(\rho^{\frac{\alpha}{2}}\nabla\rho^{\frac{\alpha}{2}}) - \nabla\cdot(\rho^{\frac{\alpha}{2}}g)$.

Weak solutions of the skeleton equation

A weak solution of the skeleton equation is a nonnegative $\rho \in \mathrm{C}([0,T];L^1(\mathbb{T}^d))$ that satisfies the following two properties.

— *The entropy estimate*: we have that

$$\rho^{\frac{\alpha}{2}} \in L^2([0,T];H^1(\mathbb{T}^d)).$$

— for every $\psi \in \mathrm{C}^\infty(\mathbb{T}^d)$ and $t \in [0,T]$,

$$\int_{\mathbb{T}^d}\rho(x,s)\psi(x)\Big|_{s=0}^{s=t} = -2\int_0^t\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}}\nabla\rho^{\frac{\alpha}{2}}\cdot\nabla\psi + \int_0^t\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}}g\cdot\nabla\psi.$$

The entropy estimate: we have that

$$\max_{t\in[0,T]}\int_{\mathbb{T}^d}\rho\log(\rho) + \int_0^T\int_{\mathbb{T}^d}|\nabla\rho^{\frac{\alpha}{2}}|^2 \lesssim \int_{\mathbb{T}^d}\rho_0\log(\rho_0) + \int_0^T\int_{\mathbb{T}^d}|g|^2.$$

An interpolation inequality: we have that

$$\left\|\rho^{\frac{\alpha}{2}}\right\|_{L_t^2L_x^2} \lesssim \|\rho_0\|_{L_x^1}^{\alpha} + \left\|\nabla\rho^{\frac{\alpha}{2}}\right\|_{L_t^2L_x^2}.$$

III. Weak solutions of the skeleton equation

The kinetic formulation: for every $\psi \in \mathrm{C}^\infty(\mathbb{T}^d)$ and $t \in [0,T]$,

$$\int_{\mathbb{T}^d} \chi\psi\Big|_{s=0}^{s=t} = -\int_0^t \int_{\mathbb{T}^d} \alpha\rho^{\alpha-1}\nabla\rho \cdot (\nabla\psi)(x,\rho) - \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{R}} (\partial_\xi\psi)(x,\xi)q$$
$$+ \int_0^t \int_{\mathbb{T}^d} (\partial_\xi\psi)(x,\rho)\rho^{\frac{\alpha}{2}} g \cdot \nabla\rho + \int_0^T \int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} g \cdot (\nabla\psi)(x,\rho).$$

The weak formulation: for every $\psi \in \mathrm{C}^\infty(\mathbb{T}^d)$ and $t \in [0,T]$,

$$\int_{\mathbb{T}^d} \rho(x,s)\psi(x)\Big|_{s=0}^{s=t} = -2\int_0^t \int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} \nabla\rho^{\frac{\alpha}{2}} \cdot \nabla\psi + \int_0^t \int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} g \cdot \nabla\psi.$$

Weak-strong continuity: does a weakly convergent sequence

$$g_n \rightharpoonup g \in (L^2_{t,x})^d,$$

induce a strongly convergent sequence of solutions

$$\rho_n \to \rho \in L^1_{t,x}?$$

— weak convergence implies that the g_n are uniformly bounded in $(L^2_{t,x})^d$

III. Weak solutions of the skeleton equation

The entropy estimate: we have that

$$\max_{t\in[0,T]}\int_{\mathbb{T}^d}\rho\log(\rho)+\int_0^T\int_{\mathbb{T}^d}|\nabla\rho^{\frac{\alpha}{2}}|^2\lesssim\int_{\mathbb{T}^d}\rho_0\log(\rho_0)+\int_0^T\int_{\mathbb{T}^d}|g|^2.$$

Spatial regularity: if $\alpha\in[1,2]$ then

$$\nabla\rho=\frac{2}{\alpha}(\rho^{1-\frac{\alpha}{2}})\nabla\rho^{\frac{\alpha}{2}}\in(L^{\frac{2}{3-\alpha}}_{t,x})^d.$$

If $\alpha\in[2,\infty)$ then

$$|\rho(x)-\rho(y)|^\alpha\leq|\rho(x)^{\frac{\alpha}{2}}-\rho(y)^{\frac{\alpha}{2}}|^2\ \text{ so that }\ \rho\in W^{\frac{2}{\alpha},1}(\mathbb{T}^d).$$

L^1-integrability of the products: we have that

$$\rho^{\frac{\alpha}{2}}\nabla\rho^{\frac{\alpha}{2}}\in(L^1_{t,x})^d\ \text{ and }\ \rho^{\frac{\alpha}{2}}g\in(L^1_{t,x})^d.$$

Regularity in time: for $s>{}^d/_2$ we have that

$$\partial_t\rho=2\nabla\cdot(\rho^{\frac{\alpha}{2}}\nabla\rho^{\frac{\alpha}{2}})-\nabla\cdot(\rho^{\frac{\alpha}{2}}g)\in H^{-s}(\mathbb{T}^d).$$

— Aubin–Lions–Simon Lemma for strong $(L^1_{t,x})^d$-compactness:

$$W^{\frac{2}{\alpha},1}(\mathbb{T}^d),W^{1,p}(\mathbb{T}^d)\subset\subset L^1(\mathbb{T}^d)\subseteq H^{-s}(\mathbb{T}^d).$$

III. Weak solutions of the skeleton equation

Equivalence of weak and kinetic solutions: for initial data with finite entropy, when is a weak solution

$$\int_{\mathbb{T}^d} \rho(x,s)\psi(x)\Big|_{s=0}^{s=t} = -2\int_0^t \int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} \nabla \rho^{\frac{\alpha}{2}} \cdot \nabla \psi + \int_0^t \int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} g \cdot \nabla \psi,$$

a kinetic solution

$$\int_{\mathbb{T}^d} \chi\psi\Big|_{s=0}^{s=t} = -\int_0^t \int_{\mathbb{T}^d} \alpha\rho^{\alpha-1}\nabla\rho \cdot (\nabla\psi)(x,\rho) - \int_0^t \int_{\mathbb{T}^d}\int_{\mathbb{R}} (\partial_\xi\psi)(x,\xi)q$$
$$+ \int_0^t \int_{\mathbb{T}^d} (\partial_\xi\psi)(x,\rho)\rho^{\frac{\alpha}{2}} g\cdot\nabla\rho + \int_0^T \int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} g \cdot (\nabla\psi)(x,\rho)?$$

Deriving the kinetic form: for $\partial_\xi \Psi(x,\xi) = \psi(x,\xi)$, for $\rho_\varepsilon = (\rho * \kappa^\varepsilon)$,

$$\partial_t \int \Psi(x,\rho_\varepsilon) = \int \psi(x,\rho_\varepsilon)\partial_t\rho_\varepsilon$$
$$= -2\int (\nabla\psi)(x,\rho_\varepsilon)\cdot(\rho^{\frac{\alpha}{2}}\nabla\rho^{\frac{\alpha}{2}})^\varepsilon - \int (\nabla\psi)(x,\rho_\varepsilon)\cdot(\rho^{\frac{\alpha}{2}} g)^\varepsilon$$
$$-2\int_{\mathbb{T}^d} (\partial_\xi\psi)(x,\rho_\varepsilon)\nabla\rho_\varepsilon\cdot(\rho^{\frac{\alpha}{2}}\nabla\rho^{\frac{\alpha}{2}})^\varepsilon - \int (\partial_\xi\psi)(x,\rho_\varepsilon)\nabla\rho_\varepsilon\cdot(\rho^{\frac{\alpha}{2}} g)^\varepsilon.$$

III. Weak solutions of the skeleton equation

DiPerna–Lions theory [DiPerna, Lions; 1989], [Ambrosio; 2004]

Let $b\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$ satisfy

$$\int_0^T \|(\nabla \cdot b)\|_{L^\infty_x} + \|b\|_{BV_x} < \infty.$$

Then, for every $\rho_0 \in L^\infty(\mathbb{T}^d)$ the continuity equation

$$\partial_t \rho = \nabla \cdot (\rho b),$$

has a unique solution in $(L^1 \cap L^\infty)(\mathbb{T}^d \times [0,T])$.

— a lower bound on $\nabla \cdot b$ is sufficient [Ambrosio; 2004]

— almost optimal conditions [Depauw; 2003]

— the skeleton equation

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

III. Weak solutions of the skeleton equation

The equation satisfied by the convolution: for $\rho = (\rho * \kappa^\varepsilon)$,

$$\partial_t \rho_\varepsilon = -(2\rho^{\frac{\alpha}{2}} \nabla \rho^{\frac{\alpha}{2}} * \nabla \kappa^\varepsilon) + (\rho^{\frac{\alpha}{2}} g * \nabla \kappa^\varepsilon).$$

Deriving the kinetic form: if ρ is a weak solution, for $\partial_\xi \Psi(x,\xi) = \psi(x,\xi)$,

$$\begin{aligned}\partial_t \int_{\mathbb{T}^d} \Psi(x,\rho_\varepsilon) &= \int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)\partial_t \rho_\varepsilon \\ &= -2\int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\rho^{\frac{\alpha}{2}} \nabla \rho^{\frac{\alpha}{2}} * \nabla \kappa^\varepsilon) + \int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\rho^{\frac{\alpha}{2}} g * \nabla \kappa^\varepsilon).\end{aligned}$$

A useful decomposition: let $\mathrm{Supp}(\psi) \subseteq \mathbb{T}^d \times [0,M]$ and let

$$A_1 = \{(x,t) \colon \rho^{\frac{\alpha}{2}}(x,t) \geq M^{\frac{\alpha}{2}} + 1\} \text{ and let } A_0 = (\mathbb{T}^d \times [0,T]) \setminus A_1.$$

We then write, for both terms,

$$\begin{aligned}\int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\rho^{\frac{\alpha}{2}} g * \nabla \kappa^\varepsilon) &= \int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_0} \rho^{\frac{\alpha}{2}} g * \nabla \kappa^\varepsilon) \\ &\quad + \int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_1} \rho^{\frac{\alpha}{2}} g * \nabla \kappa^\varepsilon).\end{aligned}$$

A useful decomposition: let $\mathrm{Supp}(\psi) \subseteq \mathbb{T}^d \times [0, M]$ and let

$$A_1 = \{(x,t)\colon \rho^{\frac{\alpha}{2}}(x,t) \geq M^{\frac{\alpha}{2}} + 1\} \text{ and let } A_0 = (\mathbb{T}^d \times [0,T]) \setminus A_1.$$

We then write

$$\int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon) = \int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_0}\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon) + \int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_1}\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon).$$

The "good" term defined by A_0: after integrating by parts,

$$\begin{aligned}\int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_0}\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon) &= \int_{\mathbb{T}^d} (\partial_\xi\psi)(x,\rho_\varepsilon)(\mathbf{1}_{A_0}\rho^{\frac{\alpha}{2}} g * \kappa^\varepsilon)\cdot\nabla\rho_\varepsilon \\ &\quad + \int_{\mathbb{T}^d} (\nabla\psi)(x,\rho_\varepsilon)\cdot(\mathbf{1}_{A_0}\rho^{\frac{\alpha}{2}} g * \kappa^\varepsilon).\end{aligned}$$

After passing $\varepsilon \to 0$, the strong $L^2_{t,x}$-convergence proves that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_0}\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon) \\ &= \int_{\mathbb{T}^d} (\partial_\xi\psi)(x,\rho)(\mathbf{1}_{A_0}\rho^{\frac{\alpha}{2}} g)\cdot\nabla\rho + \int_{\mathbb{T}^d} \mathbf{1}_{A_0}\rho^{\frac{\alpha}{2}} g\cdot(\nabla\psi)(x,\rho) \\ &= \int_{\mathbb{T}^d} (\partial_\xi\psi)(x,\rho)\rho^{\frac{\alpha}{2}} g\cdot\nabla\rho + \int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} g\cdot(\nabla\psi)(x,\rho).\end{aligned}$$

A useful decomposition: let $\mathrm{Supp}(\psi) \subseteq \mathbb{T}^d \times [0, M]$ and let

$$A_1 = \{(x,t)\colon \rho^{\frac{\alpha}{2}}(x,t) \geq M^{\frac{\alpha}{2}} + 1\} \text{ and let } A_0 = (\mathbb{T}^d \times [0,T]) \setminus A_1.$$

We then write

$$\int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon) = \int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_0}\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon) + \int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_1}\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon).$$

The "bad" term defined by A_1: in this case, using Hölder's inequality,

$$\begin{aligned}
&|\int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_1}\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon)| \\
&\leq \varepsilon^{-1}\int_{\mathbb{T}^d} |\psi(x,\rho_\varepsilon)|(\mathbf{1}_{A_1}|g|^2 * |\varepsilon\nabla\kappa^\varepsilon|)^{\frac{1}{2}}(\mathbf{1}_{A_1}\rho^\alpha * |\varepsilon\nabla\kappa^\varepsilon|)^{\frac{1}{2}} \\
&\lesssim \left(\int_{\mathbb{T}^d} |\psi(x,\rho_\varepsilon)|(\mathbf{1}_{A_1}|g|^2 * |\varepsilon\nabla\kappa^\varepsilon|)\right)^{\frac{1}{2}} \cdot \varepsilon^{-1}\left(\int_{\mathbb{T}^d} |\psi(x,\rho_\varepsilon)|(\mathbf{1}_{A_1}\rho^\alpha * |\varepsilon\nabla\kappa^\varepsilon|)\right)^{\frac{1}{2}}.
\end{aligned}$$

We first observe that

$$\lim_{\varepsilon\to 0}\left(\int_{\mathbb{T}^d} |\psi(x,\rho_\varepsilon)|(\mathbf{1}_{A_1}|g| * |\varepsilon\nabla\kappa^\varepsilon|)^2\right)^{\frac{1}{2}} \simeq \left(\int_{\mathbb{T}^d} \psi(x,\rho)\mathbf{1}_{A_1}|g|^2\right)^{\frac{1}{2}} = 0.$$

A still useful decomposition: for $A_1 = \{(x,t)\colon \rho^{\frac{\alpha}{2}}(x,t) \geq M^{\frac{\alpha}{2}} + 1\}$ we need that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1}\Big(\int_{\mathbb{T}^d} |\psi(x,\rho_\varepsilon)|(\mathbf{1}_{A_1}\rho^{\alpha} * |\varepsilon\nabla\kappa^{\varepsilon}|)\Big)^{\frac{1}{2}} < \infty.$$

We first write

$$\int_{\mathbb{T}^d} |\psi(y,\rho_\varepsilon)|(\mathbf{1}_{A_1}\rho^{\alpha} * |\varepsilon\nabla\kappa^{\varepsilon}|) = \int_{(\mathbb{T}^d)^2} |\psi(y+x,\rho_\varepsilon(y+x))|\mathbf{1}_{A_1}(x)\rho^{\alpha}(x)|\varepsilon\nabla\kappa^{\varepsilon}(y)|\,\mathrm{d}x\,\mathrm{d}y.$$

Then, for $A_k = \{(x,t)\colon \rho^{\frac{\alpha}{2}}(x,t) \geq M^{\frac{\alpha}{2}} + k\}$ and $\mathbf{1}_k = \mathbf{1}_{A_k\setminus A_{k+1}}$,

$$\begin{aligned}
&\int_{\mathbb{T}^d} |\psi(y,\rho_\varepsilon)|(\mathbf{1}_{A_1}\rho^{\alpha} * |\varepsilon\nabla\kappa^{\varepsilon}|)\,\mathrm{d}y \\
&= \sum_{k=1}^{\infty}\int_{(\mathbb{T}^d)^2} |\psi(y+x,\rho_\varepsilon(y+x))|\mathbf{1}_k(x)\rho^{\alpha}(x)|\varepsilon\nabla\kappa^{\varepsilon}(y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim \sum_{k=1}^{\infty}\int_{(\mathbb{T}^d)^2} |\psi(y+x,\rho_\varepsilon(y+x))|(M^{\frac{\alpha}{2}} + k + 1)^2\mathbf{1}_k(x)|\varepsilon\nabla\kappa^{\varepsilon}(y)|\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

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A decomposition: for $A_k = \{(x,t)\colon \rho^{\frac{\alpha}{2}}(x,t) \geq M^{\frac{\alpha}{2}} + k\}$ and $\mathbf{1}_k = \mathbf{1}_{A_k \setminus A_{k+1}}$,

$$\begin{aligned}
&\int_{\mathbb{T}^d} |\psi(y,\rho_\varepsilon)|(\mathbf{1}_{A_1}\rho^\alpha * |\varepsilon\nabla\kappa^\varepsilon|)\,\mathrm{d}y \\
&\leq \sum_{k=1}^\infty \int_{(\mathbb{T}^d)^2} |\psi(y+x,\rho_\varepsilon(y+x))|(M^{\frac{\alpha}{2}}+k+1)^2\mathbf{1}_k(x)|\varepsilon\nabla\kappa^\varepsilon(y)|\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

If $B^\varepsilon_{k,y} = \{(x,t)\colon |(\rho^{\frac{\alpha}{2}} * \kappa^\varepsilon)(x+y) - \rho^{\frac{\alpha}{2}}(x)|\mathbf{1}_k(x) \geq k\}$ then

$$\begin{aligned}
|B^\varepsilon_{k,x}| &\leq \frac{1}{k^2}\int_{\mathbb{T}^d} |(\rho^{\frac{\alpha}{2}} * \kappa^\varepsilon)(y+x) - \rho^{\frac{\alpha}{2}}(x)|^2\mathbf{1}_k(x)\,\mathrm{d}x \\
&= \frac{1}{k^2}\int_{\mathbb{T}^d} \Big|\int_{\mathbb{T}^d} \big(\rho^{\frac{\alpha}{2}}(y+x+z) - \rho^{\frac{\alpha}{2}}(x)\big)\kappa^\varepsilon(z)\,\mathrm{d}z\Big|^2\mathbf{1}_k(x)\,\mathrm{d}x \\
&\leq \frac{1}{k^2}\int_{\mathbb{T}^d} \Big(\int_{\mathbb{T}^d} |\rho^{\frac{\alpha}{2}}(y+x+z) - \rho^{\frac{\alpha}{2}}(x)|^2\kappa^\varepsilon(z)\,\mathrm{d}z\Big)\mathbf{1}_k(x)\,\mathrm{d}x \\
&= \frac{1}{k^2}\int_{\mathbb{T}^d} \Big(\int_{\mathbb{T}^d} \Big|\int_0^1 \nabla\rho^{\frac{\alpha}{2}}(x+s(y+z))\cdot(y+z)\,\mathrm{d}s\Big|^2\kappa^\varepsilon(z)\,\mathrm{d}z\Big)\mathbf{1}_k(x)\,\mathrm{d}x \\
&\lesssim \frac{|y|^2+\varepsilon^2}{k^2}\int_{(\mathbb{T}^d)^2}\int_0^1 |\nabla\rho^{\frac{\alpha}{2}}(x+s(y+z))|^2\kappa^\varepsilon(z)\mathbf{1}_k(x)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}x.
\end{aligned}$$

A decomposition: for $A_k = \{(x,t)\colon \rho^{\frac{\alpha}{2}}(x,t) \geq M + k\}$ and $\mathbf{1}_k = \mathbf{1}_{A_k \setminus A_{k+1}}$,

$$\int_{\mathbb{T}^d} |\psi(y,\rho_\varepsilon)|(\mathbf{1}_{A_1}\rho^\alpha * |\varepsilon\nabla\kappa^\varepsilon|)\,\mathrm{d}y$$
$$\leq \sum_{k=1}^{\infty} \int_{(\mathbb{T}^d)^2} |\psi(y+x,\rho_\varepsilon(y+x))|(M^{\frac{\alpha}{2}} + k + 1)^2 \mathbf{1}_k(x)|\varepsilon\nabla\kappa^\varepsilon(y)|\,\mathrm{d}x\,\mathrm{d}y.$$

If $B^\varepsilon_{k,y} = \{(x,t)\colon |(\rho^{\frac{\alpha}{2}} * \kappa^\varepsilon)(x+y) - \rho^{\frac{\alpha}{2}}(x)|\mathbf{1}_k(x) \geq k\}$ then

$$|B^\varepsilon_{k,x}| \lesssim \frac{|y|^2+\varepsilon^2}{k^2} \int_{(\mathbb{T}^d)^2} \int_0^1 |\nabla\rho^{\frac{\alpha}{2}}(x+s(y+z))|^2 \kappa^\varepsilon(z)\mathbf{1}_k(x)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}x.$$

Since, if $\alpha \in [1,2]$,

$$\rho^{\frac{\alpha}{2}}(x) - (\rho^{\frac{\alpha}{2}} * \kappa^\varepsilon)(x+y) \geq \rho^{\frac{\alpha}{2}}(x) - (\rho * \kappa^\varepsilon)^{\frac{\alpha}{2}}(y+x) \geq \rho^{\frac{\alpha}{2}}(x) - M^{\frac{\alpha}{2}},$$

for every $(y,t) \in \mathbb{T}^d \times [0,T]$,

$$\mathrm{Supp}\left(\psi(y+\cdot,\rho_\varepsilon(y+\cdot))\mathbf{1}_k(\cdot)\right) \subseteq B^\varepsilon_{k,y},$$

and we having using the boundedness of ψ that

$$\int_{\mathbb{T}^d} |\psi(y,\rho_\varepsilon)|(\mathbf{1}_{A_1}\rho^\alpha * |\varepsilon\nabla\kappa^\varepsilon|) \lesssim \sum_{k=1}^{\infty} \int_{\mathbb{T}^d} |B^\varepsilon_{k,y}|(M^{\frac{\alpha}{2}} + k + 1)^2 |\varepsilon\nabla\kappa^\varepsilon(y)|\,\mathrm{d}y.$$

III. Weak solutions of the skeleton equation

A decomposition: for $B^{\varepsilon}_{k,y} = \{(x,t) \colon |(\rho^{\frac{\alpha}{2}} * \kappa^{\varepsilon})(x+y) - \rho^{\frac{\alpha}{2}}(x)|\mathbf{1}_k(x) \geq k\}$,

$$|B^{\varepsilon}_{k,x}| \lesssim \frac{|y|^2+\varepsilon^2}{k^2}\int_{(\mathbb{T}^d)^2}\int_0^1 |\nabla \rho^{\frac{\alpha}{2}}(x+s(y+z))|^2 \kappa^{\varepsilon}(z)\mathbf{1}_k(x)\,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}x,$$

we have that

$$\begin{aligned}
&\int_{\mathbb{T}^d} |\psi(y,\rho_\varepsilon)|(\mathbf{1}_{A_1}\rho^{\alpha} * |\varepsilon\nabla\kappa^{\varepsilon}|)\,\mathrm{d}y \\
&\lesssim \sum_{k=1}^{\infty}\int_{\mathbb{T}^d} |B^{\varepsilon}_{k,y}|(M^{\frac{\alpha}{2}}+k+1)^2|\varepsilon\nabla\kappa^{\varepsilon}(y)|\,\mathrm{d}y \\
&\lesssim \sum_{k=1}^{\infty}\int_{(\mathbb{T}^d)^3}\int_0^1 \left(\frac{|y|^2+\varepsilon^2}{k^2}\right)(M^{\frac{\alpha}{2}}+k+1)^2|\varepsilon\nabla\kappa^{\varepsilon}(y)||\nabla\rho^{\frac{\alpha}{2}}(x+s(y+z))|^2\kappa^{\varepsilon}(z)\mathbf{1}_k(x) \\
&\lesssim \int_{(\mathbb{T}^d)^3}\int_0^1 (|y|^2+\varepsilon^2)|\varepsilon\nabla\kappa^{\varepsilon}(y)||\nabla\rho^{\frac{\alpha}{2}}(x+s(y+z))|^2\kappa^{\varepsilon}(z)\mathbf{1}_{A_1}(x) \\
&\lesssim \varepsilon^2\int_{(\mathbb{T}^d)^3}\int_0^1 |\varepsilon\nabla\kappa^{\varepsilon}(y)||\nabla\rho^{\frac{\alpha}{2}}(x+s(y+z))|^2\kappa^{\varepsilon}(z)\mathbf{1}_{A_1}(x) \\
&\lesssim \varepsilon^2\int_{\mathbb{T}^d}|\nabla\rho^{\frac{\alpha}{2}}|^2.
\end{aligned}$$

The conclusion: we recall that for $A_1 = \{(x,t)\colon \rho(x,t) \geq M+1\}$,

$$|\int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_1}\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon)|$$
$$\lesssim \Big(\int_{\mathbb{T}^d} |\psi(x,\rho_\varepsilon)|(\mathbf{1}_{A_1}|g|^2 * |\varepsilon\nabla\kappa^\varepsilon|)\Big)^{\frac{1}{2}} \cdot \varepsilon^{-1}\Big(\int_{\mathbb{T}^d} |\psi(x,\rho_\varepsilon)|(\mathbf{1}_{A_1}\rho^\alpha * |\varepsilon\nabla\kappa^\varepsilon|)\Big)^{\frac{1}{2}}.$$

Since we have shown that $\int_{\mathbb{T}^d} |\psi(x,\rho_\varepsilon)|(\mathbf{1}_{A_1}\rho^\alpha * |\varepsilon\nabla\kappa^\varepsilon| \lesssim \varepsilon^2 \int_{\mathbb{T}^d} |\nabla\rho^{\frac{\alpha}{2}}|^2$,

$$\lim_{\varepsilon\to 0} |\int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_1}\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon)| = 0.$$

Therefore, since

$$\int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon) = \int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_0}\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon) + \int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\mathbf{1}_{A_1}\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon),$$

we have that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{T}^d} \psi(x,\rho_\varepsilon)(\rho^{\frac{\alpha}{2}} g * \nabla\kappa^\varepsilon) = \int_{\mathbb{T}^d} (\partial_\xi\psi)(x,\rho)\rho^{\frac{\alpha}{2}} g\cdot\nabla\rho + \int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} g\cdot(\nabla\psi)(x,\rho).$$

Recovering the kinetic form: if ρ is weak solution, for $\rho_\varepsilon = (\rho * \kappa^\varepsilon)$,

$$\partial_t \int_{\mathbb{T}^d} \Psi(x, \rho_\varepsilon) = \int_{\mathbb{T}^d} \psi(x, \rho_\varepsilon)\partial_t \rho_\varepsilon$$
$$= -2 \int_{\mathbb{T}^d} \psi(x, \rho_\varepsilon)(\rho^{\frac{\alpha}{2}} \nabla \rho^{\frac{\alpha}{2}} * \nabla \kappa^\varepsilon) + \int_{\mathbb{T}^d} \psi(x, \rho_\varepsilon)(\rho^{\frac{\alpha}{2}} g * \nabla \kappa^\varepsilon)$$

and, after passing to the limit $\varepsilon \to 0$,

$$\partial_t \int_{\mathbb{T}^d} \Psi(x, \rho) = -\int_{\mathbb{T}^d} (\partial_\xi \psi)(x, \rho)\alpha \rho^{\alpha-1} |\nabla \rho|^2 - \int_{\mathbb{T}^d} \alpha \rho^{\alpha-1} (\nabla \psi)(x, \rho) \cdot \nabla \rho$$
$$+ \int_{\mathbb{T}^d} (\partial_\xi \psi)(x, \rho)\rho^{\frac{\alpha}{2}} g \cdot \nabla \rho + \int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} g \cdot (\nabla \psi)(x, \rho).$$

The kinetic equation: for $q = \delta_\rho\big(\alpha \xi^{\alpha-1} |\nabla \rho|^2\big)$,

$$\partial_t \int_{\mathbb{T}^d} \int_{\mathbb{R}} \psi(x, \rho)\chi = -\int_{\mathbb{T}^d} \int_{\mathbb{R}} (\partial_\xi \psi)(x, \xi) q - \int_{\mathbb{T}^d} \int_{\mathbb{R}} \alpha \xi^{\alpha-1} (\nabla \psi)(x, \xi) \cdot \nabla \chi$$
$$+ \int_{\mathbb{T}^d} \int_{\mathbb{R}} (\partial_\xi \psi)(x, \xi)\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi - \int_{\mathbb{T}^d} \int_{\mathbb{R}} \xi^{\frac{\alpha}{2}} (\partial_\xi \chi) g \cdot (\nabla \psi)(x, \xi),$$

and

$$\partial_t \chi = \nabla \cdot (\alpha \xi^{\alpha-1} \nabla \chi) + \partial_\xi q - \partial_\xi (\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_\xi \chi) g).$$

III. Weak solutions of the skeleton equation

Equivalence of weak and renormalized kinetic solutions [F., Gess; 2023]

Under assumptions including $\Phi(\xi) = \xi^m$ for every $m \in [1, \infty)$, a nonnegative function $\rho \in \mathrm{C}([0,T]; L^1(\mathbb{T}^d))$ that satisfies

$$\Phi^{\frac{1}{2}}(\rho) \in L^2([0,T]; H^1(\mathbb{T}^d))$$

is a renormalized kinetic solution of the skeleton equation

$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \ \text{ in } \ \mathbb{T}^d \times (0,T) \ \text{ with } \ \rho(\cdot, 0) = \rho_0,$$

for a nonnegative ρ_0 with finite entropy if and only if ρ is a weak solution. In particular, weak solutions exist and are unique.

— equivalence of renormalized and weak solutions [DiPerna, Lions; 1989], [Ambrosio; 2004].

— strong continuity with respect to weak convergence of the control

— for example, $\Phi^{\frac{1}{2}}$ convex or concave or Φ satisfies that $0 < \lambda \leq \Phi' \leq \Lambda$.

III. Weak solutions of the skeleton equation

Weak-strong continuity [F., Gess; 2023]

If ρ_n are solutions of the skeleton equation with controls $g_n \rightharpoonup g$ and initial data $\rho_{0,n} \rightharpoonup \rho_0$ with uniformly bounded entropy, then $\rho_n \to \rho$ for ρ the solution of the skeleton equation with control g and initial data ρ_0.

The entropy estimate: if $\partial_t \rho_n = \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^{\frac{\alpha}{2}} g_n)$ then

$$\max_{t\in[0,T]} \int_{\mathbb{T}^d} \rho_n \log(\rho_n) + \int_0^T \int_{\mathbb{T}^d} |\nabla \rho_n^{\frac{\alpha}{2}}|^2 \lesssim \int_{\mathbb{T}^d} \rho_{0,n} \log(\rho_{0,n}) + \int_0^T \int_{\mathbb{T}^d} |g_n|^2.$$

Compactness since the g_n are uniformly $(L^2_{t,x})^d$-bounded,

$$\rho_n \text{ is strongly compact in } L^1_{t,x} \text{ and } \rho_n^{\frac{\alpha}{2}} \text{ is weakly compact in } L^2_t H^1_x.$$

Uniqueness of the limit: We have for some ρ that, along a subsequence,

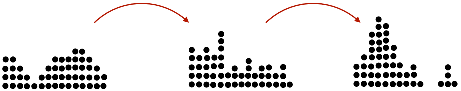
$$\rho_n \to \rho \text{ in } L^1_{t,x} \text{ and } \rho_n^{\frac{\alpha}{2}} \rightharpoonup \rho^{\frac{\alpha}{2}} \text{ in } L^2_t H^1_x,$$

from which we conclude that

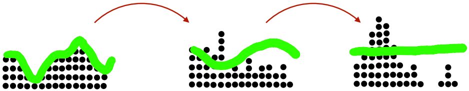
$$\partial_t \rho = 2\nabla \cdot (\rho^{\frac{\alpha}{2}} \nabla \rho^{\frac{\alpha}{2}}) - \nabla \cdot (\rho^{\frac{\alpha}{2}} g),$$

and that $\rho_n \to \rho$ along the full sequence $n \to \infty$.

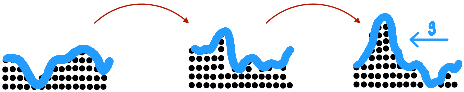
The zero range process: μ^N on $\mathbb{T}^1 \times [0,T]$ for $N = 15$ and $T(k) \sim ke^{-kt}$,



The heat equation: the hydrodynamic limit $\partial_t \bar{\rho} = \Delta \bar{\rho}$,



The skeleton equation: the controlled equation $\partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} \cdot g)$,



The rate function: we have $\mathbb{P}(\mu^N \simeq \rho) \simeq \exp(-NI(\rho))$ for

$$I(\rho) = \frac{1}{2} \inf\{\|g\|^2_{L^2_{t,x}} : \partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} g)\}.$$

IV. L.s.c. envelope of the rate function

The hydrodynamic limit: for the parabolically rescaled, mean zero particle process μ_t^N on $\mathbb{T}_N^d$, as $N \to \infty$, for $J(\overline{\rho}) = \nabla \sigma(\overline{\rho})$,

$$\mu_t^N \rightharpoonup \overline{\rho}\,\mathrm{d}x \;\text{ for }\; \partial_t \overline{\rho} = \Delta\sigma(\overline{\rho}) = \nabla \cdot J(\overline{\rho}).$$

Macroscopic fluctuation theory: the probability of observing a space-time fluctuation (ρ, j) satisfying

$$\partial_t \rho = \nabla \cdot j \;\left(\text{that is, } \partial_t \int_U \rho = \oint_{\partial U} j \cdot \nu\right),$$

satisfies the large deviations bound [Bertini et al.; 2014]

$$\mathbb{P}[\mu^N \simeq \rho] \simeq \exp\big(-NI(\rho)\big) \;\text{ for }\; I(\rho) = \int_0^T \int_{\mathbb{T}^d} (j - J(\rho)) \cdot m(\rho)^{-1}(j - J(\rho)).$$

The skeleton equation: if $(j - J(\rho)) = \sqrt{m(\rho)}g$ then $I(\rho) = \int_0^T \int_{\mathbb{T}^d} |g|^2$ and

$$\partial_t \rho = \nabla \cdot \big(J(\rho) + (j - J(\rho))\big) = \Delta\sigma(\rho) - \nabla \cdot (\sqrt{m(\rho)}g).$$

The zero range process: $\sigma(\rho) = \Phi(\rho)$ and $m(\rho) = \Phi(\rho)$ and

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho)g).$$

The exclusion process: $\sigma(\rho) = \rho$ and $m(\rho) = \rho(1-\rho)$ and

$$\partial_t \rho = \Delta\rho - \nabla \cdot (\sqrt{\rho(1-\rho)}g).$$

IV. L.s.c. envelope of the rate function

— for $I(\rho) = \frac{1}{2} \inf \left\{ \|g\|^2_{L^2(\mathbb{T}^d \times [0,T];\mathbb{R}^d)} : \partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} \cdot g) \right\}$.

Large Deviations Principle [Benois, Kipnis, Landim; 1995]

For every closed $A \subseteq D([0,T]; \mathcal{M}_+(\mathbb{T}^d))$,

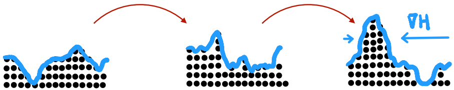
$$\limsup_{N \to \infty} \frac{1}{N} \log(\mathbb{P}[\mu^N \in A]) \leq - \inf_{m \in A} I(m).$$

For the space of smooth fluctuations

$$\mathcal{S} = \{\partial_t m = \Delta m^\alpha - \nabla \cdot (m^\alpha \nabla H) \colon H \in \mathrm{C}^{3,1}(\mathbb{T}^d \times [0,T])\},$$

for every open subset $A \subseteq D([0,T]; \mathcal{M}_+(\mathbb{T}^d))$,

$$\limsup_{N \to \infty} \frac{1}{N} \log(\mathbb{P}[\mu^N \in A]) \geq - \overline{\inf_{\rho \in (A \cap \mathcal{S})} I(\rho)}^{\mathrm{lsc}}.$$



The restricted rate function: for the space of smooth fluctuations

$$\mathcal{S} = \{\partial_t m = \Delta m^\alpha - \nabla \cdot (m^\alpha \nabla H) \colon H \in \mathrm{C}^{3,1}(\mathbb{T}^d \times [0,T])\},$$

the large deviations lower bound is defined by

$$\limsup_{N\to\infty} \frac{1}{N} \log(\mathbb{P}\big[\mu^N \in A\big]) \geq -\overline{\inf_{\rho\in(A\cap\mathcal{S})} I(\rho)}^{\mathrm{lsc}}.$$

The recovery sequence: given an arbitrary fluctuation

$$\partial_t \rho = \Delta\rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g),$$

need find a sequence $\rho_n \in \mathcal{S}$ such that

$$\rho_n \to \rho \in L^1_{t,x} \ \text{ and } \ I(\rho_n) \to I(\rho).$$

The rate function: we have

$$I(\rho) = \frac{1}{2}\inf\left\{\|g\|_{L^2}^2 : \partial_t\rho = \Delta\rho^\alpha - \nabla\cdot(\rho^{\frac{\alpha}{2}}g)\right\}.$$

The Hilbert space: $H^1_{\rho^\alpha}$ is the strong closure w.r.t. the inner product

$$\langle\nabla\psi,\nabla\phi\rangle = \int_0^T\int_{\mathbb{T}^d}\rho^\alpha\nabla\psi\cdot\nabla\phi \ \text{ for } \ \phi,\psi\in \mathrm{C}^\infty.$$

Unique minimizer: the equation defines

$$\partial_t\rho - \Delta\rho^\alpha = -\nabla\cdot(\rho^{\frac{\alpha}{2}}g)\in H^{-1}_{\rho^\alpha},$$

and if $I(\rho)<\infty$ then the minimizer $g=\rho^{\frac{\alpha}{2}}\nabla H$ for $H\in H^1_{\rho^\alpha}$ and

$$I(\rho) = \frac{1}{2}\int_0^T\int_{\mathbb{T}^d}\rho^\alpha|\nabla H|^2 = \frac{1}{2}\|H\|^2_{H^1_{\rho^\alpha}} = \frac{1}{2}\|\partial_t\rho - \Delta\Phi(\rho)\|^2_{H^{-1}_{\rho^\alpha}}.$$

The "ill-posed" equation: we have the formally "supercritical" equation

$$\partial_t\rho = \Delta\rho^\alpha - \nabla\cdot(\rho^\alpha\nabla H).$$

The recovery sequence: given an arbitrary fluctuation

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g),$$

need find a sequence $\rho_n \in \mathcal{S}$ such that

$$\rho_n \to \rho \in L^1_{t,x} \text{ and } I(\rho_n) \to I(\rho).$$

A first attempt: there exists $H \in H^1_{\rho^\alpha}$ such that

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^\alpha \nabla H) \text{ and } I(\rho) = \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \rho^\alpha |\nabla H|^2.$$

Let ρ_ε solve

$$\partial_t \rho_\varepsilon = \Delta \rho_\varepsilon^\alpha - \nabla \cdot (\rho_\varepsilon^\alpha \nabla H^\varepsilon).$$

Passing $\varepsilon \to 0$?

— supercritical with no stable estimates with respect to ∇H

— the Hilbert space framework is too rigid

A second attempt: for some $g \in L^2_{t,x}$ and ρ_0 with finite entropy,

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g) \text{ with } \rho(\cdot, 0) = \rho_0.$$

Regularizing the data: we consider

$$\rho_{0,n} = \Big((\rho_0 \wedge n) \vee \frac{1}{n}\Big) * \kappa_x^{\frac{1}{n}} \text{ and } g_n = g * \kappa_{t,x}^{\frac{1}{n}},$$

and solve

$$\partial_t \rho_n = \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^{\frac{\alpha}{2}} g_n) \text{ with } \rho_n(\cdot, 0) = \rho_{0,n}.$$

There exists $H_n \in H^1_{\rho_n^\alpha}$ such that

$$\partial_t \rho_n = \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^\alpha \nabla H_n) \text{ with } \rho_n(\cdot, 0) = \rho_{0,n}.$$

Deducing the regularity of H_n: we have the elliptic equation

$$-\nabla \cdot (\rho_n^\alpha \nabla H_n) = \partial_t \rho_n - \Delta \rho_n^\alpha.$$

— not necessarily uniformly elliptic

— is ρ_n regular?

The final attempt: for some $g \in L^2_{t,x}$ and ρ_0 with finite entropy,

$$\partial_t \rho = \Delta \rho^{\alpha} - \nabla \cdot (\rho^{\frac{\alpha}{2}} g) \text{ with } \rho(\cdot, 0) = \rho_0.$$

Regularizing the data: we consider

$$\rho_{0,n} = \left((\rho_0 \wedge n) \vee \frac{1}{n}\right) * \kappa_x^{\frac{1}{n}} \text{ and } g_n = g * \kappa_{t,x}^{\frac{1}{n}},$$

"Turning off" the control: for $\sigma_n(\xi) = 0$ if $\xi \leq \frac{1}{n}$ or $\xi \geq n$, solve

$$\begin{aligned} \partial_t \rho_n &= \Delta \rho_n^{\alpha} - \nabla \cdot (\rho_n^{\frac{\alpha}{2}} \sigma_n(\rho_n) g_n) \\ &= \Delta \rho_n^{\alpha} - \nabla \cdot (\rho_n^{\frac{\alpha}{2}} \tilde{g}_n), \end{aligned}$$

for the control $\tilde{g}_n = \sigma_n(\rho_n) g_n$.

Regularity of ρ_n: we have that $\frac{1}{n} \leq \rho_n \leq n$ and $\rho_n \in \mathrm{C}^{\infty}(\mathbb{T}^d \times [0, T])$.

Deducing the regularity of H_n: There exists $H_n \in H^1_{\rho_n^{\alpha}}$ such that

$$\partial_t \rho_n = \Delta \rho_n^{\alpha} - \nabla \cdot (\rho_n^{\alpha} \nabla H_n) \text{ and } -\nabla \cdot (\rho_n^{\alpha} \nabla H_n) = \partial_t \rho_n - \Delta \rho_n^{\alpha}.$$

— in general, $\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g)$ for $\Phi \in \mathrm{C}^2_{\mathrm{loc}}((0, \infty))$

IV. L.s.c envelope of the rate function

The fluctuation: for some $g \in L^2_{t,x}$ and ρ_0 with finite entropy,

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g) \text{ with } \rho(\cdot, 0) = \rho_0.$$

The recovery sequence: for $\sigma_n(\xi) = 0$ if $\xi \leq \frac{1}{n}$ or $\xi \geq n$, solve

$$\partial_t \rho_n = \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^{\frac{\alpha}{2}} \sigma_n(\rho_n) g_n) = \Delta \rho_n^\alpha - \nabla \cdot (\rho_n^{\frac{\alpha}{2}} \tilde{g}_n),$$

for the control $\tilde{g}_n = \sigma_n(\rho_n) g_n$ and with $\rho_n(\cdot, 0) = \rho_{0,n}$.

Compactness: the ρ_n satisfy uniformly the entropy estimate and

$$\rho_n \to \rho \text{ and } \sigma(\rho_n) g_n \mathbf{1}_{\{\rho > 0\}} \to g \mathbf{1}_{\{\rho > 0\}} \text{ and } I(\rho_n) \leq \|\sigma(\rho_n) g_n\|_2^2 \to \|g\|_2^2 .$$

[F., Gess; 2023]

For the space of smooth fluctuations

$$\mathcal{S} = \{\partial_t m = \Delta m^\alpha - \nabla \cdot (m^\alpha \nabla H) \colon H \in \mathrm{C}^{3,1}(\mathbb{T}^d \times [0, T])\},$$

we have that

$$\overline{I(\rho)\big|_{\mathcal{S}}}^{\mathrm{lsc}} = I(\rho) = \frac{1}{2} \inf\{\|g\|_2^2 : \partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g)\}.$$

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