2012 LSU Math Contest
Open Session

Questions 1 - 14 are worth 1 point each and questions 15 - 25 are worth 2 points each.

No calculators are allowed.

Pictures are only sketches and are not necessarily drawn to scale or proportion.

You have one hour and twenty minutes to complete the entire morning exam.

Questions 1 - 14 Multiple Choice

Please:
- *Use the answer sheet for your answers.*
- *Answer only one choice A, B, C, D, or E for each question by circling your answer on the answer sheet.*
- *Completely erase any answer you wish to change.*
- *Do not make stray marks on the answer sheet.*

1

Let a be an irrational number. Which of the following is necessarily an irrational number?

A a^3 B $\sqrt{|a|}$ C $\sin a$ D $\ln|a|$ E None of these

2

The equation $\cos x = \tan x$ has

A No roots

B A positive root

C Two different roots

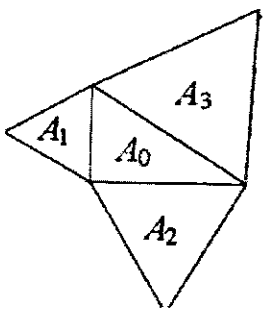
D A negative root

E None of the above

on the interval$(-\pi/2, \pi/2)$.

3

On the picture below



$A_0, \ldots, A_3$ denote the areas of the corresponding triangles . The triangle with area A_0 is a right triangle, all the other triangles are equilateral. Which of the following equations is true?

A $A_1 + A_2 = A_3$ B $(A_1)^2 + (A_2)^2 = (A_3)^2$

C $A_1 + A_2 + A_3 = 3A_0$

D $A_1 + A_2 = \sqrt{2}A_3$ E $A_1 + A_2 = \sqrt{3}A_3$

4

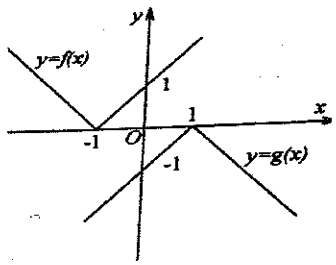
Eight spheres, each of radius 1/2, are centered at each of the vertices of a cube of edge length 1. What is the diameter of the smaller sphere of the two that are tangent to all eight spheres?

A $\sqrt{2}-1$ B $\sqrt{3}/4$ C 1/2 D $\sqrt{3}-1$

E such a sphere does not exist

5

The graphs of two functions $f, g : \mathbb{R} \to \mathbb{R}$ are pictured below.



Which of the following identities is true for all real x?

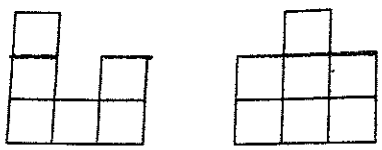
A $f(x) = -g(x) + 2$ B $f(x) = -g(x) - 2$

C $f(x) = -g(x+2)$ D $f(x+2) = -g(x)$

E $f(x+1) = -g(x-1)$

6

The picture shows respectively the views from the left and front of a construction made of unit cube blocks.

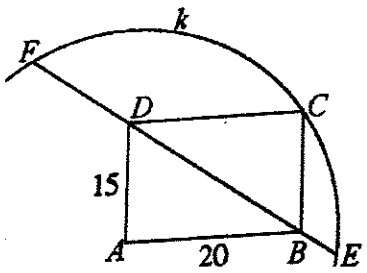


What is the minimum and the maximum number of blocks that were used if each block either lays on the ground or on top of another block?

A 7 & 13 B 7 & 15 C 7 & 16 D 8 & 13 E 8 & 16

7

Let $ABCD$ be a rectangle with side lengths 15 and 20 as shown. Let k be a circle centered at A passing through C.



Find the length of a chord EF.

A 4 B 25 C $2\sqrt{37 \cdot 13}$ D $2\sqrt{20 \cdot 25}$ E 50

8

If $\log_7 98$, k, $\log_7 0.5$ form an arithmetic sequence, then $k =$

A 1 B 2 C 3 D 4 E 5

9

$ABCD$ is a square. Three parallel lines ℓ_1, ℓ_2 and ℓ_3 pass through A, B and C respectively. The distance between ℓ_1 and ℓ_2 is 5 inches, and the distance between ℓ_2 and ℓ_3 is 7 inches. The area, in square inches, of $ABCD$ is

A 35 B 70 C 74 D 144 E none of these

10

Consider the statement:"If a circle is tangent to n sides (not their extensions) of regular polygon, then it is tangent to all sides." The smallest value of n for which this statement is true is

A 1 B 2 C 3 D 4 E 5

11

If $f(x) = 2^x$, then 4^8 is equal to

A $f(f(2))$ B $f(f(f(2)))$ C $f(f(f(f(2))))$

D $f(f(f(f(f(2)))))$ E none of these

12

Two statements are made about four points in space.

P: "All four points lie in a plane."

Q: "Three of the four points lie on a line."

Which of the following is true

A P and Q are equivalent.

B P implies Q but not vice versa.

C Q implies P but not vice versa.

D Neither P nor Q implies the other and both can be true.

E Neither P nor Q implies the other and both can be false.

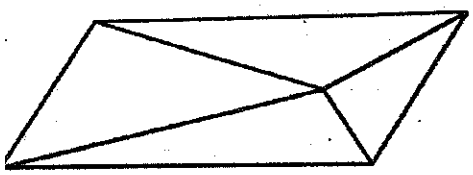
13

A two-person game is played in which each player, in turn, removes 1, 2, 3, 4, or 5 matches from a common pile, until the pile is exhausted. The player taking the LAST MATCH LOSES. If the starting pile contains 2012 matches, how many matches must the first player take on her first turn to guarantee a win with a perfect subsequent play?

A 1 B 2 C 3 D 4 E 5

14

A parallelogram is divided into four triangles with common vertex. The sides of the parallelogram are bases of these triangles - see the picture below.



Which of the following quadruplets of numbers can express the areas of these four triangles?

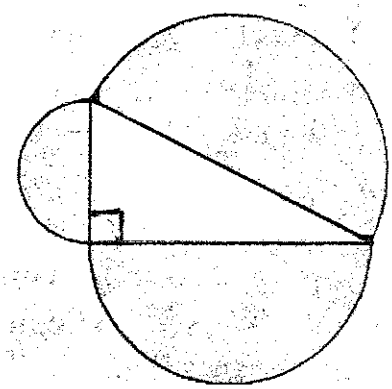
A 3, 5, 6, 7 B 4, 5, 8, 9 C 5, 6, 7, 12

D 5, 6, 8, 10 E 10, 11, 12, 19

Questions 15 - 25 Exact Answers

These next eleven questions require exact numerical or algebraic answers. Hand written exact answers must be written on the answer sheet with fractions reduced, radicals simplified, and denominators rationalized (Improper fractions can be left alone or changed to mixed fractions). Do not make an approximation for π or other irrational numbers. Answers must be exact. Large numbers should not be multiplied out, i.e., do not try to multiply out 20! *or* 6^{40}.

15 Three semicircles are attached to three sides of a right triangle, as shown.



Knowing that the sum of the areas of all these semicircles is ten times the area of this triangle, find the sum of the tangents of the acute angles of the triangle.

16 Find the largest positive integer that always divides (exactly) the expression $n^2(n^2-1)(n^2-4)$ for any integer n.

17

$$f(x) = x^2 + \sqrt{x^4+1} + \frac{1}{x^2 - \sqrt{x^4+1}}.$$

Find $f\left(2011^{2012}\right)$.

18 Find all the solutions to the equation $x^{x\sqrt{x}} = (x\sqrt{x})^x$, $x > 0$.

19 Find positive integers u and v satisfying

$$\sqrt{18 - 2\sqrt{65}} = \sqrt{u} - \sqrt{v}.$$

20 Two quadratic equations $2011x^2 + 2012x + 1 = 0$ and $x^2 + 2012x + 2011 = 0$ have a root in common. Find the product of the roots that are not in common.

21 Find three prime numbers less than 100 which are factors of $3^{32} - 2^{32}$.

22 If $x_1, x_2, \ldots, x_5$ are the distinct roots of the equation $3x^5 + 8x^4 + 3x^3 + x^2 - 4x + 1 = 0$, find the numerical value of $(1 + x_1)(1 + x_2)\ldots(1 + x_5)$.

23 Find all positive real values of x that satisfy

$$\frac{1}{x + \sqrt{x}} + \frac{1}{x - \sqrt{x}} \leq 1.$$

24 Solve for x: $|x|x - 5x - 6 = 0$.

25 [*This is also a tie breaker question - see below.*] Determine all angles θ with $0 \leq \theta < 2\pi$ such that $\sin^6\theta + \cos^3\theta = 1$.

Tie Breaker requiring Full Solution

Please give a **detailed explanation** on the answer sheet **of** your **solution to Question 25** above.

This tie breaker question is graded as an essay question, i.e. it is graded for the clarity of explanation and argument as well as correctness.

It is the only question graded for partial credit. Do not hesitate to write your thoughts even if your solution is not rigorous!

It is graded only to separate first, second, and third place ties.