Questions 1 - 21 are worth 1 point each and questions 22 - 28 are worth 2 points each.

No calculators are allowed.

Pictures are only sketches and are not necessarily drawn to scale or proportion.

You have one hour and twenty minutes to complete the entire morning exam.

Questions 1 - 21 Multiple Choice

Please:
- *Use the answer sheet for your answers.*
- *Answer only one choice A, B, C, D, or E for each question by writing your answer on the answer sheet.*
- *Completely erase any answer you wish to change.*
- *Do not make stray marks on the answer sheet.*

1. The expression
$$\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\right)^{-1}$$
simplifies to

A $\frac{12}{7}$ B 7 C 12 D $\frac{7}{3}$ E $\frac{1}{3}$

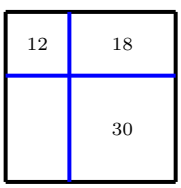
2. A floor measures 9 feet by 12 feet. How many 9 inch by 9 inch tiles are needed to cover the floor? (Reminder: 12 inches is 1 foot).

A 96 B 144 C 192 D 256 E 81

3. A is proportional to B and inversely proportional to the square of C. Suppose $A = 8$ when $B = 16$ and $C = 2$. What is A when $B = 6$ and $C = 3$?

A 2 B $\frac{2}{3}$ C 1 D $\frac{4}{3}$ E 6

4. A horizontal and vertical line divide a large rectangle into four smaller rectangles with areas as indicated in the illustration below. Find the area of the remaining rectangle.



A 6 B 15 C 18 D 20 E none of these

5. If the repeated decimal $0.\overline{216}$ is expressed as a reduced fraction what is the denominator?

A 13 B 37 C 99 D 111 E 999

6. How many integers, x, are there so that
$$\left(x-\frac{1}{2}\right)\left(x-\frac{2}{3}\right)\left(x-\frac{3}{4}\right)\cdots\left(x-\frac{2018}{2019}\right)<0?$$

A 0 B 1 C 2018 D 2019 E none of these

7. Find a positive integer M so that the equation
$$(x-12)(x+M)=M-51$$
has exactly one solution.

A 1 B 2 C $\frac{5}{2}$ D 4 E 30

8. What is the unit digit for the number $9^{20}-6^{18}$?

A 2 B 3 C 5 D 7 E 8

9. Two sides of a triangle have lengths 6 and 13. What is the smallest possible length of the third side if it is an integer?

A 2 B 3 C 7 D 8 E 19

10. Suppose A and B are two points in a plane. The collection of points P such that $|AP|+|BP|=k$, where k is a constant greater that $|AB|$ is what planar figure?

A line B circle C ellipse D hyperbola E ray

11. What is the greatest possible area of a triangle if two of its sides have length 6 and 13?

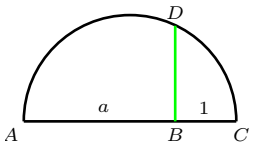
A 18 B 24 C 39 D 42 E 78

12. Suppose a and b are positive integers whose sum is divisible by 3. Which of the following statements are true?

I a^2+b^2 is divisible by 3
II a^2-b^2 is divisible by 3
III a^3+b^3 is divisible by 3

A I only B II only C III only D I and II only E II and III only

13. In the semicircle in the illustration below AC is perpendicular to BD, $|AB|=a$, and $|BC|=1$.



Find $|BD|$.

A a B $\sqrt{a}$ C $\sqrt{a}-1$ D $a-1$ E $\sqrt{a}+1$

14. Suppose x and y are positive numbers such that
$$\log_x y-\log_y x=3.$$
What is $(\log_x y)^2+(\log_y x)^2$?

A 5 B 7 C 9 D 11 E 13

15. There are 6^3 unit cubes that are put together to form a $6 \times 6 \times 6$ cube. It is then painted red on all sides. How many of the unit cubes have paint on them?

A 64 B 128 C 152 D 125 E 216

16. Suppose f is a real valued function defined on all the real numbers and

$$f(x) + 2f(2 - x) = x^2.$$

What is $f(3)$?

A $\frac{-7}{3}$ B $\frac{-5}{3}$ C 0 D $\frac{5}{3}$ E $\frac{7}{3}$

17. Suppose x_1 and x_2 are roots of the equation $x^2 - 96x + 2018 = 0$. What is $(x_1 - 1)(x_2 - 1)$?

A 98 B 94 C 2115 D 1923 E 2020

18. There are 30 students in Mr Moore's discrete mathematics class: 23 of his students major in mathematics, 12 major in physics, and 3 major in subjects outside of math and physics. How many double major in mathematics and physics?

A 2 B 3 C 5 D 7 E 8

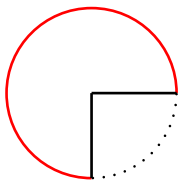
19. How many 0's appear at the end of the product of the first 2018 primes?

A 0 B 1 C 22 D 3 E 2018

20. The lower vertices of a square lie on the x-axis while the upper vertices lie on the parabola given by $y = 35 - x^2$. What is the area of the square?

A 25 B 49 C 98 D 100 E none of these

21. Begin with a paper disk of radius 1. Cut away a quarter of it as shown in the illustration below. You now have $3/4$ of the disk remaining. Now connect the two radii together to form a cone. What is its height?



A $\frac{\sqrt{7}}{4}$ B $\frac{1}{2}$ C $\frac{\sqrt{2}}{2}$ D $\frac{\sqrt{7}}{8}$ E $\frac{3}{4}$

Questions 22 - 28 Exact Answers

These next seven questions require exact numerical or algebraic answers. Hand-written exact answers must be written on the answer sheet with fractions reduced, radicals simplified, and denominators rationalized (Improper fractions can be left alone or changed to mixed fractions). Do not make an approximation for π or other irrational numbers. Answers must be exact. Large numbers should not be multiplied out, i.e., do not try to multiply out 20! or 6^{40}.

22. How far from a 30 ft lamppost should a 6 ft man stand in order to cast a 10 ft shadow? Specify your answer in feet.

23. Find an integer between 400 and 1000 that is divisible by 7 and 13 and the sum of its digits is a prime number.

24. Every day Jill leaves home to pick up her husband, Jack, at the train station. They arrive simultaneously at 5:00 pm and immediately drive home together. Jill always drives at a constant speed. However, one day, Jack catches an earlier train and arrives at 4:00 pm. He then starts to walk home. On the way he meets Jill, who is on her way to meet the 5:00 pm train. He gets into the car and they arrive home 20 minutes earlier than usual. How many minutes had Jack been walking?

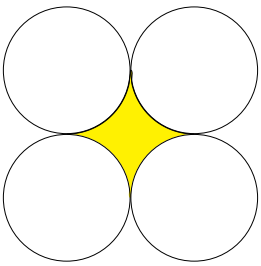
25. Suppose m and n are positive integers not divisible by 10 and such that $mn = 2000$. What is $m + n$?

26. What is the smallest positive integer p for which

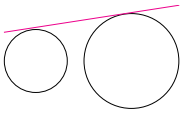
$$7^{n+p} = 7^n \mod 100,$$

for all positive n?

27. In the illustration below the centers of the circles are at the vertices of a square of side length 2. What is the area of the yellow shaded region?



28. Consider the circle with radius 2 and center $(0, 0)$ and the circle of radius 3 with center $(6, 0)$. There is a unique line that lies above the circles and tangent to them. Find the slope of this line. The following diagram may be helpful.



Tie Breaker requiring Full Solution

Please give a **detailed explanation** of your solution to **Question 28**. Write your explanation on the **reverse side** of your answer sheet. *This tie breaker question is graded as an essay question, i.e. it is graded for the clarity of explanation and argument as well as correctness.*

It is the only question graded for partial credit. Do not hesitate to write your thoughts even if your solution is not rigorous!

It is graded only to separate first, second, and third place ties.