- No calculators are allowed.
- Pictures are only sketches and are not necessarily drawn to scale or proportion.
- You have one hour and fifteen minutes to complete the entire team session.

These 10 problems require exact numerical or algebraic answers. Exact answers must be written with fractions reduced, radicals simplified, and denominators rationalized. Do not make an approximation for π or other irrational numbers.

The tiebreaker for the team competition is time. *If your team reaches a point where you are satisfied or expect that you will not have more solutions in the allotted time, then you may wish to turn in your paper a little early to get a time advantage.*

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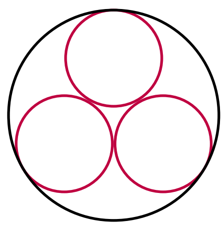
1. Three circles of radius r can be inscribed in a circle of radius 1 so that they are tangent to each other and tangent to the circle of radius 1. (See the diagram.) It turns out that r can be written in the form $r = a\sqrt{3}+b$, where a and b are integers. Find $a^2 + b^2$.



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2. Let N be the integer

$$N = 123456789101112\ldots2016.$$

In other words, N is the number obtained by adjoining $1, 2, 3, \ldots, 2016$ together and in order. What is the remainder when N is divided by 75?

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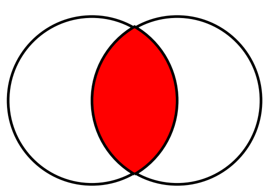
3. In the figure given the two circles have radius r. Each circle goes through the center of the other. Find the area of the shaded region, that is, the intersection of the two circles.



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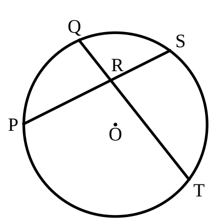
4. Two lines intersect at R within a circle with center O. The points of intersection on the circle are P, Q, S, and T as illustrated. Suppose $\angle SOT = \theta$ and $\angle QOP = \phi$. Find $\angle SRT$ in terms of θ and ϕ.



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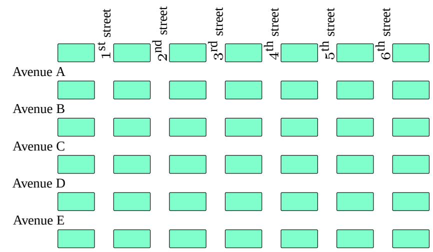
5. The street layout in a small Louisiana town is as in the diagram below. There is a coffee shop at the intersection of Avenue E and each numbered street. Joe is staying in a small B&B at the corner of Avenue A and 1st street. When he wakes up the urge to get coffee is strong. If he only travels through town South or East in how many ways can he get to a coffee shop?



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6. Three positive integers, a, b, c, have product 1386. Further,

$$\frac{a}{3} = b + 1 = c - 4.$$

Find $a + b + c$.

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7. Suppose n and m are positive integers. Mike and Mary each have a bag of $N = nm$ marbles labelled 1 to N. Mike randomly takes out a marble from his bag. Suppose a is the sum of the numbers on the remaining marbles in his bag. Similarly, Mary randomly takes out a marble from her bag. Suppose b is the sum of the numbers on the remaining marbles in her bag. Find the probability that $a - b$ is a multiple of n. Express your answer as a reduced rational function in the variables n and m.

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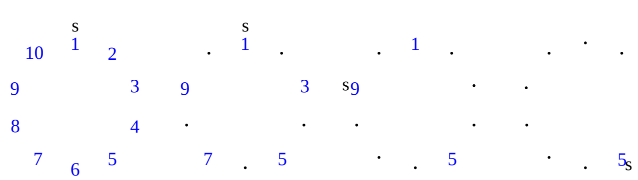
8. A series of n light poles are arranged in a circle and numbered, clockwise, from 1 to n. All the lights are initially on. You start at light pole #1 and begin walking clockwise around the circle of lights turning off every second **lighted** pole doing so until there is only one pole that remains lighted. (To be clear, the first light turned off is #2.) Let $f(n)$ be the number of the last remaining lighted pole. For example, below we illustrate that $f(10) = 5$. In each circle, the 's' represents your standing position and it is the next light you turn off. In the first circle all lights are on and you are standing at light #1. In your first walk around you turn off the even numbered lights. In your second walk around you turn off light #3 and #7. You are now standing at light #9. Continuing, you next turn off #1 and #9. The last light on is #5, so $f(10) = 5$.



Find $f(2016)$.

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9. A $5 \times 5 \times 5$ woooden cube is painted on all 6 faces and then cut up into $unit$ cubes. One unit is randomly selected and rolled. What is the probability that **exactly one** of the five visible faces is painted?

10. At a picnic there are c children, m mothers, and f fathers, with $2 \leq f < m < c$. Every person shakes hand with every other person. The sum of the number of handshakes amongst the children, amongst the mothers, and amongst the fathers is 80. How many persons attended the picnic?