The kinetic formulation of the skeleton equation

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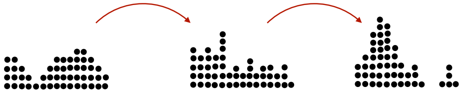
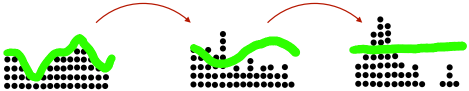
13 August 2024

I. Macroscopic fluctuation theory

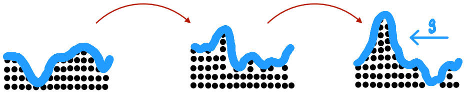
The zero range process: μ^N on $\mathbb{T}^1 \times [0,T]$ for $N = 15$ and $T(k) \sim ke^{-kt}$,



The heat equation: the hydrodynamic limit $\partial_t \overline{\rho} = \Delta \overline{\rho}$,



The skeleton equation: the controlled equation $\partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} \cdot g)$,



The rate function: we have $\mathbb{P}(\mu^N \simeq \rho) \simeq \exp(-NI(\rho))$ for

$$I(\rho) = \frac{1}{2} \inf\{\|g\|^2_{L^2_{t,x}} : \partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho} g)\}.$$

The hydrodynamic limit: for the parabolically rescaled, mean zero particle process μ_t^N on $\mathbb{T}_N^d$, as $N \to \infty$, for $J(\overline{\rho}) = \nabla \sigma(\overline{\rho})$,

$$\mu_t^N \rightharpoonup \overline{\rho}\,\mathrm{d}x \;\text{ for }\; \partial_t \overline{\rho} = \Delta \sigma(\overline{\rho}) = \nabla \cdot J(\overline{\rho}).$$

Macroscopic fluctuation theory: the probability of observing a space-time fluctuation (ρ, j) satisfying

$$\partial_t \rho = \nabla \cdot j,$$

satisfies the large deviations bound [Bertini et al.; 2014]

$$\mathbb{P}[\mu^N \simeq \rho] \simeq \exp\big(-NI(\rho)\big) \;\text{ for }\; I(\rho) = \int_0^T \int_{\mathbb{T}^d} (j - J(\rho)) \cdot m(\rho)^{-1} (j - J(\rho)).$$

The skeleton equation: if $(j - J(\rho)) = \sqrt{m(\rho)}g$ then $I(\rho) = \int_0^T \int_{\mathbb{T}^d} |g|^2$ and

$$\partial_t \rho = \nabla \cdot \big(J(\rho) + (j - J(\rho))\big) = \Delta \sigma(\rho) - \nabla \cdot (\sqrt{m(\rho)}g).$$

The zero range process: $\sigma(\rho) = \Phi(\rho)$ and $m(\rho) = \Phi(\rho)$ and

$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g).$$

The exclusion process: $\sigma(\rho) = \rho$ and $m(\rho) = \rho(1-\rho)$ and

$$\partial_t \rho = \Delta \rho - \nabla \cdot (\sqrt{\rho(1-\rho)}g).$$

The skeleton equation: in the case of the zero range process,

$$\partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \text{ in } \mathbb{T}^d \times (0, T),$$

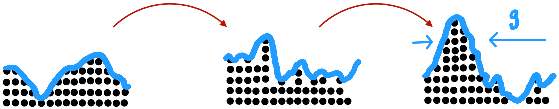
for an L^2-control $g \in (L^2_{t,x})^d$. We specialize to the case, for some $\alpha \in (0, \infty)$,

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

— energy critical in L^1

— supercritical in L^p for every $p \in (1, \infty)$

— nonnegative integrable initial data



The skeleton equation: we have tha $\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g)$.

Zooming in: for $\lambda, \eta, \tau \to 0$ the rescaling $\tilde{\rho}(x,t) = \lambda \rho(\eta x, \tau t)$ solves

$$\partial_t \tilde{\rho} = \left(\frac{\tau}{\eta^2 \lambda^{\alpha-1}}\right) \Delta \left(\tilde{\rho}^\alpha\right) - \nabla \cdot \left(\tilde{\rho}^{\frac{\alpha}{2}} \tilde{g}\right)$$

for $\tilde{g}$ defined by

$$\tilde{g}(x,t) = \left(\frac{\tau}{\eta \lambda^{\frac{\alpha}{2}-1}}\right) g(\eta x, \tau t).$$

Preserving the diffusion and the L^r-norm of the initial data,

$$\left(\frac{\tau}{\eta^2 \lambda^{\alpha-1}}\right) = 1 \text{ and } \lambda = \eta^{\frac{d}{r}}.$$

We then have that

$$\|\tilde{g}\|_{L_t^p L_x^q} = \eta^{1 - \frac{d}{p} + \frac{2}{q} + \frac{d}{r}\left(\frac{\alpha}{2} - \frac{\alpha}{q} + \frac{1}{q}\right)} \|g\|_{L_t^p L_x^q}.$$

We require $1 + \frac{d}{r}\left(\frac{\alpha}{2} + \frac{1}{q}\right) \geq \frac{2}{q} + \frac{d}{p} + \frac{d\alpha}{rq}$, and if $p = q = 2$,

$$d/2r \geq d/2 \text{ and } r = 1.$$

If $r = 1$ then

$$1 + d\left(\frac{\alpha}{2} + \frac{1}{q}\right) \geq \frac{2}{q} + d\left(\frac{1}{p} + \frac{\alpha}{q}\right) \text{ and } p = q = 2.$$

II. The kinetic formulation of the skeleton equation

A formal uniqueness proof: if ρ_1 and ρ_2 solve

$$\partial_t \rho = \Delta \rho^{\alpha} - \nabla \cdot (\rho^{\frac{\alpha}{2}} g),$$

we have using the distributional equalities

$$|\xi|' = \operatorname{sgn}(\xi) \ \text{ and } \ \operatorname{sgn}'(\xi) = 2\delta_0(\xi),$$

that, integrating on the torus $\mathbb{T}^d$,

$$\begin{aligned}
\partial_t \Big(\int |\rho_1 - \rho_2| \Big) &= \int \operatorname{sgn}(\rho_1 - \rho_2) \Delta(\rho_1^{\alpha} - \rho_2^{\alpha}) - \int \operatorname{sgn}(\rho^1 - \rho^2) \nabla \cdot \Big((\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}}) g \Big) \\
&= \int \operatorname{sgn}(\rho_1^{\alpha} - \rho_2^{\alpha}) \Delta(\rho_1^{\alpha} - \rho_2^{\alpha}) - \int \operatorname{sgn}(\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}}) \nabla \cdot \big(\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}} \big) g \big) \\
&= -\int 2\delta_0(\rho_1^{\alpha} - \rho_2^{\alpha}) |\nabla \rho_1^{\alpha} - \nabla \rho_2^{\alpha}|^2 - \int \nabla \cdot \Big(|\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}}| g \Big) \leq 0.
\end{aligned}$$

We therefore have that

$$\max_{t \in [0,T]} \|\rho_1 - \rho_2\|_{L^1(\mathbb{T}^d)} = \|\rho_{1,0} - \rho_{2,0}\|_{L^1(\mathbb{T}^d)} \,.$$

A slightly less naive uniqueness proof: for $f^\delta = (f * \kappa^\delta)$, and ρ_1, ρ_2 solving

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g),$$

we have that

$$\partial_t \Big(\int |\rho_1 - \rho_2|^\delta \Big) = \int \operatorname{sgn}^\delta(\rho_1 - \rho_2) \Delta(\rho_1^\alpha - \rho_2^\alpha) - \int \operatorname{sgn}^\delta(\rho^1 - \rho^2) \nabla \cdot \big((\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}}) g \big)$$
$$= - \int 2\delta_0^\delta(\rho_1 - \rho_2) \nabla(\rho_1 - \rho_2) \cdot \nabla(\rho_1^\alpha - \rho_2^\alpha) + \int 2\delta_0^\delta(\rho_1 - \rho_2)(\rho_1^{\frac{\alpha}{2}} - \rho_2^{\frac{\alpha}{2}}) \nabla(\rho_1 - \rho_2) \cdot g.$$

If $\alpha = 1$ then using Hölder's and Young's inequalities, for every $\varepsilon \in (0,1)$,

$$\partial_t \Big(\int |\rho_1 - \rho_2|^\delta \Big) + \int 2\delta_0^\delta(\rho_1 - \rho_2) |\nabla(\rho_1 - \rho_2)|^2$$
$$\leq \varepsilon \int \delta_0^\delta(\rho_1 - \rho_2) |\nabla(\rho_1 - \rho_2)|^2 + \frac{1}{\varepsilon} \int \delta_0^\delta(\rho_1 - \rho_2) (\sqrt{\rho_1} - \sqrt{\rho_2})^2 |g|^2.$$

Therefore, using that $\delta_0^\delta(\rho_1 - \rho_2) \lesssim \delta^{-1} \mathbf{1}_{\{|\rho_1 - \rho_2| < \delta\}}$,

$$\partial_t \Big(\int |\rho_1 - \rho_2|^\delta \Big) + \int \delta_0^\delta(\rho_1 - \rho_2) |\nabla(\rho_1 - \rho_2)|^2 \lesssim \int \mathbf{1}_{\{0 < |\rho_1 - \rho_2| < \delta\}} |g|^2.$$

Preservation of nonnegativity: if $\rho_0 \geq 0$ then at the first time ρ hits zero,

$$0 \geq \partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g) = \Delta \rho^\alpha - \rho^{\frac{\alpha}{2}} \nabla \cdot g - \nabla \rho^{\frac{\alpha}{2}} \cdot g \geq 0,$$

from which we conclude that $\partial_t \rho = 0$ and that $\rho \geq 0$ if $\rho_0 \geq 0$.

Conservation of mass: solutions preserve mass if $\rho_0 \geq 0$,

$$\partial_t \Big(\int_{\mathbb{T}^d} \rho(x,t) \Big) = \int_{\mathbb{T}^d} \partial_t \rho = \int_{\mathbb{T}^d} \nabla \cdot \big(\nabla \rho^\alpha - \rho^{\frac{\alpha}{2}} g \big) = 0.$$

An a priori estimate: for an arbitrary nonlinearity Ψ with $\psi = \Psi'$,

$$\partial_t \Big(\int_{\mathbb{T}^d} \Psi(\rho) \Big) = -\alpha \int_{\mathbb{T}^d} \rho^{\alpha-1} \psi'(\rho) |\nabla \rho|^2 + \int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} g \cdot \psi'(\rho) \nabla \rho.$$

Therefore, using Hölder's and Young's inequalities, for every $\varepsilon \in (0,1)$,

$$\partial_t \Big(\int_{\mathbb{T}^d} \Psi(\rho) \Big) + \alpha \int_{\mathbb{T}^d} \rho^{\alpha-1} \psi'(\rho) |\nabla \rho|^2 \leq \frac{1}{2\varepsilon} \int_{\mathbb{T}^d} |g|^2 + \frac{\varepsilon}{2} \int_{\mathbb{T}^d} \rho^\alpha \psi'(\rho)^2 |\nabla \rho|^2.$$

To close the estimate, we require that

$$\rho^\alpha \psi'(\rho)^2 \lesssim \psi'(\rho) \rho^{\alpha-1} \ \text{ so } \ \psi'(\xi) \lesssim \frac{1}{\xi}.$$

The equation: we have that

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

A priori estimates: for $\Psi'(\xi) = \psi(\xi)$, to close the estimate

$$\max_{t\in[0,T]} \int_{\mathbb{T}^d} \Psi(\rho) + \int_0^T \int_{\mathbb{T}^d} \psi'(\xi)\rho^{\alpha-1}|\nabla \rho|^2 \lesssim \int_{\mathbb{T}^d} \Psi(\rho_0) + \int_0^T \int_{\mathbb{T}^d} |g|^2,$$

we require that $\psi'(\xi) \lesssim \frac{1}{\xi}$.

Entropy dissipation: if $\psi(\xi) = \log(\xi)$ then $\Psi(\xi) = \xi \log(\xi) - \xi$ and using

$$\rho^{\alpha-2}|\nabla \rho|^2 = |\rho^{\frac{\alpha-2}{2}} \nabla \rho|^2 = \frac{4}{\alpha^2}|\nabla \rho^{\frac{\alpha}{2}}|^2,$$

we have using the preservation of the L^1-norm that

$$\max_{t\in[0,T]} \int_{\mathbb{T}^d} \rho \log(\rho) + \int_0^T \int_{\mathbb{T}^d} |\nabla \rho^{\frac{\alpha}{2}}|^2 \lesssim \int_{\mathbb{T}^d} \rho_0 \log(\rho_0) + \int_0^T \int_{\mathbb{T}^d} |g|^2.$$

— fluctuation-dissipation relation $\Psi''(\xi) \simeq \frac{\Phi'(\xi)}{\Phi(\xi)}$ for $\Phi(\xi) = \xi^\alpha$

A local H^1-estimate: for $M < N \in (0,\infty)$ let $\psi'_{M,N} = (N-M)^{-1}\mathbf{1}_{\{M<\xi<N\}}$ so that

$$\psi_{M,N}(\xi) = \frac{(x-M)_+}{N-M} \wedge 1 \quad \text{and} \quad \Psi_{M,N}(\xi) = \begin{cases} \frac{(x-M)_+^2}{2(N-M)} & \xi \in [0,N] \\ \frac{1}{2}(N-M) + (x-N) & \xi \in [N,\infty). \end{cases}$$

Since for a constant depending on M, N we have $\psi'(\xi) \lesssim \frac{1}{\xi}$,

$$\max_{t\in[0,T]} \int_{\mathbb{T}^d} \Psi_{M,N}(\rho) + \int_0^T \int_{\mathbb{T}^d} \psi'_{M,N}(\xi)\rho^{\alpha-1}|\nabla\rho|^2 \lesssim \int_{\mathbb{T}^d} \Psi_{M,N}(\rho_0) + \int_0^T \int_{\mathbb{T}^d} |g|^2,$$

and, therefore, for a constant depending on M, N,

$$\int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M\le\rho\le N\}}\rho^{\alpha-1}|\nabla\rho|^2 \lesssim \int_{\mathbb{T}^d} \rho_0 + \int_0^T \int_{\mathbb{T}^d} |g|^2.$$

Local regularity: for every $K \in (1,\infty)$,

$$\left((\rho \wedge K) \vee K^{-1}\right) \in L^2_t H^1_x.$$

The skeleton equation: for an $(L^2_{t,x})^d$-valued control g,

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

L^1**-contraction**: if ρ_1 and ρ_2 are solutions,

$$\max_{t\in[0,T]} \|\rho_1 - \rho_2\|_{L^1(\mathbb{T}^d)} = \|\rho_{1,0} - \rho_{2,0}\|_{L^1(\mathbb{T}^d)}.$$

Preservation of nonnegativity and mass: if $\rho_0 \geq 0$ then $\rho \geq 0$ with

$$\|\rho(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_0\|_{L^1(\mathbb{T}^d)}.$$

The entropy estimate: if ρ_0 is nonnegative with finite entropy then

$$\max_{t\in[0,T]} \int_{\mathbb{T}^d} \rho \log(\rho) + \int_0^T \int_{\mathbb{T}^d} |\nabla \rho^{\frac{\alpha}{2}}|^2 \lesssim \int_{\mathbb{T}^d} \rho_0 \log(\rho_0) + \int_0^T \int_{\mathbb{T}^d} |g|^2.$$

The local H^1**-estimate**: for every $M < N \in (0,\infty)$,

$$\int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<N\}} \rho^{\alpha-1} |\nabla \rho|^2 \lesssim \int_{\mathbb{T}^d} \rho_0 + \int_0^T \int_{\mathbb{T}^d} |g|^2.$$

A renormalized equation: for $\eta \in (0,1)$ we consider the regularized equation

$$\partial_t \rho = \Delta \rho^{\alpha} + \eta \Delta \rho - \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

Then, for a smooth $S \colon \mathbb{R} \to \mathbb{R}$ and $\phi \in \mathrm{C}^{\infty}(\mathbb{T}^d)$,

$$\begin{aligned}
\partial_t \Big(\int S(\rho)\phi(x) \Big) &= \int S'(\rho)\phi(x) \big(\Delta \rho^{\alpha} + \eta \Delta \rho - \nabla \cdot (\rho^{\frac{\alpha}{2}} g) \big) \\
&= - \int S'(\rho) \nabla \phi(x) \cdot (\alpha \rho^{\alpha-1} \nabla \rho + \eta \nabla \rho) \\
&\quad - \int S''(\rho)\phi(x) \big(\alpha \rho^{\alpha-1} |\nabla \rho|^2 + \eta |\nabla \rho|^2 \big) \\
&\quad + \int S''(\rho)\phi(x) \rho^{\frac{\alpha}{2}} g \cdot \nabla \rho + S'(\rho) \nabla \phi(x) \cdot (\rho^{\frac{\alpha}{2}} g).
\end{aligned}$$

If $S'' \geq 0$ and $\phi \geq 0$ then the entropy formulation yields that, after passing $\eta \to 0$,

$$\begin{aligned}
\partial_t \Big(\int S(\rho)\phi(x) \Big) \leq - \int S'(\rho) \nabla \phi(x) \cdot (\alpha \rho^{\alpha-1} \nabla \rho) - \int S''(\rho)\phi(x) (\alpha \rho^{\alpha-1} |\nabla \rho|^2) \\
+ \int S''(\rho)\phi(x) \rho^{\frac{\alpha}{2}} g \cdot \nabla \rho + S'(\rho) \nabla \phi(x) \cdot (\rho^{\frac{\alpha}{2}} g).
\end{aligned}$$

II. The kinetic formulation of the skeleton equation

The kinetic function: for $\chi(x,\xi,t) = \mathbf{1}_{\{0<\xi<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\xi<0\}}$ we have

$$\partial_\xi \chi = \delta_0 - \delta_\rho \ \text{ and } \ \nabla_x \chi = \delta_\rho \nabla \rho,$$

for $\delta_\rho = \delta_0(\xi - \rho)$.

A renormalized equation: since we have that

$$\partial_t\Big(\int S(\rho)\phi(x)\Big) = -\int S'(\rho)\nabla\phi(x)\cdot(\alpha\rho^{\alpha-1}\nabla\rho + \eta\nabla\rho)$$
$$-\int S''(\rho)\phi(x)\big(\alpha\rho^{\alpha-1}|\nabla\rho|^2 + \eta|\nabla\rho|^2\big) + \int S''(\rho)\phi(x)\rho^{\frac{\alpha}{2}} g\cdot\nabla\rho + S'(\rho)\nabla\phi(x)\cdot(\rho^{\frac{\alpha}{2}} g),$$

we have using the equality $\int_{\mathbb{R}} S'(\xi)\chi(x,\xi,t)\,\mathrm{d}\xi = S(\rho)$ that

$$\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi S'(\xi)\phi(x)\Big) = -\int\int \nabla(S'(\xi)\phi(x))\cdot(\alpha\xi^{\alpha-1}\nabla\chi + \eta\nabla\chi)$$
$$-\int\int \partial_\xi(S'(\xi)\phi(x))\delta_\rho\big(\alpha\xi^{\alpha-1}|\nabla\rho|^2 + \eta|\nabla\rho|^2\big)$$
$$+\int\int \partial_\xi(S'(\xi)\phi(x))\xi^{\frac{\alpha}{2}} g\cdot\nabla\chi - \nabla(S'(\rho)\phi(x))\cdot(\xi^{\frac{\alpha}{2}}\partial_\xi\chi g).$$

— the test function $\psi(x,\xi) = S'(\xi)\phi(x)$

A renormalized equation: for the regularized skeleton equation

$$\partial_t \rho = \Delta \rho^{\alpha} + \eta \Delta \rho - \nabla \cdot (\rho^{\frac{\alpha}{2}} g),$$

we have for $\chi = \mathbf{1}_{\{0<\xi<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\xi<0\}}$, for every $\psi \in \mathrm{C}_c^\infty(\mathbb{T}^d \times (0,\infty))$,

$$\partial_t \Big(\int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi \psi \Big) = - \int \int \nabla \psi \cdot (\alpha \xi^{\alpha-1} \nabla \chi + \eta \nabla \chi) - \int \int \partial_\xi \psi \delta_\rho \big(\alpha \xi^{\alpha-1} |\nabla \rho|^2 + \eta |\nabla \rho|^2 \big)$$
$$+ \int \int \partial_\xi \psi \xi^{\frac{\alpha}{2}} g \cdot \nabla \chi - \nabla \psi \cdot (\xi^{\frac{\alpha}{2}} \partial_\xi \chi g),$$

or, distributionally, for the measure $q^\eta = \delta_\rho \big(\alpha \xi^{\alpha-1} |\nabla \rho|^2 + \eta |\nabla \rho|^2 \big)$,

$$\partial_t \chi = \alpha \xi^{\alpha-1} \Delta \chi + \eta \Delta \chi + \partial_\xi q^\eta - \partial_\xi (\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} \partial_\xi \chi g).$$

Using the distributional equalities $\nabla \chi = \delta_\rho \nabla \rho$ and $\partial_\xi \chi = \delta_0 - \delta_\rho$,

$$\partial_t \Big(\int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi \psi \Big) = - \int \int (\alpha \rho^{\alpha-1} + \eta)(\nabla \psi)(x,\rho) \cdot \nabla \rho - \int \int \partial_\xi \psi q^\eta$$
$$+ \int \int (\partial_\xi \psi)(x,\rho) \rho^{\frac{\alpha}{2}} g \cdot \nabla \rho + \rho^{\frac{\alpha}{2}} g \cdot (\nabla \psi)(x,\rho).$$

- $(\nabla \psi)(x,\rho)$ is the derivative of ψ evaluated at (x,ρ)
- for example, [Perthame; 1998], [Chen, Perthame; 2003]

The kinetic formulation of the skeleton equation: for the regularized equation

$$\partial_t \rho = \Delta \rho^{\alpha} + \eta \Delta \rho - \nabla \cdot (\rho^{\frac{\alpha}{2}} g),$$

for the defect measure $q^{\eta} = \delta_{\rho}(\alpha \xi^{\alpha-1} |\nabla \rho|^2 + \eta |\nabla \rho|^2)$, the kinetic formulation is

$$\partial_t \chi = \alpha \xi^{\alpha-1} \Delta \chi + \eta \Delta \chi + \partial_{\xi} q^{\eta} - \partial_{\xi}(\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_{\xi} \chi) g).$$

Passing $\eta \to 0$, if weakly in the sense of measures,

$$q^{\eta} \rightharpoonup q \geq \delta_{\rho}(\alpha \xi^{\alpha-1} |\nabla \rho|^2),$$

and if $\rho^{\eta} \to \rho$ strongly then the kinetic function χ of ρ solves

$$\partial_t \chi = \alpha \xi^{\alpha-1} \Delta \chi + \partial_{\xi} q - \partial_{\xi}(\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_{\xi} \chi) g),$$

for a locally finite, nonnegative measure q on $\mathbb{T}^d \times \mathbb{R} \times [0, T]$ with

$$q \geq \delta_{\rho}(\alpha \xi^{\alpha-1} |\nabla \rho|^2).$$

— the kinetic formulation exactly quantifies this "entropy inequality"

The skeleton equation: for $g \in (L^2_{t,x})^d$ and $\alpha \in (0,\infty)$,

$$\partial_t \rho = \Delta\rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g)$$

for $\chi = \mathbf{1}_{\{0<\xi<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\xi<0\}}$ the kinetic formulation is

$$\partial_t \chi = \alpha\xi^{\alpha-1}\Delta\chi + \partial_\xi q - \partial_\xi(\xi^{\frac{\alpha}{2}} g \cdot \nabla\chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}}(\partial_\xi \chi) g),$$

for a locally finite, nonnegative measure $q \geq \delta_\rho(\alpha\xi^{\alpha-1}|\nabla\rho|^2)$.

The defect measure: if ρ solves the porous media equation $\partial_t\rho = \Delta\rho^\alpha$,

$$q(x,\xi,t) = \delta_\rho \alpha\xi^{\alpha-1}|\nabla\rho|^2 = \delta_\rho \frac{4\alpha}{(\alpha+1)^2}|\nabla\rho^{\frac{\alpha+1}{2}}|^2,$$

and for $\chi = \mathbf{1}_{\{0<\xi<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\xi<0\}}$,

$$\partial_t \chi = \alpha\xi^{\alpha-1}\Delta\chi + \partial_\xi q.$$

An L^2-estimate: testing the equation witch $\psi(x,\xi) = \xi$, since $\int_{\mathbb{R}} \chi\xi = \frac{1}{2}\rho^2$,

$$\frac{1}{2}\int_{\mathbb{T}^d} \rho^2(x,t) = \int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi(x,\xi,t)\xi = \frac{1}{2}\int_{\mathbb{T}^d}\rho_0^2 - \int_0^t\int_{\mathbb{T}^d} q.$$

and, therefore,

$$\max_{t\in[0,T]} \|\rho\|^2_{L^2} + \int_0^T\int_{\mathbb{T}^d} |\nabla\rho^{\frac{\alpha+1}{2}}|^2 \lesssim \max_{t\in[0,T]} \|\rho\|^2_{L^2} + \int_0^T\int_{\mathbb{T}^d} q \lesssim \|\rho_0\|_2^2.$$

The equation: for $\chi = \mathbf{1}_{\{0<\xi<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\xi<0\}}$,

$$\partial_t \chi = \alpha \xi^{\alpha-1} \Delta \chi + \partial_\xi q - \partial_\xi (\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_\xi \chi) g).$$

Preservation of mass: since $\int_{\mathbb{R}} \chi(x,\xi,t)\,\mathrm{d}\xi = \rho(x,t)$,

$$\partial_t \Big(\int_{\mathbb{T}^d} \rho \Big) = \partial_t \Big(\int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi \Big) = \int_{\mathbb{T}^d} \int_{\mathbb{R}} \partial_t \chi = 0.$$

The entropy estimate: for test function $\psi(\xi) = \log(\xi)$, since $\nabla \chi = \delta_\rho \nabla \rho$,

$$\int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi \log(\xi) \Big|_{s=0}^{s=T} = -\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}} \frac{1}{\xi} q + \int_0^T \rho^{\frac{\alpha}{2}-1} g \cdot \nabla \rho$$

$$= -\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}} \frac{1}{\xi} q + \frac{2}{\alpha} \int_0^T g \cdot \nabla \rho^{\frac{\alpha}{2}}.$$

Since we have that

$$\frac{1}{\xi} q \geq \frac{1}{\xi} \cdot \delta_\rho \big(\xi^{\alpha-1} |\nabla \rho|^2 \big) = \rho^{\alpha-2} |\nabla \rho|^2 \simeq |\nabla \rho^{\frac{\alpha}{2}}|^2,$$

using the preservation of mass and $\int_{\mathbb{R}} \chi \log(\xi) = \rho \log(\rho) - \rho$,

$$\max_{t \in [0,T]} \int_{\mathbb{T}^d} \rho \log(\rho) + \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}} \frac{1}{\xi} q \lesssim \int_{\mathbb{T}^d} \rho_0 \log(\rho_0) + \int_0^T \int_{\mathbb{T}^d} |g|^2.$$

II. The kinetic formulation of the skeleton equation

The equation: for $\chi = \mathbf{1}_{\{0<\xi<\rho(x,t)\}} - \mathbf{1}_{\{\rho(x,t)<\xi<0\}}$,

$$\partial_t \chi = \alpha \xi^{\alpha-1}\Delta\chi + \partial_\xi q - \partial_\xi(\xi^{\frac{\alpha}{2}} g \cdot \nabla\chi) + \nabla\cdot(\xi^{\frac{\alpha}{2}}(\partial_\xi\chi)g).$$

The local H^1-estimate: for $\psi'_M(\xi) = \mathbf{1}_{\{M<\xi<M+1\}}$ with $\psi_M(0)=0$,

$$\begin{aligned}\int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi\psi_M\Big|_{s=0}^{s=T} &= -\int_0^T\int_{\mathbb{T}^d}\int_M^{M+1} q + \int_0^T\int_{\mathbb{T}^d}\int_{\mathbb{R}} \mathbf{1}_{\{M<\xi<M+1\}}\xi^{\frac{\alpha}{2}} g\cdot\delta_\rho\nabla\rho \\ &= -\int_0^T\int_{\mathbb{T}^d}\int_M^{M+1} q + \int_0^T\int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<M+1\}}\rho^{\frac{\alpha}{2}} g\cdot\nabla\rho \\ &= -\int_0^T\int_{\mathbb{T}^d}\int_M^{M+1} q + \frac{2}{\alpha+1}\int_0^T\int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<M+1\}}\rho^{\frac{1}{2}} g\cdot\nabla\rho^{\frac{\alpha+1}{2}}.\end{aligned}$$

Since we have that

$$q \geq \delta_\rho\big(\alpha\xi^{\alpha-1}|\nabla\rho|^2\big) \simeq \delta_\rho|\nabla\rho^{\frac{\alpha+1}{2}}|^2,$$

we have that

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d}\int_M^{M+1} q &\lesssim \int_{\mathbb{T}^d}(\rho_0 - M)_+ + \int_0^T\int_{\mathbb{T}^d}\mathbf{1}_{\{M<\rho<M+1\}}\rho|g|^2 \\ &\lesssim \int_{\mathbb{T}^d}(\rho_0 - M)_+ + (M+1)\int_0^T\int_{\mathbb{T}^d}\mathbf{1}_{\{M<\rho<M+1\}}|g|^2.\end{aligned}$$

The kinetic formulation of the skeleton equation: for $q \geq \delta_\rho \alpha \xi^{\alpha-1}|\nabla \rho|^2$,

$$\partial_t \chi = \alpha \xi^{\alpha-1} \Delta \chi + \partial_\xi q - \partial_\xi (\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_\xi \chi) g).$$

Preservation of nonnegativity and mass: if $\rho_0 \geq 0$ then $\rho \geq 0$ with

$$\|\rho(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_0\|_{L^1(\mathbb{T}^d)}.$$

The entropy estimate: if ρ_0 is nonnegative with finite entropy then

$$\max_{t \in [0,T]} \int_{\mathbb{T}^d} \rho \log(\rho) + \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{R}} \frac{1}{\xi} q \lesssim \int_{\mathbb{T}^d} \rho_0 \log(\rho_0) + \int_0^T \int_{\mathbb{T}^d} |g|^2.$$

The local H^1-estimate: for every $M < N \in (0,\infty)$,

$$\alpha \int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<N\}} \rho^{\alpha-1} |\nabla \rho|^2 \lesssim \int_0^T \int_{\mathbb{T}^d} \fint_M^N q$$

$$\lesssim \int_{\mathbb{T}^d} (\rho_0 - M)_+ + \frac{N}{N-M} \int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<N\}} |g|^2.$$

— local H^1-regularity of ρ away from $\{\rho \simeq 0\}$ and $\{\rho \simeq \infty\}$: for every $K \in \mathbb{N}$,

$$\left((\rho \wedge K) \vee \frac{1}{K}\right) \in L^2_t H^1_x.$$

The local H^1-estimate: if $\psi'(\xi) = \mathbf{1}_{\{M<\xi<M+1\}}$,

$$\int_0^T \int_{\mathbb{T}^d} \int_M^{M+1} q \lesssim \int_{\mathbb{T}^d} (\rho_0 - M)_+ + (M+1) \int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<M+1\}} |g|^2.$$

A real analysis lemma: if a_k are nonnegative with $\sum_{k=1}^\infty a_k < \infty$ then

$$\liminf_{k\to\infty} k a_k = 0.$$

If not, for k large $\frac{1}{k} \lesssim a_k$ and $\sum_{k=1}^\infty a_k$ diverges logarithmically.

Vanishing of the defect measure at infinity: we claim that

$$\liminf_{M\to\infty} \int_0^T \int_{\mathbb{T}^d} \int_M^{M+1} q = 0.$$

If $\rho_0 \in L^1(\mathbb{T}^d)$ then $\lim_{M\to\infty} \int_{\mathbb{T}^d} (\rho_0 - M)_+ = 0$. For the control term, for $k \in \mathbb{N}$,

$$a_k = \int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{k-1<\rho<k\}} |g|^2.$$

Then $\sum_{k=1}^\infty a_k \le \int_0^T \int_{\mathbb{T}^d} |g|^2 < \infty$ and

$$\liminf_{k\to\infty} k a_k = \liminf_{M\to\infty} (M+1) \int_0^T \int_{\mathbb{T}^d} \mathbf{1}_{\{M<\rho<M+1\}} |g|^2 = 0.$$

II. The kinetic formulation of the skeleton equation

A renormalized kinetic solution of the skeleton equation

A nonnegative function $\rho \in \mathrm{C}([0,T];L^1(\mathbb{T}^d))$ is a renormalized kinetic solution of the skeleton equation if there exists a nonnegative, locally finite measure q on $\mathbb{T}^d \times \mathbb{R} \times [0,T]$ such that ρ and q satisfy the following four properties.

— *Preservation of mass*: $\|\rho(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_0\|_{L^1(\mathbb{T}^d)}$ for every $t \in [0,T]$.

— *Local H^1-regularity*: $\left((\rho \wedge K) \vee \frac{1}{K}\right) \in L^2([0,T];H^1(\mathbb{T}^d))$ for every $K \in \mathbb{N}$.

— *Regularity and vanishing of the measure at infinity*: we have that

$$\delta_\rho\left(\alpha\xi^{\alpha-1}|\nabla\rho|^2\right) \leq q \quad \text{and} \quad \liminf_{M\to\infty} q(\mathbb{T}^d \times [M,M+1] \times [0,T]) = 0.$$

— *The equation*: for every $\psi \in \mathrm{C}_c^\infty(\mathbb{T}^d \times (0,\infty))$ and $t \in [0,T]$,

$$\int_{\mathbb{T}^d} \chi\psi\Big|_{s=0}^{s=t} = -\int_0^t\int_{\mathbb{T}^d} \alpha\rho^{\alpha-1}\nabla\rho \cdot (\nabla\psi)(x,\rho) - \int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}} (\partial_\xi\psi)(x,\xi)q$$
$$+ \int_0^t\int_{\mathbb{T}^d} (\partial_\xi\psi)(x,\rho)\rho^{\frac{\alpha}{2}} g \cdot \nabla\rho + \int_0^T\int_{\mathbb{T}^d} \rho^{\frac{\alpha}{2}} g \cdot (\nabla\psi)(x,\rho).$$

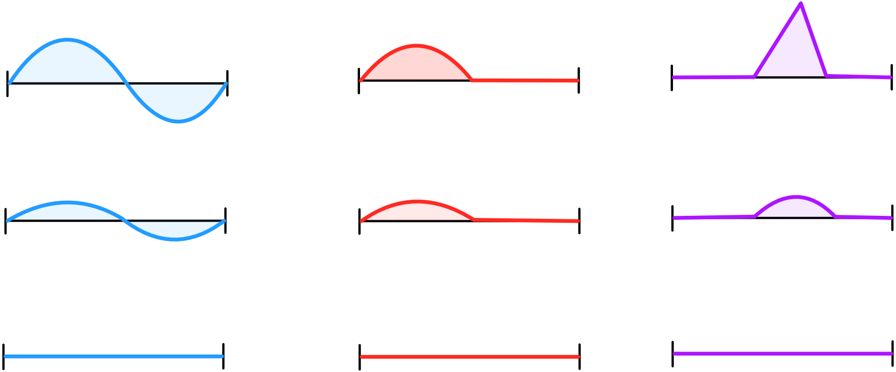
— the equation is not enforced on the set $\{\rho = 0\}$! Why are solutions unique?

A source of nonuniqueness: take the positive part of a signed solution to the heat equation

$$\partial_t \rho = \Delta \rho,$$

Or, for $U \subseteq \mathbb{T}^d$, extend by zero the solution to the Dirichlet problem

$$\partial_t \rho = \Delta \rho \ \text{ in } \ U \times (0, T).$$



Flux through zero: for $\alpha \in (0,1)$ consider a signed solution of the equation

$$\partial_t \rho = \Delta \rho^\alpha.$$

At what rate does "positive mass" change? Precisely, for $\rho_+ = (\rho \vee 0)$,

$$\begin{aligned}\partial_t\Big(\int_{\mathbb{T}^d} \rho_+\Big) = \int_{\mathbb{T}^d} \mathbf{1}_{\{\rho \geq 0\}} \partial_t \rho &= \int_{\mathbb{T}^d} \mathbf{1}_{\{\rho\geq 0\}} \Delta \rho^\alpha \\ &= -\int_{\mathbb{T}^d} \delta_0(\rho) \nabla \rho \cdot \nabla \rho^\alpha \\ &= -\int_{\mathbb{T}^d} \delta_0(\rho)\big(\alpha \rho^{\alpha-1} |\nabla \rho|^2\big)\end{aligned}$$

For the defect measure

$$q(x,\xi,t) = \delta_0(\rho - \xi)\big(\alpha \xi^{\alpha-1}|\nabla \rho|^2\big),$$

we have that

$$\int_0^t \int_{\mathbb{T}^d} q(x,0,s)\,\mathrm{d}x\,\mathrm{d}s = \int_0^t \int_{\mathbb{T}^d} \delta_0(\rho)\big(\alpha\rho^{\alpha-1}|\nabla\rho|^2\big) = \int_{\mathbb{T}^d} \big(\rho_{0,+}(x) - \rho_+(x,t)\big).$$

Change of "positive mass": "positive mass" lost at the rate of

$$\int_{\mathbb{T}^d} q(x,0,t).$$

Vanishing of the defect measure: for the equation

$$\partial_t\chi = \alpha\xi^{\alpha-1}\Delta\chi + \partial_\xi q - \partial_\xi(\xi^{\frac{\alpha}{2}} g\cdot\nabla\chi) + \nabla\cdot(\xi^{\frac{\alpha}{2}}(\partial_\xi\chi)g),$$

for the test functions $\psi'_\beta = \frac{2}{\beta}\mathbf{1}_{\{\frac{\beta}{2}<\xi<\beta\}}$ and $\zeta'_M = -\mathbf{1}_{\{M<\xi<M+1\}}$,

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi\psi_\beta\zeta_M\Big|_{s=0}^{s=t} = -\frac{2}{\beta}\int_0^t\int_{\mathbb{T}^d}\int_{\frac{\beta}{2}}^{\beta} q + \int_0^t\int_{\mathbb{T}^d}\int_M^{M+1} q$$
$$+\frac{2}{\beta}\int_0^t\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}} g\cdot\nabla\rho\mathbf{1}_{\{\frac{\beta}{2}<\rho<\beta\}} + \int_0^t\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}} g\cdot\nabla\rho\mathbf{1}_{\{M<\rho<M+1\}}.$$

We have using $\rho^{\frac{\alpha}{2}}\nabla\rho = \rho^{\frac{1}{2}}\cdot\rho^{\frac{\alpha-1}{2}}\nabla\rho$ and Hölder's and Young's inequalities that

$$\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi\psi_\beta\zeta_M\Big|_{s=0}^{s=t} + \frac{2}{\beta}\int_0^t\int_{\mathbb{T}^d}\int_{\frac{\beta}{2}}^{\beta} q$$
$$\lesssim \int_0^t\int_{\mathbb{T}^d}\int_M^{M+1} q + \frac{1}{\beta}\int_0^t\int_{\mathbb{T}^d}\rho|g|^2\mathbf{1}_{\{\frac{\beta}{2}<\rho<\beta\}} + \int_0^t\int_{\mathbb{T}^d}\rho|g|^2\mathbf{1}_{\{M<\rho<M+1\}}.$$

The righthand side vanishes as $M\to\infty$ and $\beta\to 0$. Therefore,

$$\text{“}\int_0^t\int_{\mathbb{T}^d} q(x,0,s)\text{”} = \lim_{\beta\to 0}\frac{2}{\beta}\int_0^t\int_{\mathbb{T}^d}\int_{\frac{\beta}{2}}^{\beta} q = \int_{\mathbb{T}^d}\big(\rho_0(x)-\rho(x,t)\big) = 0.$$

A renormalized kinetic solution of the skeleton equation

A nonnegative function $\rho \in \mathrm{C}([0,T];L^1(\mathbb{T}^d))$ is a renormalized kinetic solution of the skeleton equation if there exists a nonnegative, locally finite measure q on $\mathbb{T}^d \times \mathbb{R} \times [0,T]$ such that ρ and q satisfy the following four properties.

- *Preservation of mass*: $\|\rho(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_0\|_{L^1(\mathbb{T}^d)}$ for every $t \in [0,T]$.
- *Local H^1-regularity*: $\left((\rho \wedge K) \vee \frac{1}{K}\right) \in L^2([0,T];H^1(\mathbb{T}^d))$ for every $K \in \mathbb{N}$.
- *Regularity and vanishing of the measure at infinity*: we have that

$$\delta_\rho\left(\alpha\xi^{\alpha-1}|\nabla\rho|^2\right) \leq q \ \text{ and } \ \liminf_{M\to\infty} q(\mathbb{T}^d \times [M, M+1] \times [0,T]) = 0.$$

- *The equation*: for every $\psi \in \mathrm{C}_c^\infty(\mathbb{T}^d \times (0,\infty))$ and $t \in [0,T]$,

$$\int_{\mathbb{T}^d} \chi\psi\Big|_{s=0}^{s=t} = -\int_0^t\int_{\mathbb{T}^d} \alpha\rho^{\alpha-1}\nabla\rho\cdot(\nabla\psi)(x,\rho) - \int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{R}}(\partial_\xi\psi)(x,\xi)q$$
$$+\int_0^t\int_{\mathbb{T}^d}(\partial_\xi\psi)(x,\rho)\rho^{\frac{\alpha}{2}}g\cdot\nabla\rho + \int_0^T\int_{\mathbb{T}^d}\rho^{\frac{\alpha}{2}}g\cdot(\nabla\psi)(x,\rho).$$

- we have that $\lim_{\beta\to 0}\left(\beta^{-1}q\left(\mathbb{T}^d \times (\frac{\beta}{2},\beta) \times [0,T]\right)\right) = 0.$

A useful identity: if ρ_1 and ρ_2 are kinetic solutions, for

$$\chi_i(x,\xi,t) = \mathbf{1}_{\{0<\xi<\rho_i(x,t)\}} - \mathbf{1}_{\{\rho_i(x,t)<\xi<0\}},$$

we have

$$\begin{aligned}\int_{\mathbb{T}^d} |\rho_1-\rho_2| = \int_{\mathbb{T}^d}\int_{\mathbb{R}} |\chi_1-\chi_2|^2 &= \int_{\mathbb{T}^d}\int_{\mathbb{R}} (\chi_1)^2+(\chi_2)^2-2\chi_1\chi_2 \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi_1\mathrm{sgn}(\xi)+\chi_2\mathrm{sgn}(\xi)-2\chi_1\chi_2 \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi_1+\chi_2-2\chi_1\chi_2.\end{aligned}$$

The cutoff functions: the cutoff at zero, for $\beta\in(0,1)$,

$$\psi_\beta(0)=0 \ \text{ and } \ \psi_\beta' = \frac{2}{\beta}\mathbf{1}_{\{\frac{\beta}{2}<\xi<\beta\}},$$

and the cutoff at infinity, for $M\in(1,\infty)$,

$$\zeta_M(0)=1 \ \text{ and } \ \zeta_M' = -\mathbf{1}_{\{M<\xi<M+1\}}.$$

The essential identity: we will use that

$$\int_{\mathbb{T}^d}|\rho_1-\rho_2| = \lim_{\beta\to 0}\lim_{M\to\infty}\left(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\left(\chi_1+\chi_2-2\chi_1\chi_2\right)\psi_\beta\zeta_M\right).$$

The equation: we have that

$$\partial_t \chi = \alpha \xi^{\alpha-1} \Delta \chi + \partial_\xi q - \partial_\xi (\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_\xi \chi) g),$$

and we use that

$$\int_{\mathbb{T}^d} |\rho_1 - \rho_2| = \lim_{\beta \to 0} \lim_{M \to \infty} \Big(\int_{\mathbb{T}^d} \int_{\mathbb{R}} \big(\chi_1 + \chi_2 - 2\chi_1 \chi_2\big) \psi_\beta \zeta_M \Big).$$

The singletons: we have that

$$\begin{aligned}
\partial_t \Big(\int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi_i \psi_\beta \zeta_M \Big) &= -\int_{\mathbb{T}^d} \int_{\mathbb{R}} q_i \partial_\xi (\psi_\beta \zeta_M) + \int_{\mathbb{T}^d} \big(\partial_\xi (\psi_\beta \zeta_M)\big)(\rho_i) \rho_i^{\frac{\alpha}{2}} g \cdot \nabla \rho_i \\
&= -\frac{2}{\beta} q_i \big(\mathbb{T}^d \times (\frac{\beta}{2}, \beta) \times (0,t)\big) + q_i \big(\mathbb{T}^d \times (M, M+1) \times (0,t)\big) \\
&\quad + \frac{2}{\beta} \int_{\mathbb{T}^d} \mathbf{1}_{\{\frac{\beta}{2} < \rho_i < \beta\}} \zeta_M(\rho_i) \rho_i^{\frac{1}{2}} g \cdot \rho_i^{\frac{\alpha-1}{2}} \nabla \rho_i + \int_{\mathbb{T}^d} \mathbf{1}_{\{M < \rho_i < M+1\}} \psi_\beta(\rho_i) \rho_i^{\frac{1}{2}} g \cdot \rho_i^{\frac{\alpha-1}{2}} \nabla \rho_i \\
&\lesssim \int_0^t \int_{\mathbb{T}^d} \int_M^{M+1} q + \frac{1}{\beta} \int_0^t \int_{\mathbb{T}^d} \rho |g|^2 \mathbf{1}_{\{\frac{\beta}{2} < \rho < \beta\}} + \int_0^t \int_{\mathbb{T}^d} \rho |g|^2 \mathbf{1}_{\{M < \rho < M+1\}}.
\end{aligned}$$

These terms vanish in the limit $M \to \infty$ and $\beta \to 0$.

The mixed term: we have that

$$\partial_t \chi = \alpha \xi^{\alpha-1} \Delta \chi + \partial_\xi q - \partial_\xi (\xi^{\frac{\alpha}{2}} g \cdot \nabla \chi) + \nabla \cdot (\xi^{\frac{\alpha}{2}} (\partial_\xi \chi) g),$$

and, therefore,

$$\begin{aligned}
&\partial_t \Big(\int_{\mathbb{T}^d} \int_{\mathbb{R}} \chi_1 \chi_2 \psi_\beta \zeta_M \Big) \\
&= - \int_{\mathbb{T}^d} \int_{\mathbb{R}} q_1 (\partial_\xi \chi_2) \psi_\beta \zeta_M - \int_{\mathbb{T}^d} \int_{\mathbb{R}} q_2 (\partial_\xi \chi_1) \psi_\beta \zeta_M - 2 \int_{\mathbb{T}^d} \int_{\mathbb{R}} \alpha \xi^{\alpha-1} \nabla \chi_1 \cdot \nabla \chi_2 \\
&\quad + \int_{\mathbb{T}^d} \int_{\mathbb{R}} \xi^{\frac{\alpha}{2}} \partial_\xi \chi_2 g \cdot \nabla \chi_1 \psi_\beta \zeta_M + \int_{\mathbb{T}^d} \int_{\mathbb{R}} \xi^{\frac{\alpha}{2}} \partial_\xi \chi_1 g \cdot \nabla \chi_2 \psi_\beta \zeta_M \\
&\quad - \int_{\mathbb{T}^d} \int_{\mathbb{R}} \xi^{\frac{\alpha}{2}} \partial_\xi \chi_1 g \cdot \nabla \chi_2 \psi_\beta \zeta_M - \int_{\mathbb{T}^d} \int_{\mathbb{R}} \xi^{\frac{\alpha}{2}} \partial_\xi \chi_2 g \cdot \nabla \chi_1 \psi_\beta \zeta_M \\
&\quad + \int_{\mathbb{T}^d} \int_{\mathbb{R}} \partial_\xi (\psi_\beta \zeta_M) \chi_2 \xi^{\frac{\alpha}{2}} g \cdot \nabla \chi_1 + \int_{\mathbb{T}^d} \int_{\mathbb{R}} \partial_\xi (\psi_\beta \zeta_M) \chi_1 \xi^{\frac{\alpha}{2}} g \cdot \nabla \chi_2 \\
&\quad - \int_{\mathbb{T}^d} \int_{\mathbb{R}} q_1 \chi_2 \partial_\xi (\psi_\beta \zeta_M) - \int_{\mathbb{T}^d} \int_{\mathbb{R}} q_2 \chi_1 \partial_\xi (\psi_\beta \zeta_M).
\end{aligned}$$

In comparison to the skeleton equation

$$\partial_t \rho = \Delta \rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g) + \text{"cutoff error"}.$$

The dissipative error: for $\partial_t \rho = \Delta\rho^\alpha - \nabla\cdot(\rho^{\frac{\alpha}{2}} g) +$ "cutoff error",

$$\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}} \chi_1\chi_2\psi_\beta\zeta_M\Big)$$

$$= -\int_{\mathbb{T}^d}\int_{\mathbb{R}} q_1(\partial_\xi\chi_2)\psi_\beta\zeta_M - \int_{\mathbb{T}^d}\int_{\mathbb{R}} q_2(\partial_\xi\chi_1)\psi_\beta\zeta_M - 2\int_{\mathbb{T}^d}\int_{\mathbb{R}} \alpha\xi^{\alpha-1}\nabla\chi_1\cdot\nabla\chi_2\psi_\beta\zeta_M\ldots$$

$$= \int_{\mathbb{T}^d}\int_{\mathbb{R}} \delta_{\rho_2}q_1\psi_\beta\zeta_M + \int_{\mathbb{T}^d}\int_{\mathbb{R}} \delta_{\rho_1}q_2\psi_\beta\zeta_M - 2\int_{\mathbb{T}^d}\int_{\mathbb{R}} \alpha\xi^{\alpha-1}\nabla\rho^1\cdot\nabla\rho^2\delta_{\rho_1}\delta_{\rho_2}\psi_\beta\zeta_M\ldots$$

$$\geq \int_{\mathbb{T}^d}\int_{\mathbb{R}} \delta_{\rho_1}\delta_{\rho_2}\alpha\xi^{\alpha-1}\big(|\nabla\rho_1|^2+|\nabla\rho_2|^2\big)\psi_\beta\zeta_M - 2\int_{\mathbb{T}^d}\int_{\mathbb{R}} \alpha\xi^{\alpha-1}\nabla\rho^1\cdot\nabla\rho^2\delta_{\rho_1}\delta_{\rho_2}\psi_\beta\zeta_M\ldots$$

Local regularity: after regularizing $\chi_i^\delta = (\chi * \kappa^\delta)$, for $\overline{\kappa}_i^\delta = \kappa^\delta(\rho_i - \xi)$,

$$\nabla\chi_i^\delta(x,\xi,t) = (\nabla\chi\kappa^\delta)(x,\xi,t) = (\delta_{\rho_i}\nabla\rho * \kappa^\delta)(x,\xi,t) = \nabla\rho(x,t)\kappa^\delta(\rho_i-\xi),$$

and

$$\begin{aligned}2\int_{\mathbb{T}^d}\int_{\mathbb{R}} \alpha\xi^{\alpha-1}\nabla\chi_1^\delta\cdot\nabla\chi_2^\delta\psi_\beta\zeta_M &= \int_{\mathbb{T}^d}\int_{\mathbb{R}}(\alpha\rho_1^{\alpha-1}+\alpha\rho_2^{\alpha-1})\nabla\rho_1\cdot\nabla\rho_2\delta_{\rho_1}^\delta\delta_{\rho_2}^\delta\psi_\beta\zeta_M \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{R}}\alpha\Big(\rho_1^{\frac{\alpha-1}{2}} - \rho_2^{\frac{\alpha-1}{2}}\Big)^2\nabla\rho_1\cdot\nabla\rho_2\overline{\kappa}_1^\delta\overline{\kappa}_2^\delta\psi_\beta\zeta_M \\ &\quad + 2\int_{\mathbb{T}^d}\int_{\mathbb{R}}\alpha\rho_1^{\frac{\alpha-1}{2}}\rho_2^{\frac{\alpha-1}{2}}\nabla\rho_1\cdot\nabla\rho_2\overline{\kappa}_1^\delta\overline{\kappa}_2^\delta\psi_\beta\zeta_M.\end{aligned}$$

The conservative error: for $\partial_t\rho = \Delta\rho^\alpha - \nabla\cdot(\rho^{\frac{\alpha}{2}}g) +$ "cutoff error",

$$\begin{aligned}\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_1\chi_2\psi_\beta\zeta_M\Big) &\geq \dots \\ &+\int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}\partial_\xi\chi_2 g\cdot\nabla\chi_1\psi_\beta\zeta_M + \int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}\partial_\xi\chi_1 g\cdot\nabla\chi_2\psi_\beta\zeta_M \\ &-\int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}\partial_\xi\chi_1 g\cdot\nabla\chi_2\psi_\beta\zeta_M - \int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}\partial_\xi\chi_2 g\cdot\nabla\chi_1\psi_\beta\zeta_M + \dots\end{aligned}$$

Local regularity of $\xi^{\frac{\alpha}{2}}$: after regularizing $\chi_i^\delta = (\chi * \kappa^\delta)$, for $\overline{\kappa}_i^\delta = \kappa^\delta(\rho_i - \xi)$,

$$\partial_\xi\chi_i^\delta(x,\xi,t) = (\partial_\xi\chi * \kappa^\delta)(x,\xi,t) = \kappa^\delta(\xi) - \kappa^\delta(\rho_i - \xi),$$

and

$$\begin{aligned}&\int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}(\partial_\xi\chi_2^\delta)g\cdot\nabla\chi_1^\delta\psi_\beta\zeta_M - \int_{\mathbb{T}^d}\int_{\mathbb{R}}\xi^{\frac{\alpha}{2}}(\partial_\xi\chi_2^\delta)g\cdot\nabla\chi_1^\delta\psi_\beta\zeta_M \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{R}}\big(\rho_2^{\frac{\alpha}{2}} - \rho_1^{\frac{\alpha}{2}}\big)g\cdot\nabla\rho_1\overline{\kappa}_1^\delta\overline{\kappa}_2^\delta\psi_\beta\zeta_M \\ &\simeq \delta^{-1}\int_{\mathbb{T}^d}\int_{\mathbb{R}}\mathbf{1}_{\{|\rho_1-\rho_2|<\delta\}}\big(\rho_2^{\frac{\alpha}{2}} - \rho_1^{\frac{\alpha}{2}}\big)\psi_\beta(\rho_1)\zeta_M(\rho_1).\end{aligned}$$

The cutoff error: for $\partial_t \rho = \Delta\rho^\alpha - \nabla\cdot(\rho^{\frac{\alpha}{2}} g) + \text{"cutoff error"}$,

$$\begin{aligned}\partial_t\Big(\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_1\chi_2\psi_\beta\zeta_M\Big) &\geq \dots \\ &+\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\xi(\psi_\beta\zeta_M)\chi_2\xi^{\frac{\alpha}{2}}g\cdot\nabla\chi_1+\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\xi(\psi_\beta\zeta_M)\chi_1\xi^{\frac{\alpha}{2}}g\cdot\nabla\chi_2 \\ &-\int_{\mathbb{T}^d}\int_{\mathbb{R}}q_1\chi_2\partial_\xi(\psi_\beta\zeta_M)-\int_{\mathbb{T}^d}\int_{\mathbb{R}}q_2\chi_1\partial_\xi(\psi_\beta\zeta_M).\end{aligned}$$

We have that

$$\begin{aligned}&\int_{\mathbb{T}^d}\int_{\mathbb{R}}\partial_\xi(\psi_\beta\zeta_M)\chi_2\xi^{\frac{\alpha}{2}}g\cdot\nabla\chi_1=\int_{\mathbb{T}^d}\rho_1^{\frac{\alpha}{2}}g\cdot\nabla\rho_1\chi_2(x,\rho_1,t)\partial_\xi(\psi_\beta\zeta_M)(\rho_1) \\ &\lesssim\int_{\mathbb{T}^d}\rho_1^{\frac{1}{2}}g\cdot\rho_1^{\frac{\alpha-1}{2}}\nabla\rho_1\Big(\frac{2}{\beta}(\mathbf{1}_{\{\frac{\beta}{2}<\rho_1<\beta\}}+\mathbf{1}_{\{M<\rho_1<M+1\}}\Big) \\ &\lesssim\int_{\mathbb{T}^d}\rho_1|g|^2\Big(\frac{2}{\beta}(\mathbf{1}_{\{\frac{\beta}{2}<\rho_1<\beta\}}+\mathbf{1}_{\{M<\rho_1<M+1\}}\Big)+\int_{\mathbb{T}^d}\int_{\frac{\beta}{2}}^{\beta}\frac{1}{\xi}q_1+\int_{\mathbb{T}^d}\int_M^{M+1}q_1.\end{aligned}$$

Conclusion: we have that $\partial_t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\chi_1\chi_2\geq 0$ and, therefore,

$$\partial_t\int_{\mathbb{T}^d}|\rho_1-\rho_2|=\partial_t\int_{\mathbb{T}^d}\int_{\mathbb{R}}\big(\chi_1+\chi_2-2\chi_1\chi_2\big)\leq 0.$$

II. The kinetic formulation of the skeleton equation

Well-posedness of renormalized kinetic solutions [F., Gess; 2023]

Let $T \in (0,\infty)$, $d \in \mathbb{N}$, and let $\Phi \in \mathrm{C}^1_{\mathrm{loc}}((0,\infty)) \cap \mathrm{C}([0,\infty))$ satisfy that

— $\Phi(0) = 0$ with $\Phi' > 0$ on $(0,\infty)$,

— Φ' is locally $1/2$-Hölder continuous on $(0,\infty)$,

— and $\max_{\{0<\xi\leq M\}} \frac{\Phi(\xi)}{\Phi'(\xi)} \leq cM$.

Then for every nonnegative $\rho_0 \in L^1(\mathbb{T}^d)$ there exists a unique renormalized kinetic solution of the equation

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\Phi^{\frac{1}{2}}(\rho) g) \ \text{ in } \ \mathbb{T}^d \times (0,T) \ \text{ with } \ \rho(\cdot,0) = \rho_0.$$

Furthermore, if ρ_1 and ρ_2 are two solutions with initial data $\rho_{1,0}$ and $\rho_{2,0}$, then

$$\max_{t\in[0,T]} \|\rho_1(x,t) - \rho_2(x,t)\|_{L^1(\mathbb{T}^d)} = \|\rho_{1,0} - \rho_{2,0}\|_{L^1(\mathbb{T}^d)}.$$

— including $\Phi(\xi) = \xi^\alpha$ for every $\alpha \in (0,\infty)$, for which

$$\partial_t \rho = \Delta\rho^\alpha - \nabla \cdot (\rho^{\frac{\alpha}{2}} g).$$

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