Questions 1 - 13 are worth 1 point each and questions 14 - 24 are worth 2 points each.

No calculators are allowed.

Pictures are only sketches and are not necessarily drawn to scale or proportion.

The people supervising this test are not permitted to explain to you the meaning of any question.

You have one hour and twenty minutes to complete the entire morning exam.

### Questions 1 - 13  Multiple Choice

Please:
- *Use the answer sheet for your answers.*
- *Answer only one choice A, B, C, D, or E for each question by circling your answer on the answer sheet.*
- *Completely erase any answer you wish to change.*
- *Do not make stray marks on the answer sheet.*

1. Find the smallest real solution $x$ to the equation

$$\ln 12 - \ln(x - 3) = \ln(x + 1).$$

A $-5$     B $-3$     C 3     D 5     E none of these

2. Find the sum of all real solutions to the equation

$$(x^3 + x^2 + 3x + 6)^2 - (3x^2 + x + 10)^2 = 0.$$

A $-2$     B 0     C 2     D $-4$     E 4

3. One of the roots to the polynomial

$$x^4 - x^3 - 2x^2 - 2x + 4$$

is $-1 + i$, where $i = \sqrt{-1}$. Find the sum of all the real roots.

A 1     B $-1$     C 3     D $-3$     E 0

4. When a positive integer $n$ is divided by 2, 3, 4, 5, and 6 the remainders are always 1. However, $n$ is divisible by 7. What is the least $n$ with this property?

A 7     B 61     C 77     D 301     E 721

5. The integers

$$-4, -3, -2, -1, 0, 1, 2, 3, 4$$

can be inserted into a $3 \times 3$ square to form a 'magic square', where the sum of the rows, columns and diagonals all add up to the same number. If the numbers $-4$, $-2$, and 4 are as shown in a magic square what is $x$?

|     | $-4$ |     |
| --- | ---  | --- |
| $-2$ |     | 2  |
| $x$ |     |     |

A $-3$     B 0     C $-1$     D 3     E 4

6. A square of side $x$ is removed from the interior of a square of side $X$ resulting in an $a\%$ loss of area to the square of side $X$. Find a formula for $x$ in terms of $a$ and $X$.

A $x = \frac{\sqrt{100-a}}{10} X$     B $x = \frac{\sqrt{a}}{10} X$     C $x = \frac{100-a}{100} X$

D $x = aX$     E none of these

7. What integer is the following expression:

$$\frac{3}{\log_9 36} + \frac{8}{\log_{\sqrt{3}} 36} + \frac{15}{\log_{\sqrt[3]{4}} 36}.$$

A 1     B 2     C 3     D 4     E 5

8. $\sin^{-1} \frac{1}{3} + \cos^{-1} \frac{1}{3}$ simplifies to

A 0     B $\frac{\pi}{4}$     C $\frac{\pi}{3}$     D $\frac{\pi}{2}$     E $\pi$

9. Simplify $(2 + i)^5 + (2 - i)^5$, where $i^2 = -1$

A $-76$     B $-62i$     C $76 - 62i$     D $62i$     E 76

10. An operation on a row of seven circle lights (either purple or yellow) changes the color of two of the lights (from yellow to purple or purple to yellow). Suppose you start with the following light pattern:



Which of the following patterns cannot be obtained by repeated applications of this operation?
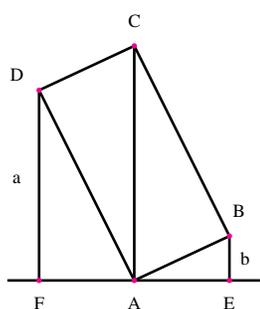
A 

B 

C 

D 

E 

11. The sum

$$\frac{1}{3 \cdot 6} + \frac{1}{6 \cdot 9} + \frac{1}{9 \cdot 12} + \cdots + \frac{1}{2013 \cdot 2016}$$

simplifies to:

A 1     B $\frac{1}{3}$     C $\frac{2013}{2016}$     D $\frac{671}{2016}$     E none of these

12. In the figure $ABCD$ is a rectangle. The points $A$, $F$, and $E$ are collinear. The line segments $DF$, $BE$, and $CA$ are all perpendicular to $FE$. Suppose the length of $DF$ is $a$ and the length of $BE$ is $b$. Find the area of the rectangle $ABCD$.



A $a + b$ B $\sqrt{ab}(a + b)$ C $a^2 + b^2$ D $2ab$ E $2\sqrt{a^2 + b^2}$

13. The number of integers between 1 and 2016, inclusive, that are divisible neither by 7 nor by 9 is

A 480 B 512 C 1504 D 1536 E 1984

### Questions 14 - 24 Exact Answers

*These next ten questions require exact numerical or algebraic answers. Hand written exact answers must be written on the answer sheet with fractions reduced, radicals simplified, and denominators rationalized (Improper fractions can be left alone or changed to mixed fractions). Do not make an approximation for $\pi$ or other irrational numbers. Answers must be exact. Large numbers should not be multiplied out, i.e., do not try to multiply out $20!$ or $6^{40}$.*

14. A chest of drawers contains exactly two red socks, two blue socks, two yellow socks, and a single black sock. If two socks are taken from the drawer at random, what is the probability that they match?

15. Two line segments, each parallel to the diameter of a semi-circle of radius 10, have endpoints on the semi-circle. The lengths of the line segments are 16 and 12. What is the distance between the line segments?
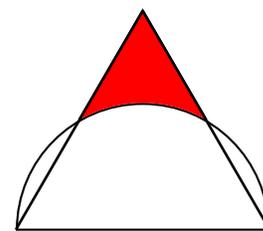
16. For how many primes $p$ is $p^2 + 4p - 1$ also a prime?

17. The product, $P$, of three consecutive non-zero positive integers is 56 times the sum of the three integers. What is the sum of the digits in $P$?

18. Solve the following equation for $x$:

$$\left(\frac{1}{x} - \frac{1}{4}\right)^3 = \frac{3}{16x} - \frac{1}{64}.$$
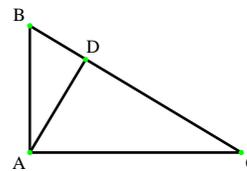
19. An equilateral triangle shares its base with a semi-circle that has a diameter of 12 units. What is the area of that portion of the triangle that is outside of the semicirle, ie. the area of the shaded region in the diagram provided?



20. Suppose $d$ and $n$ are positive integers. We say $d$ is a divisor of $n$ if $\frac{n}{d}$ is an integer. How many divisors of $n = 2^2 3^3 4^4 5^2 6^3 7^4 8^2$ are perfect cubes? (Write your answer in decimal form.)

21. Five couples (husband and wife) are to be seated in a row of 11 chairs. How many ways can they be seated if a husband must sit next to his wife? (Write your answer in decimal form.)

22. In triangle $\triangle ABC$, $BD = 3$, $DC = 10$, $\angle BAC = 90°$, and $\angle BDA = 90°$. Find $AD$.



23. Upon entering a tall building, you can either take the stairs or the elevator. If you take the stairs, it will take you 20 seconds to walk up the first flight of stairs, 21 seconds to walk up the second flight of stairs, 22 seconds to walk up the third flight of stairs, and so forth. If you take the elevator, you have to wait 3 minutes for the elevator to arrive and then it will take 4 seconds to move up each flight of stairs. Compute the minimum number of flights of stairs for which taking the elevator takes strictly less time than taking the stairs.

24. Suppose $N$ is the number of digits in $2^{2016}$, written in decimal form. How many digits does $N$ have?

### Tie Breaker requiring Full Solution

Please give a **detailed explanation** of your solution to **Question 24**. Write your explanation on the **reverse side** of your answer sheet.

*This tie breaker question is graded as an essay question, i.e. it is graded for the clarity of explanation and argument as well as correctness.*

It is the only question graded for partial credit. Do not hesitate to write your thoughts even if your solution is not rigorous!

**It is graded only to separate first, second, and third place ties.**