Questions 1 - 13 are worth 1 point each and questions 14 - 24 are worth 2 points each.

No calculators are allowed.

Pictures are only sketches and are not necessarily drawn to scale or proportion.

The people supervising this test are not permitted to explain to you the meaning of any question.

You have one hour and twenty minutes to complete the entire morning exam.

Questions 1 - 13 Multiple Choice

Please:

- *Use the answer sheet for your answers.*
- *Answer only one choice A, B, C, D, or E for each question by writing your answer on the answer sheet.*
- *Completely erase any answer you wish to change.*
- *Do not make stray marks on the answer sheet.*

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1. How many 3 digit integers are there (between 100 and 999) in which no two adjacent digits are equal?

A 900 B 648 C 729 D 171 E none of these

2. Find a positive number k so that the parabola and line given by

$$y = x^2 - 5x + 10 \qquad \text{and} \qquad y = k(x-2),$$

respectively, intersect at exactly one point.

A 3 B 5 C 0 D 2 E 4

3. The perimeter of a right triangle is $7 + \sqrt{29}$. The sum of the squares of all three sides is 58. Find the area of the triangle.

A 2 B 5 C $3 + 2\sqrt{29}$ D 4 E 10

4. Seven different playing cards, with values from ace to seven, are shuffled and placed in a row on a table to form a seven-digit number. What is the probability that this seven-digit number is divisible by 11?

A 4/35 B 1/7 C 8/35 D 2/7 E 12/35

5. If n is a positive integer, write $s(n)$ for the sum of the digits of n. What is $s(1) + s(2) + s(3) + \cdots + s(1000)$?

A 9991 B 10000 C 13501 D 14999 E 15000

6. Find $\arctan(1/3) + \arctan(1/5) + \arctan(1/7) + \arctan(1/8)$.

A $\pi/6$ B $\pi/5$ C $\pi/4$ D $\pi/3$ E $\pi/2$

7. Compute the minimum value of the expression $\sin^{1000}\alpha + \cos^{1000}\alpha$ for a real number α.

A 2^{-999} B 2^{-1000} C 2^{-1001} D 2^{-499} E 2^{-500}

8. Consider an infinite supply of cardboard equilateral triangles and squares, all of them with side length of one inch. What would be the convex polygon (without holes) with the largest number of sides that you could construct with these pieces, if the pieces are not allowed to overlap?

A hexagon B octagon C 10-gon D 12-gon E 16-gon

9. Let R be the circle $x^2 + (y+2)^2 = 9$. Let S be the set of all circles in the plane such that for each circle C in S, we have:

(a) C lies in the first quadrant outside R.

(b) C is tangent to R and to the x-axis.

On what geometric object must the center of the circles in S lie?

A an ellipse that is not a circle B a circle C a straight line D a hyperbola E a parabola

10. Which of the following numbers is the largest?

A $\sin 10^\circ$ B $\cos 10^\circ$ C $\tan 10^\circ$ D $\dfrac{1}{\sin 10^\circ}$ E $\dfrac{1}{\cos 10^\circ}$

11. How many polynomials $p(x)$ satisfy $p(10) = 10!$ and

$$xp(x-1) = (x-10)p(x),$$

for all $x \in \mathbb{R}$?

A 0 B 1 C 10 D infinitely many E none of these

12. Let a, b, and c be the three roots of $x^3 - 48x + 7$. What is the value of $a^3 + b^3 + c^3$?

A -7 B -21 C 21 D 48 E -48

13. Let S be the set of all positive integers whose prime divisors can only be 2 or 3. Thus $1, 2, 3, 4, 6, 8, 9, 12$, are the smallest of the elements in S. What is the sum of the reciprocals of the elements of S?

A 1 B 2 C 3 D 4 E 6

Questions 14 - 24 Exact Answers

These next ten questions require exact numerical or algebraic answers. Hand written exact answers must be written on the answer sheet with fractions reduced, radicals simplified, and denominators rationalized (Improper fractions can be left alone or changed to mixed fractions). Do not make an approximation for π or other irrational numbers. Answers must be exact. Large numbers should not be multiplied out, i.e., do not try to multiply out $20!$ or 6^{40}.

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14. Suppose $0 \le \theta \le 90^\circ$ and $\tan 2\theta = 4/3$. Compute $\cos\theta$.

15. Three distinct prime numbers are randomly selected from the first ten prime numbers. What is the probability that their sum is even.

16. If a and b are positive real numbers satisfying $(a-b)^2 = 4(ab)^3$, what is the smallest possible value of $\frac{1}{a} + \frac{1}{b}$?

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17. Find

$$\sin^2(1^\circ) + \sin^2(2^\circ) + \cdots + \sin^2(89^\circ) + \sin^2(90^\circ).$$

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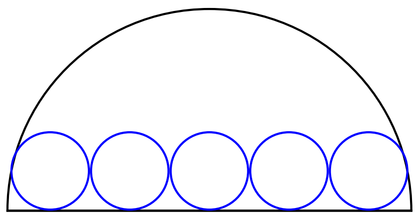
18. Five small circles, each of radius 1, lie inside a larger semicircle. The centers of each small circle lie on a line parallel to the diameter of the semicircle. Adjacent small circles are tangent, the leftmost and rightmost small circles are tangent to the semicirle, and all are tangent to the diameter of the semicircle as in the diagram below. What is the radius of the semicircle?



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19. Suppose $mn = 9009$ where both m and n are positive two digit integers. What is $m + n$?

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20. Let N be a 3-digit positive number with distinct, nonzero digits, and let S be the sum of its digits. What is the smallest possible value of N/S?

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21. A deck of cards consists of four suits with 13 distinct cards in each suit. A "hand" is a subset of these cards consisting of exactly 6 cards. Let N be the number of hands containing at least one card of each suit. What is the largest prime factor of N ?

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22. Find the largest possible integer n such that $1+2+3+\cdots+n \leq 2017$.

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23. An urn contains balls numbered from 1 to 11. You select 3 balls at random. What is the probabilty that the sum of the numbers on the chosen balls is divisible by 3?

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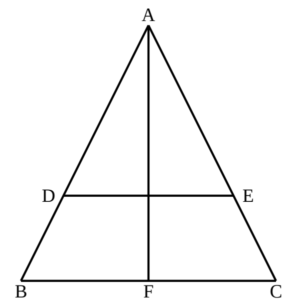
24. Suppose that $\triangle ABC$ is isosceles with $AB = AC$. Let D be a point on the side $\overline{AB}$, and let E be a point on the side $\overline{AC}$ for with the following are true:

    (a) $\overline{DE}$ is parallel to $\overline{BC}$.

    (b) $\triangle ADE$ and trapezoid $BCED$ have the same area and the same perimeter.

    (c) F is the midpoint of $\overline{BC}$.

    Find $\sin(\angle BAF)$.



Tie Breaker requiring Full Solution

Please give a **detailed explanation** of your solution to **Question 24**. Write your explanation on the **reverse side** of your answer sheet.

This tie breaker question is graded as an essay question, i.e. it is graded for the clarity of explanation and argument as well as correctness.

It is the only question graded for partial credit. Do not hesitate to write your thoughts even if your solution is not rigorous!

It is graded only to separate first, second, and third place ties.